INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 12 (2023), No. 4, 89 - 100.

ISOTOMIC CONJUGATE AND PARALLELISM

FRANCISCO JAVIER GARCÍA CAPITÁN

Abstract. We show some nice results of parallelism related to isotomic conjugates and investigate several correspondences bewteen points at line on infinite [1].

1. Cevian triangle and Anticevian triangle

Let ABC be a triangle and P a point. Lines AP, BP, CP meet lines BC, CA, AB at points A', B', C' that is called the *cevian triangle* of P. If A'', B'', C'' are the harmonic conjugates of P with respect to AA',BB', CC', then $A''B''C''$ form the *anticevian triangle* of P, that is, a triangle whose cevian triangle of P is ABC (See Figure 1).

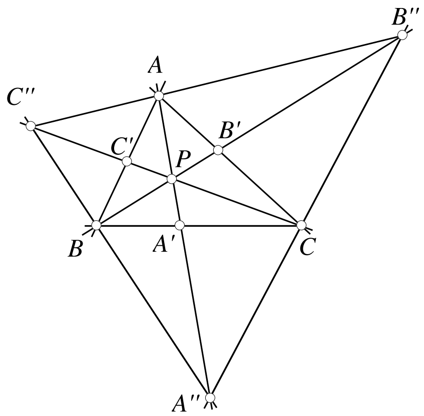

Figure 1

These are two important properties of cevian and anticevian triangles:

- Any cevian triangle is perspective with any anticevian triangle.
- If $A'B'C'$ and $A''B''C''$ are the cevian and anticevian triangles of P then A'', B'', C'' are the harmonic conjugates of P with respect to AA', BB', CC' respectively.

Keywords and phrases: isotomic conjugate, paralelism, McCay cubic,
(2020) Mathematics Subject Classification: 51-02
Received: 05.06.2023. In revised form: 26.08.2023. Accepted: 11.06.2023

[1]We are very thankful to Ivan Pavlov and Elias Hagos [4] for bringing this topic to our attention.

2. Barycentric coordinates

We assume some knowledge about Triangle Geometry and barycentric coordinates. More precisely, we will make use of the following facts:

- If $P = (u_1 : v_1 : w_1)$ and $Q = (u_2 : v_2 : w_2)$ have coordinates with the same sum (weight), then the point J that divides PQ in the ratio $m : n$ has coordinates
$$J = (nu_1 + mu_2 : nv_1 + mv_2 : nw_1 + mw_2).$$
- If $P = (u_1 : v_1 : w_1)$ and $Q = (u_2 : v_2 : w_2)$ have coordinates with the same sum (weight), then the infinite point J of line PQ has coordinates
$$J = (u_1 - u_2 : v_1 - v_2 : w_1 - w_2).$$
- If $P = (u : v : w)$ has cevian triangle $A'B'C'$ and A'', B'', C'' are the reflection of A', B', C' with respect to the midpoints of the corresponding sides then AA'', BB'', CC'' concur and the intersection point P^* is called the *isotomic conjugate* of P, whose coordinates are $P^* = (vw : wu : uv)$.

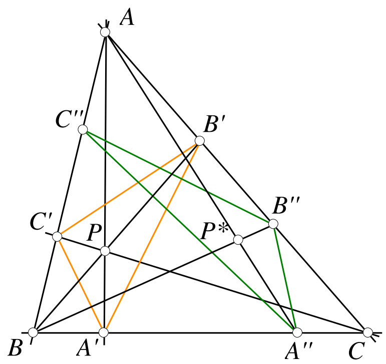

Figure 2

- The locus of points P such that P, the isotomic conjugate P^* of P and another point Q are collinear is a cubic, called the isocubic with pole the centroid G and pivot the point Q. If $Q = (p : q : r)$, then the equation of the cubic $\mathcal{C}(Q)$ is
$$\begin{vmatrix} x & y & z \\ p & q & r \\ qr & rp & pq \end{vmatrix} = 0$$
or
$$p(y^2 - z^2)x + q(z^2 - x^2)y + r(x^2 - y^2)z = 0. \tag{1}$$
We will follow the procedure given by Bernard Gibert in [3] to construct points of a such cubic.

3. A parallelism

Proposition 3.1. *Let ABC be a triangle, P a point with cevian triangle $A'B'C'$, and P^* the isotomic conjugate of P. If G' is the centroid of $A'B'C'$ then lines PP^* and GG' are parallel.*

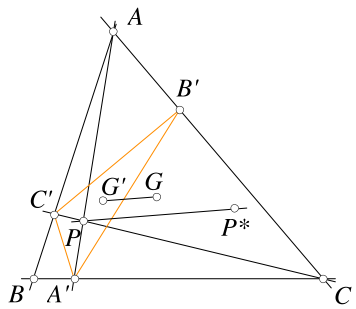

FIGURE 3

Solution. If $P = (u : v : w)$ in homogeneous barycentric coordinates then the vertices of the cevian triangle of P are

$$A' = (0 : v : w)\,, B' = (u : 0 : w)\,, C' = (u : v : 0)\,.$$

We write the B' and C' in the form

$$B' = ((u + v)u : 0 : (u + v)w)\,, C' = ((w + u)u : (w + u)v : 0)$$

with the same sum (weight) $(w + u)(u + v)$. Hence the sum of these coordinates gives the midpoint M' of $B'C'$ as

$$M' = ((2u + v + w)u : (w + u)v : (u + v)w)$$

the weight being $2(w + u)(u + v)$. Now we consider

$$\begin{aligned} A' &= (\mathbf{0} : 2v(w + u)(u + v) : 2w(w + u)(u + v))\,, \\ M' &= (\boldsymbol{u(v + w)(2u + v + w)} : (v + w)(w + u)v : (v + w)(u + v)w)\,, \end{aligned}$$

both with weight $2(v + w)(w + u)(u + v)$, the sum $2M' + A'$ gives the coordinates of the centroid G' of $A'B'C'$, the first of which is obviously $2u(v + w)(2u + v + w)$. Therefore the centroid G' is the point

$$u(v + w)(2u + v + w) : v(u + w)(u + 2v + w) : (u + v)w(u + v + 2w)$$

with sum $3(v + w)(w + u)(u + v)$.

Thus the first coordinate of the infinite point J of GG' can be calculated as

$$\begin{aligned} &(v + w)(w + u)(u + v) - u(v + w)(2u + v + w) \\ =&(v + w)(u^2 + uv + uw + vw - 2u^2 - uv - uw) \\ =&(v + w)(vw - u^2). \end{aligned}$$

and we get

$$J = \big((v + w)(u^2 - vw) : (u + w)(v^2 - uw) : (u + v)(w^2 - uv)\big)\,.$$

On the other hand, if $P = (u : v : w)$, then $P^* = (vw : wu : uv)$, we can consider them in the form

$$\begin{aligned} P &= (\boldsymbol{(vw + wu + uv)u} : (vw + wu + uv)v : (vw + wu + uv)w)\,, \\ P^* &= (\boldsymbol{(u + v + w)vw} : (u + v + w)wu : (u + v + w)uv)\,, \end{aligned}$$

both with sum $(u+v+w)(vw+wu+uv)$. The first coordinate of the infinite point of PP^* is

$$\begin{aligned}&(vw+wu+uv)u-(u+v+w)vw\\=&u^2w+u^2v-v^2w-vw^2\\=&(v+w)(u^2-vw),\end{aligned}$$

that is, this infinite point is J as well and lines PP^* and GG' are parallel.

By symmetry, the centroid G'' of the cevian triangle $A''B''C''$ of P^* also lies on line GG'. Moreover, we have the following nice property:

Proposition 3.2. *The centroid G of ABC is the midpoint of $G'G''$.*

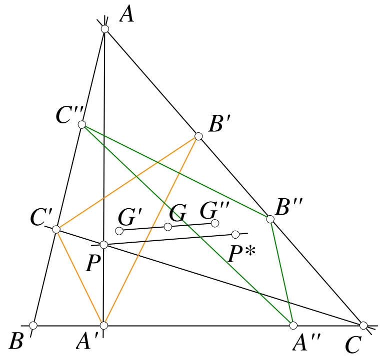

Figure 4

Solution. We substitue $(u:v:w)$ by $(vw:wu:uv)$ in the coordinates of G', and we get the coordinates of G'':

$$\begin{aligned}&vw(wu+uv)(2vw+wu+uv):\ldots:\ldots\\=&uvw(v+w)(2vw+wu+uv):\ldots:\ldots\\=&(v+w)(2vw+wu+wv):\ldots:\ldots\end{aligned}$$

Since the coordinates of G' have sum $3(v+w)(w+u)(u+v)$, the coordinates of G'' have the same sum (the coordinates have been divided by uvw).

Since G' lies on the line through G parallel to PP^*, then, by symmetry, G'' also lies on this line.

Morevoer, we have

$$\begin{aligned}&u(v+w)(2u+v+w)+(v+w)(2vw+wu+uv)\\=&(v+w)(2u^2+2uv+2uw+2vw),\\=&2(v+w)(w+u)(u+v),\end{aligned}$$

therefore, the midpoint of G' and G'' is $(1:1:1)$, the centroid.

4. Locus

According to Proposition 1 we can associate an infinite point J to any point P, namely the infinite point of line PP^*, where P^* is the isotomic conjugate of P.

Conversely, if J is a given infinite point, as we have seen in section 1, P such that J is the infinite point of line joining P and P^* is the cubic $\mathcal{C}(J)$.

These are the steps to construct the cubic $\mathcal{C}(J)$ by points.

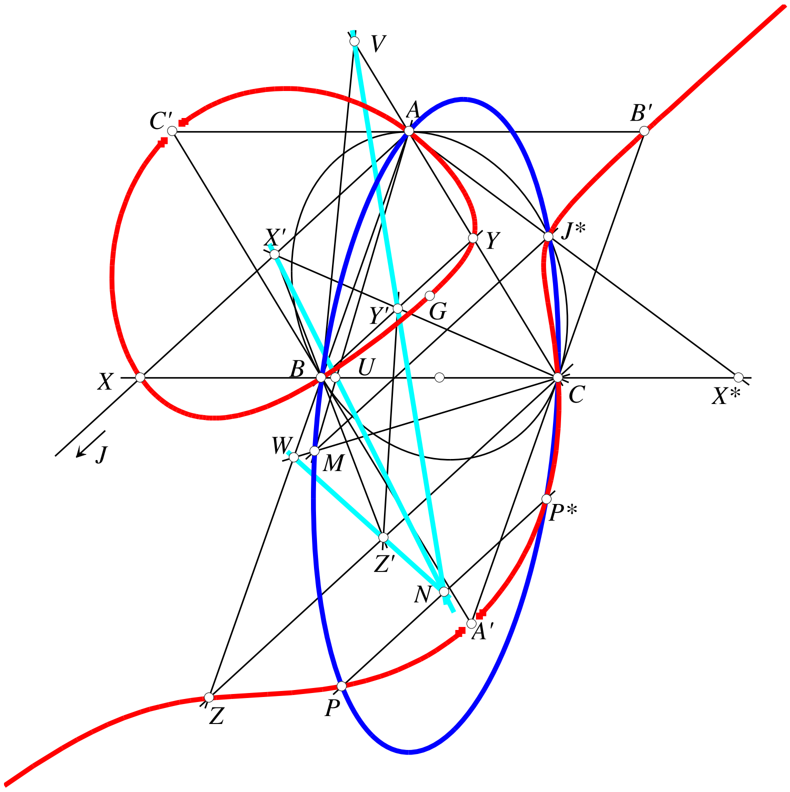

Figure 5

- XYZ is the cevian triangle of J.
- X', Y', Z' are the midpints AX, BY, CZ respectively. Since J is infinite, $X'Y'Z'$ is the anticevian triangle of J.
- We construct the isotomic conjugate J^* of J. In the figure we see J^* lying on Steiner circumellipse and AJ^* intersecting BC at X^*, the reflection of X in the midpoint of BC.
- For any point M on the line through JJ^* we consider the cevian triangle UVW of M and the perspector N of UVW with $X'Y'Z'$.
- The circumconic $ABCMJ^*$ intersects the parallel to JJ^* through N at two points P, P^* on the cubic.

Remarks. 1. The cubic $\mathcal{C}(J)$ always goes through the centroid G and the vertices A', B', C' of the anticomplementary triangle. In all cases the tangent at these points G go through J.

2. In the preceding figure we can observe that N seems to be the midpoint of the two isotomic points P and P^*. We state this property and an interesting locus for N in the following proposition.

We denote by $\mathbb{S}$ the Steiner circumellipse of ABC.

Proposition 4.1. *Let ABC be a triangle and $J = (p : q : r)$ a point. The locus of the midpoint N of pairs of isotomic conjugates P, P^* collinear with J is a cubic with equation*

$$(2) \qquad \sum_{cyclic} \left(p^2(y^2 - z^2)(y + z - x) + 2\left((p + q)ry - (p + r)qz\right) yz\right) = 0.$$

If J is infinite, this cubic factors as

$$(x+y+z)\left((q^2-r^2)x^2+(r^2-p^2)y^2+(p^2-q^2)z^2\right)=0.$$

Proof. If $P=(u:v:w)$ and $P^*=(vw:wu:uv)$ then the midpoint of PP^* is $M=(x:y:z)$ where

$$\begin{aligned} x &= vw(u+v+w)+u(uv+uw+vw)\\ y &= wu(u+v+w)+v(uv+wu+vw)\,.\\ z &= uv(u+v+w)+w(uv+ww+vw) \end{aligned}$$

The cubic equation (2) follows from eliminating u, v, w from these equations together with the condition (1).

We denote by $\mathcal{H}(J)$ the conic with equation

$$(q^2-r^2)x^2+(r^2-p^2)y^2+(p^2-q^2)z^2=0. \tag{3}$$

We highlight some interesting properties of $\mathcal{H}(J)$.

Proposition 4.2. *Let ABC be a triangle and $J=(p:q:r)$ an infinite point. Then we have:*

(1) *$\mathcal{H}(J)$ goes through J, G and it is circumscribed to the anticomplementary triangle of $A'B'C'$ and to the anticevian triangle $X'Y'Z'$ of J.*
(2) *The second infinite point of $\mathcal{H}(J)$ is $J'=(q-r:r-p:p-q)$.*
(3) *The center S of $\mathcal{H}(J)$ lies on $\mathbb{S}$ and the isotomic conjugate S^* of S is the isotomic conjugate of the infinite point of the trilinear polar of J.*

Proof. The anticomplementary triangle $A'B'C'$ has vertices

$$A'=(-1:1:1), B'=(1:-1:1), C'=(1:1:-1).$$

The vertices of the anticevian triangle of J are the points

$$X''=(-p:q:r), Y''=(p:-q:r), Z''=(p:q:-r).$$

We six points easily satisfy (3).

In order to check that $(q-r:r-p:p-q)$ also lies on the conic, we consider the transformation $J\to J'$, $r\to p-q$ and we get the same equation for $\mathcal{H}(J')$:

$$\begin{aligned} &(r-p)^2-(p-q)^2=r^2+p^2-2rp-p^2-q^2+2pq\\ =&r^2-q^2-2p(r-q)=r^2-q^2+2(r+q)(r-q)\\ =&3r^2-3q^2=-3(q^2-r^2). \end{aligned}$$

In this way, by symmetry since J lies on $\mathcal{H}(J)$, then J' lies on $\mathcal{H}(J')$, but $\mathcal{H}(J)$ and $\mathcal{H}(J')$ are the same, therefore J' lies on $\mathcal{H}(J)$.

In coordinates we have

$$S^*=(p(q-r):q(r-p):r(p-q))\,,$$

the infinite point of $qrx+rpy+pqz=0$, the trilinear polar of J.

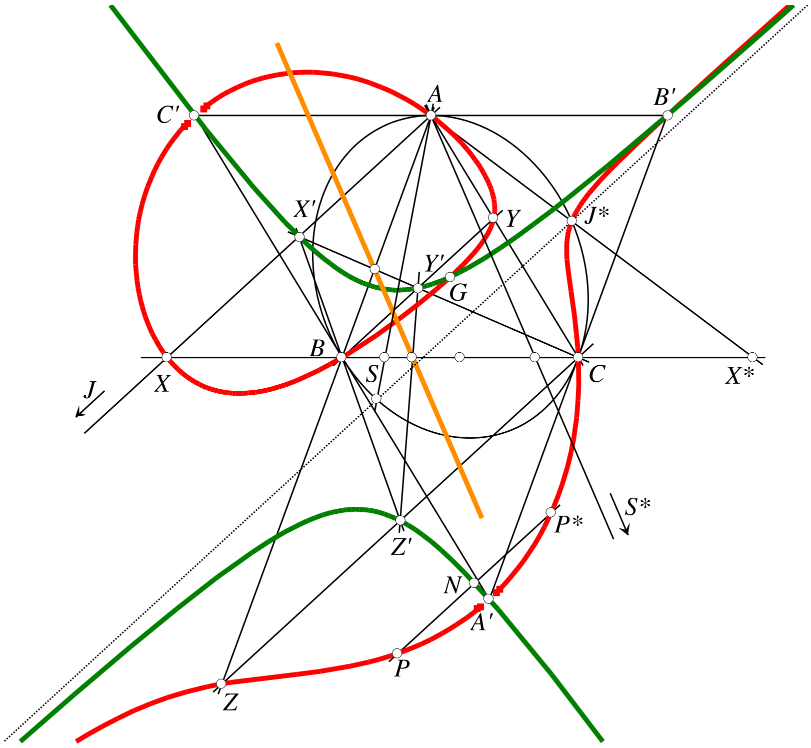

Figure 6

Note that the trilinear polar of J goes through the intersection of $Y''Z''$, $Z''X''$, $X''Y''$ with BC, CA, AB respectively. In the following figure we have drawn the intersection D of BC with a parallel through A to the trilinear polar of J, then we find the reflection D' of D with respect to the midpoint of BC and we get S as the second intersection of $\mathbb{S}$ and AD'.

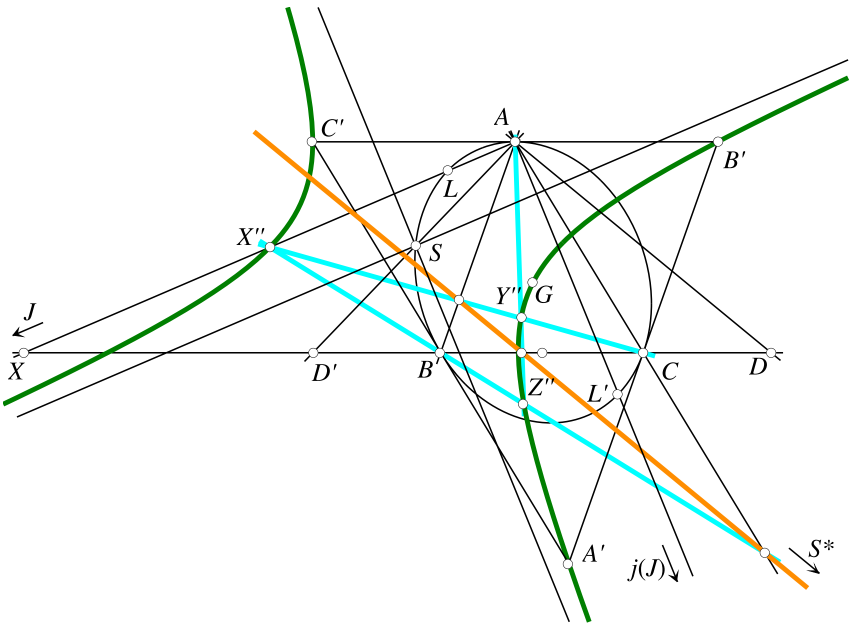

Figure 7

5. Looking for a structure

We have just seen that an infinite point $J = (p : q : r)$ leads to another infinite point S^* whose isotomic conjugate S is the center of a conic whose infinite points are J and $J' = (q-r : r-p : p-r)$. What type of transformation in the line at infinity is $J \to J'$?

Proposition 5.1. *Let ABC be a triangle. The map j sending J to J' is the homography from the line at infinite to itself sending the infinite points of the sidelines to the infinite points of the corresponding medians.*

Proof. We have $j(\infty_{BC}) = j(0:1:-1) = (2:-1:-1) = \infty_{AG}$. Conversely we have $j(\infty_{AG}) = j(2:-1:-1) = (0:-3:3) = \infty_{BC}$, and the same for the other sides.

The following proposition shows a simple construction of j:

Proposition 5.2. *If AJ intersects again $\mathbb{S}$ at L and L' is the antipode of L on $\mathbb{S}$, then J' is the infinite point of AL'.*

Proof. If $J = (p:q:r)$, then $AJ : ry - qz = 0$. Therefore from $r(pq) - q(pr) = pqr - pqr = 0$ and $(pq)(rp)+(rp)(qr)+(qr)(pq) = pqr(p+q+r) = 0$, $L = (qr : rp : pr)$ is the second intersection of AJ and $\mathbb{S}$, with equation $yz + zx + xy = 0$.

We write G and L as $G = (qr+rp+pq : qr+rp+pq : qr+rp+pq)$ and $L = (3qr : 3pq : 3rp)$ and we get

$$\begin{aligned} L' = 2G - L &= (-qr + 2rp + 2pq : 2qr + 2rp - pq : 2qr + rp + 2pq) \\ &= ((2q+r)(q+2r) : -((q-r)(q+2r)) : (q-r)(2q+r)) \end{aligned}$$

thus AL' has equation $(2q+r)y + (q+2r)z = 0$, the infinite point of AL' being $(q-r : q+2r : -2q-r) = (q-r : r-p : p-q) = j(J)$.

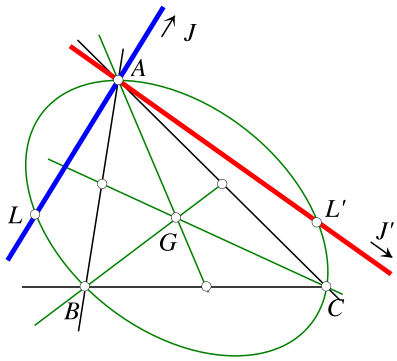

Figure 8

Remarks. 1. The homography j is an involution.

2. J and J' are the same if and only if $q^2 + qr + r^2 = 0$. In other words, the fixed points of j are the (imaginary) infinite points of $\mathbb{S}$.

3. As a result, $\mathcal{H}(J)$ is always a hyperbola.

6. Starting from S

In the preceding proposition we have constructed S for a given J. In fact, since $\mathcal{H}(J) = \mathcal{H}(j(J))$, we would rather name it as $\mathcal{H}(S)$.

We give the following construction for the asymptotes of $\mathcal{H}(S)$ for a given point S on $\mathbb{S}$.

Proposition 6.1. *Let ABC be a triangle and S a point on $\mathbb{S}$. If V and W are the intersections of $\mathbb{S}$ and the polar of the infinite point of the isotomic conjugate of S, then SV and SW are the asymptotes of $\mathcal{H}(S)$, that is the infinite points of $\mathcal{H}(S)$ are the infinite points of lines SV and SW.*

Proof. In barycentric coordinates, if $S^* = (u : v : w)$ is a infinite point, then we look for infinite points $(p : q : r)$ such that

$$p(q-r) : q(r-p) : r(p-q) = u : v : w.$$

We find the solutions

$$J_1 = \left(u : w + \sqrt{u^2+uv+v^2} : v - \sqrt{u^2+uv+v^2}\right),$$
$$J_2 = \left(u : w - \sqrt{u^2+uv+v^2} : v + \sqrt{u^2+uv+v^2}\right).$$

The line joining the isotomic conjugates of J_1 and J_2 is $ux+vy+wz=0$.

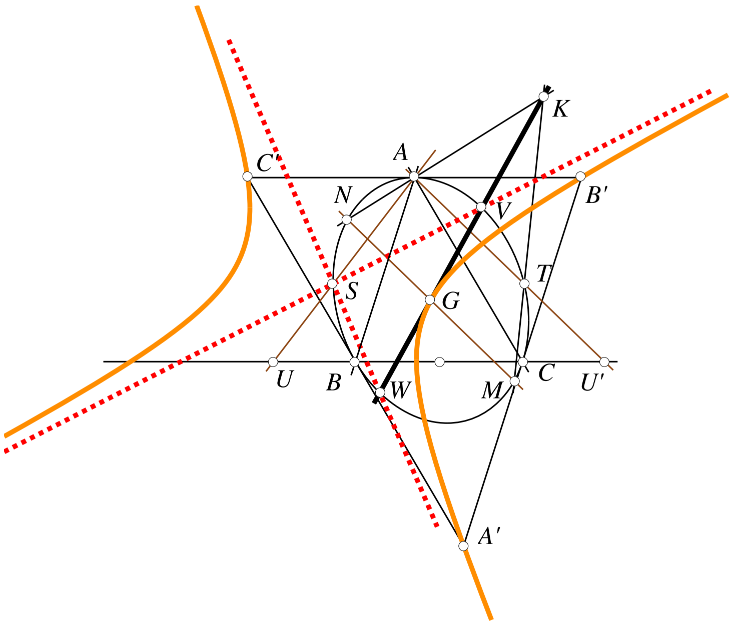

Figure 9

Construction. Join AS intersecting BC at U and construct the reflection U' of U with respect to the midpoint of BC. Join AU', intersecting $\mathbb{S}$ at T. If M, N are the extremities of the diameter of $\mathbb{S}$ parallel to AU', call K the intersection of MT and NA. Then GK is the polar of the infinite point of AU', that is the polar of the isotomic conjugate of S. Let this line intersect $\mathbb{S}$ at V, W. Then SV and SW are the asymptotes of $\mathcal{H}(S)$.

In fact V and W are the isotomic conjugates of the infinite points of $\mathcal{H}(S)$.

We can also construct the reflection T' of T in G and the line VW as the parallel through G to AT'.

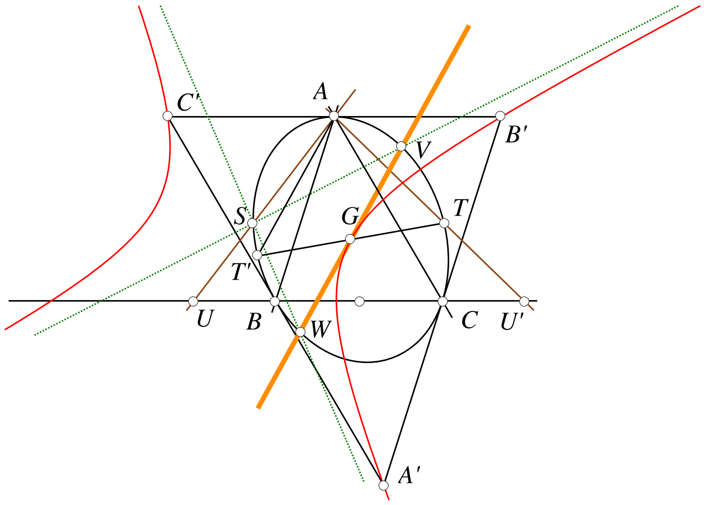

Figure 10

7. Barycenter square

This is another point of view showing the relationship between infinite points and their barycentric squares on the Steiner inellipse.

Proposition 7.1. *Let ABC be a triangle and J an infinite point. If AJ intersects BC at X, draw parallels through X to AB and CA intersecting CA and AB at Y, Z respectively, then find $M = BY \cap CZ$. We have:*

(1) *The midpoint of AM is the barycentric square of J. We denote it by J^2.*

(2) *The point J^2 lies on the Steiner inellipse and the tangent at J^2 is the trilinear polar of J.*

(3) *The center S of $\mathcal{H}(J)$ is the isotomic conjugate of the infinite point of the tangent at J^2 to the Steiner inellipse of ABC.*

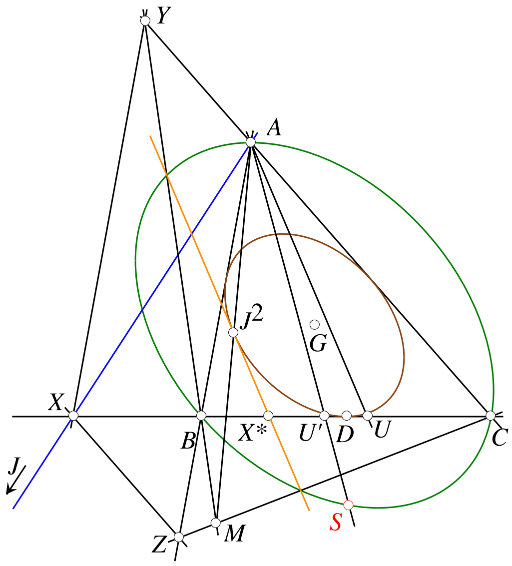

Figure 11

Proof. If $J = (p : q : r)$ then we have the intersection points

$$X = (0 : q : r), \quad Y = (q : 0 : r), \quad Z = (r : q : 0).$$

Then $M = (qr : q^2 : r^2)$. If we write $A = (q^2 + qr + r^2 : 0 : 0)$, then the midpoint of AM is $((q + r)^2 : q^2 : r^2) = (p^2 : q^2 : r^2) = J^2$.

The Steiner inellipse has equation

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0.$$

A substitution $(x : y : z) \to (p^2 : q^2 : r^2)$ gives

$$\begin{aligned} &p^4 + q^4 + r^4 - 2q^2r^2 - 2p^2r^2 - 2p^2q^2 \\ =&(p + q + r)(-p + q + r)(p - q + r)(p + q - r), \end{aligned}$$

therefore J^2 lies on Steiner inellipse when J is infinite.

Construction. Call X^* the harmonic conjugate of X with respect to B, C. Draw a parallel through A to X^*J^2, intersecting BC at U and find the reflection U' of U with respect to the midpoint of BC. Then AU' intersects again $\mathbb{S}$ at S.

8. Another parallelism

This is another result about parallelism and isotomic conjugates that can be easily proved with barycentric coordinates:

Proposition 8.1. *Let ABC be a triangle, P a point and P^* its isotomic conjugate. Let A'', B'', C'' the third intersections of the cubic $\mathcal{C}(P)$ and lines AP, BP, CP, and A''', B''', C''' the third intersections of the cubic $\mathcal{C}(P)$ and lines AP^*, BP^*, CP^*. Then lines $A''A'''$, $B''B'''$, $C''C'''$ are parallel to PP^*.*

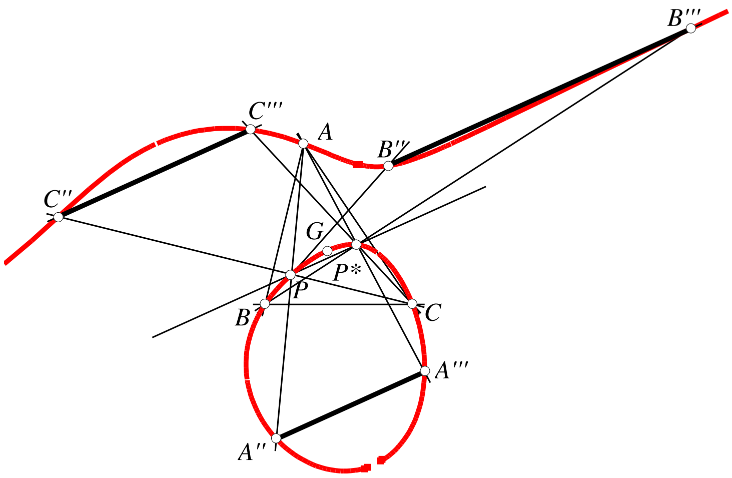

Figure 12

Proof. We find the coordinates

$$\begin{aligned} A'' &= (-vw(u + v + w) : v(uv + uw + vw) : w(uv + uw + vw)), \\ A''' &= (-(uv + uw + vw) : w(u + v + w) : v(u + v + w)) \end{aligned}$$

with sums $u(v^2 + vw + w^2)$ and $v^2 + vw + w^2$ respectively.

We have

$$\begin{aligned}
&u(vw+wu+uv)-vw(u+v+w)\\
=&u^2(v+w)-vw(v+w)=(v+w)(u^2-vw),\\
&v(vw+wu+uv)-uw(u+v+w)\\
=&v^2(w+u)-wu(w+u)=(w+u)(v^2-wu),\\
&w(vw+wu+uv)-uv(u+v+w)\\
=&w^2(u+v)-uv(u+v)=(u+v)(w^2-uv).
\end{aligned}$$

Therefore the point

$$J=\left((v+w)(u^2-vw):(u+w)(v^2-uw):(u+v)(w^2-uv)\right),$$

that is, the infinite point of PP^* is also the infinite point of $A''A'''$.

References

[1] García Capitán, F. J. *Barycentric Coordinates*, pp.32-48. International Journal of Computer Discovered Mathematics, Volume 0, 2015.

[2] García Capitán, F. J. *Posing and Solving Problems with Barycentric Coordinates*, pp.36-44, International Journal of Computer Discovered Mathematics, Volume 1, 2016.

[3] Gibert, B. and Ehrmann, J. P. *Special Isocubics in the Triangle Plane.* `bernard-gibert.pagesperso-orange.fr/files/isocubics.html`

[4] Hagos, E. and Pavlov, I. *Romantics of Geometry.* Problem 12615. `facebook.com/groups/parmenides52/posts/6215597515220569`

DEPARTAMENT OF MATHEMATICS
I.E.S. ÁLVAREZ CUBERO,
14800 PRIEGO DE CÓRDOBA, SPAIN
E-mail address: `garciacapitan@gmail.com`