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PROJECTIVITIES AND CIRCUMCONICS OF A TRIANGLE

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Abstract. In this article we study some conics defined by a projectivity of the plane, which has precisely three ordinary fixed points. We discuss the mutual relations between these conics and the defining them projectivity and prove a characterization of these projectivities reducing their definition to a circle and five distinct points on it. The subject represents a link between projective geometry of the plane and Euclidean geometry, showing in particular an intimate relation between conics circumscribing a triangle and generic projectivities.

1. Introduction

"Projectivities" or *"Projective transformations"* of the real plane are continuous invertible transformations of the plane into itself preserving collinearity ([6, p.52]). The aim of this article is to study the relations between *"generic"* projectivities and *"circumconics"* of triangles, i.e. conics passing through the vertices of the triangle ([9, p.332], [24, p.109]).

By saying *"generic"* we exclude the subgroup of affinities, studied in an earlier article [15] and consider projectivities of the real plane which have precisely three real fixed points, none of them lying at infinity. The three fixed points of the projectivity f define the triangle ABC and its circumcircle κ and we'll see that f defines also a unique pair of points $\{D, E = f(D)\}$ on κ, which, together with the fixed points suffice to completely determine f (see Figure 1).

By the *"fundamental theorem of projectivities of the plane"* ([23, I, p.96]), according to which *"a projectivity is uniquely defined by giving four points and their images"*, the above configuration certainly defines a projectivity fixing the three vertices of the triangle ABC and mapping D to E. The discussion expanded below shows that the converse is also true, i.e. given a

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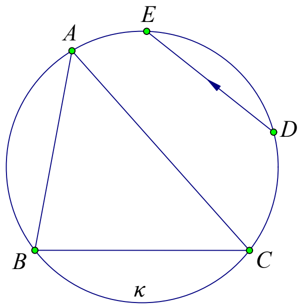

FIGURE 1. The basic configuration defining a *"generic"* projectivity

generic projectivity f, possessing three ordinary fixed points forming a triangle ABC, we can find on the circumcircle, and more general on a circumconic κ of the triangle, a unique pair of points $\{D,E\}$ with the property $f(D)=E$.

Representing the given projectivity f with a matrix, its fixed points correspond to the eigenvectors of the matrix and are readily determined using elementary linear algebra. This defines the triangle ABC and its circumcircle. In the configuration of figure 1 the circle could be replaced by an arbitrary circumconic of the triangle ABC of fixed points of f. The detection of the other two points $\{D,E\}$ for a given projectivity f by our method, involves the consideration of the so-called *"conic of intersections $CoI_f(P)$ "* of f relative to a non-fixed point $P \neq f(P)$ of the projectivity. Figure 2 suggests their definition: As the line ε revolves about a point P, the image-line $\varepsilon' = f(\varepsilon)$ revolves about the image point $P' = f(P)$ and the intersection of the two lines $Q = \varepsilon \cap f(\varepsilon)$ describes a conic $\mu_P = CoI_f(P)$ circumscribing the triangle ABC.

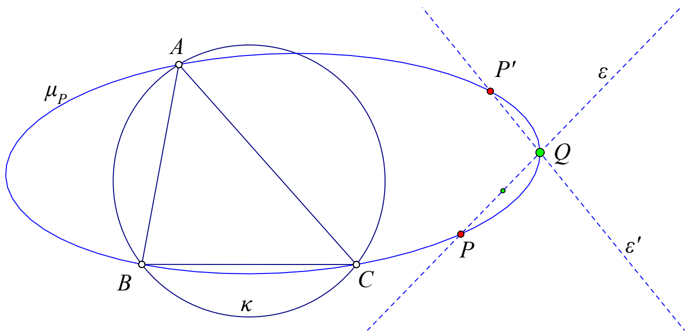

FIGURE 2. A conic of intersections $\mu_P = CoI_f(P)$

The shape of the conic μ_P depends on the selected point P, and varying the location of P we obtain all kinds of conics, genuine, as well as degenerate.

More precisely, it turns out and will be proved below, that the kind of the conic μ_P depends on the relative location of P w.r.t. to a parabola κ_f associated with the projectivity. This parabola is the tangent to the three side-lines of the triangle and the line ε_f, which maps to infinity by

f (see Figure 3). For ordinary points P of the plane, not lying on a side-line of the triangle ABC, the conic μ_P is an ellipse/parabola/hyperbola when P is respectively inside/on/outside the parabola. Given the generic projectivity f, the precious point D on the circumcircle κ, establishing the configuration of figure 1, is the focus of the parabola κ_f, the circumcircle κ coinciding then with $\mu_D = CoI_f(D)$.

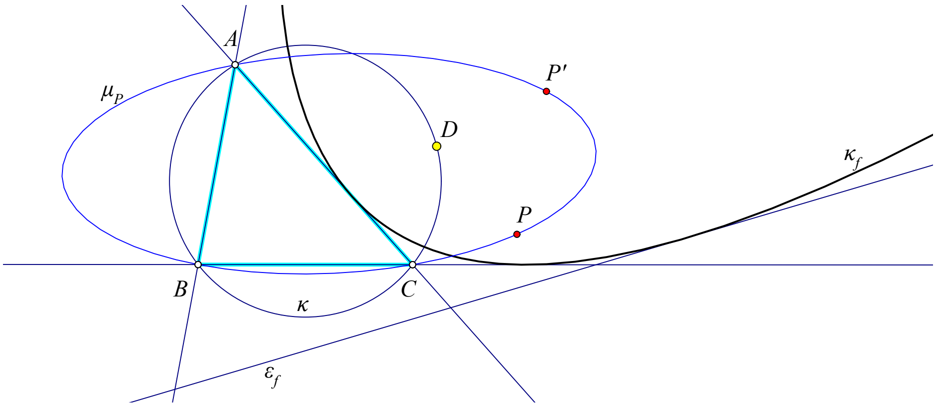

FIGURE 3. The parabola κ_f of the projectivity f

Because of the facts: (i) that every conic of intersections $CoI_f(P)$ passes through the vertices of the triangle ABC, and (ii) that given a generic projectivity f and a particular circumconic κ, every other circumconic can be represented in the form of a $CoI_f(E)$ for an appropriate point $E \in \kappa$, the whole subject represents a link between the projective geometry of the plane and the advanced Euclidean geometry involving *"circumconics"*.

Regarding the organization of the material, we start in section 2 with a short review of projectivities of the projective plane, later in the form of the extended Euclidean plane. In section 3 we take a closer look at the conics of intersection, discussing their basic properties. In section 4 we make a digression in an other non-generic kind of projectivities, the *"homologies"*, and justify why we exclude them in connection with pairs of circumconics.

In section 5 we study the parabola κ_f of a generic projectivity controlling the shape of the various $\{\mu_P = CoI_f(P)\}$. In section 6 we study the particular case in which we consider the circumcircle κ as the fundamental circumconic and relate any other circumconic to a projectivity and vice-versa. In section 7 we fix a generic projectivity f and study the shape of the various circumconics $\mu_P = CoI_f(P)$ of the triangle ABC of its fixed points, in dependence of the location of the point P relative to the hyperbola κ_f associated with f.

In section 8 we throw a glance at projectivities different from affinities, having again precisely three fixed points, but one of them lying at infinity. Two fixed points at infinity, would imply that the line at infinity is invariant and the projectivity reduces to an affinity, which we have excluded. With one fixed point at infinity the triangle ABC becomes *infinite* and all conics $\mu_P = CoI_f(P)$ become hyperbolas and in some cases also parabolas.

The remaining sections 9, 10, 11 and 12 are devoted to applications of the discussion in some concrete examples, which are respectively the *"Steiner ellipse"*, some formulas in computing with *"barycentrics"*, the *"Jerabek hyperbola"* and the general parabolas circumscribing a triangle.

2. Projective plane and generic projectivities

When one considers relations between Euclidean and projective geometry it is natural to use the model of the projective plane resulting from the extension of the Euclidean one through the addition of the *"line at infinity"* ℓ_∞. In this model, the *"ordinary"* points of the Euclidean plane are represented w.r.t. a Cartesian coordinate system by constant multiples of triples of real numbers of the form $(x : y : 1)$ and we denote them by capital letters $\{A, B, \ldots\}$. The *"points at infinity"* making up the *"line at infinity"* ℓ_∞ and extending the Euclidean plane to the projective one are represented by triples of the form $(x : y : 0)$ and, if we want to emphasize their special location, we denote them by $\{[A], [B], \ldots\}$. Triples $(x' : y' : z') = k(x : y : z) = (kx : ky : kz)$ differing by a non-zero multiplicative constant represent the same point. Equation $z = 0$ represents the line ℓ_∞ and its points $(x : y : 0)$ can be identified with pairs $(x : y)$ defining *"direction"* vectors of lines of the Euclidean plane. Thus, for a point at infinity $[A]$ we may speak of the *"direction"* of $[A]$ or the point at infinity where two parallel lines meet. Analogously, the line $B[A]$, i.e. the line through B and $[A]$ is meant to be the line through B in the direction determined by the point at infinity $[A]$.

In this model of the projective plane, a projectivity is represented by an invertible 3×3 real matrix M. Two quadruples of points *"in general position"*, i.e. no three of them collinear, define such a matrix up to a non-zero multiplicative constant, representing a projectivity mapping the ordered points of the first quadruple to corresponding points of the second. In fact, denoting the points by $P_i(x_i : y_i : z_i)$ and their images by $Q_i(x'_i : y'_i : z'_i)$ for $i = 1..4$, the matrix M of the corresponding projectivity is defined by the matrix equations $MP = QK$ and $MP_4 = k_4 Q_4$ with

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \; P = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix},$$

$$Q = \begin{pmatrix} x'_1 & x'_2 & x'_3 \\ y'_1 & y'_2 & y'_3 \\ z'_1 & z'_2 & z'_3 \end{pmatrix}, \; K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \; P_4 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \; Q_4 = \begin{pmatrix} x'_4 \\ y'_4 \\ z'_4 \end{pmatrix}.$$

The matrices $\{M, K, P\}$ are by assumption invertible, this leading to equations

$$M = QKP^{-1} \quad \text{and} \quad (QKP^{-1})P_4 - k_4 Q_4 = 0 .$$

Last equation defines a homogeneous linear system of three equations and four unknowns: the numbers $\{k_1, k_2, k_3, k_4\}$. Its solutions are constant multiples of one particular solution: $\{(k_1, k_2, k_3, k_4) = k(a, b, c, d), k \in \mathbb{R}\}$. This determines the matrix M up to a multiplicative constant, which is compatible with the determination of the projectivity, since points of the projective plane, as we noticed, are represented by coordinate triples $(x : y : z)$

defined up to a multiplicative constant. We denote the projectivity determined through such a matrix by f and also by f_M, if we aim to stress its dependence from the matrix. The line ε_f, sent to infinity by f_M, is determined by the linear equation of the last row of M :

$$\varepsilon_f \ : \ a_3x + b_3y + c_3z = 0 \ .$$

By saying *"generic"*, we assume that the matrix M has three distinct real eigenvalues $\{k_1, k_2, k_3\}$ and corresponding eigenvectors $\{e_1, e_3, e_3\}$ with corresponding triples of coordinates $\{(x_i : y_i : z_i), \ z_i \neq 0, i = 1, 2, 3\}$, i.e. defining *"ordinary"* points of the plane, which are fixed by the projectivity. Next lemma guarantees that such a projectivity f never reduces to an *"affinity"*, later characterized by the fact, that the line at infinity remains invariant under f, equivalently $\varepsilon_f = \ell_\infty$ ([5, p.191], [6, p.98]).

Lemma 2.1. *If the invertible matrix M has three distinct independent eigenvectors corresponding to ordinary points of the plane, i.e.*

$$P_i(x_i : y_i : z_i) \quad \text{with} \quad z_i \neq 0 \quad \text{and} \quad MP_i = k_iP_i \quad \text{for} \quad i = 1, 2, 3 \ ,$$

then the corresponding projectivity f_M is an affinity $(\varepsilon_\infty = \ell_\infty)$, if and only if the three corresponding eigenvalues $\{k_1, k_2, k_3\}$ of M are equal.

Proof. We may pass to the eigenvectors $\{Q_i(s_i : t_i : 1) = (x_i/z_i : y_i/z_i : 1)\}$. Denoting by Q the matrix with columns $\{Q_i\}$ and the diagonal matrix of the $\{k_i\}$ with K, the matrix M, satisfies

$$MQ = QK \quad \Rightarrow \quad M = QKQ^{-1} \ .$$

Doing the matrix multiplications, the third row of the matrix M is found to be

$$(0, 0, 1)M = ((0, 0, 1)Q)KQ^{-1} = (1, 1, 1)KQ^{-1} = (k_1, k_2, k_3)Q^{-1} = V \ .$$

Should V coincide with $(0, 0, k)$ (the coefficients of the line $z = 0$), then we would have $(k_1, k_2, k_3) = (0, 0, k)Q = (k, k, k)$, which proves the lemma.

3. The conics of intersection

As we noticed in the introduction, given a generic projectivity f of the plane, a conic of intersection $\mu_P = CoI_f(P)$ for a non-fixed by f point P, is defined by considering the lines λ through the point P and their images $\lambda' = f(\lambda)$ through $P' = f(P)$. This correspondence of lines $f \ : \ \lambda \mapsto \lambda'$ is a *"homography"* between the pencils $\{P^*, P'^*\}$ of lines through P and of lines through P'. By the Chasles-Steiner principle of generation of conics ([4, p.5], [2, p.72], [10, p. 259]), this implies that the intersections $\{Q = \lambda \cap \lambda'\}$ describe a conic passing through $\{P, P'\}$. Obviously if λ passes through a fixed point X of f, then its image $\lambda' = f(\lambda)$ passes also through X. Thus all conics $CoI_f(P)$ pass also through all the fixed points of f. We formulate these simple facts in the form of a theorem (see Figure 4).

Theorem 3.1. *Given a generic projectivity f with fixed points $\{A, B, C\}$ and a non-fixed point P, the conic of intersections $\mu = CoI_f(P)$ passes through the fixed points and also through the points $\{P, P' = f(P)\}$.*

The following propositions formulate basic properties of the conics of intersections.

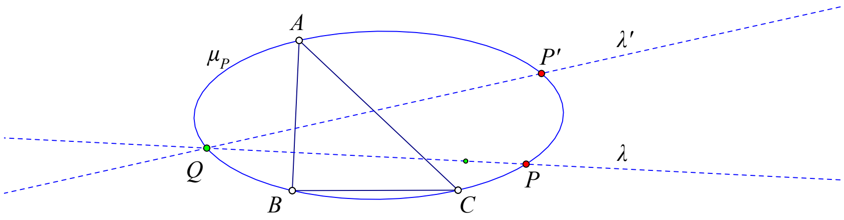

FIGURE 4. The conic $\mu_P = CoI_f(P)$ passing through $\{P, P'\}$

Lemma 3.1. *Given four points $\{A, B, C, D\}$ in general position, every projectivity f fixing the points $\{A, B, C\}$ corresponds bijectively to one point E with the property $E = f(D)$. For such a projectivity f, the conic μ passing through the five points $\{A, B, C, D, E\}$ coincides with $CoI_f(D)$.*

Proof. These are trivial consequences, the first resulting from the *"fundamental theorem of projectivities"* alluded to in the introduction, and the second from the fact, that five points in general position determine a unique conic passing through them.

Lemma 3.2. *Given the generic projectivity f with fixed points $\{A, B, C\}$, a conic μ can be represented as $\mu = CoI_f(P)$ for at most a unique point P.*

Proof. This follows immediately from the property of cross ratio of a pencil $P(UV; WZ)$ of four lines $\{PU, PV, PW, PZ\}$ passing through these five points $\{P, U, V, W, Z\}$ of the conic ([1, p.352]): *" for fixed positions of the points $\{U, V, W, Z\}$ on the conic, the cross ratio $P(UV; WZ)$ is independent of the position of P on the conic"*. Having that, assume that two points $\{P, P'\}$ produce the same conic $CoI_f(P) = CoI_f(P') = \mu$ with image points $\{f(P) = Q, f(P') = Q' \in \mu\}$. Then, since projectivities preserve the cross ratio, we have

$$P'(AB; CP) = Q'(AB; CQ) = P'(AB; CQ) \quad \Rightarrow \quad P = Q$$

and the projectivity fixes four points, hence is the identity and fixes all points of the plane, contradicting the hypothesis.

Corollary 3.1. *A conic $CoI_f(P)$ of a generic projectivity f cannot be invariant under f.*

Proof. In fact, if $\mu_P = CoI_f(P)$ is invariant with $P' = f(P) \in \mu_P = f(\mu_P)$, then for two lines $\{\lambda \ni P, \lambda' = f(\lambda) \ni P'\}$ generating by their intersection $Q = \lambda \cap \lambda'$ the conic μ_P, we'll have

$$\mu_P \ni f(Q) = f(\lambda) \cap f(\lambda') \quad \Rightarrow \quad CoI_f(P) = f(CoI_f(P)) = CoI_f(P').$$

Thus μ_P is represented as $CoI_f(P)$ for two different points P and P', which by the preceding lemma is not possible, for $P' \neq P$. Thus it must be $P' = P$, implying that f fixes $\{A, B, C, P\}$, hence is the identity which is impossible for a conic of intersections.

Lemma 3.3. *The conic $\mu = CoI_f(D)$ passing through the three fixed points $\{A, B, C\}$ of the projectivity f and the points $\{D, E = f(D)\}$ has the following properties (see Figure 5).*

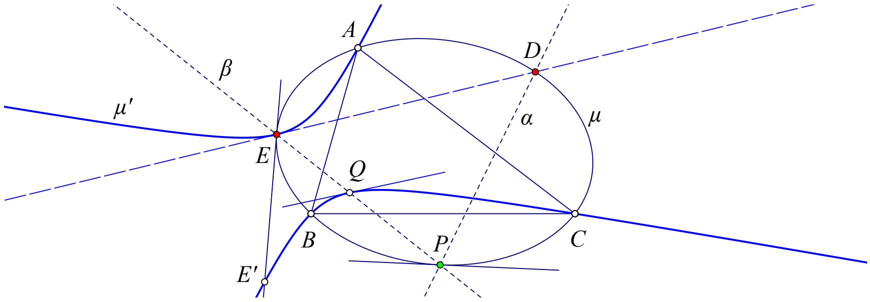

FIGURE 5. Tangents of $\mu = CoI_f(D)$ and $\mu' = f(\mu)$

(1) *$\mu = CoI_f(D)$ maps via f to a conic μ' passing through the four points $\{A, B, C, E\}$ and it is $\mu' = CoI_f(E)$, i.e. the "operators" commute: $f(CoI_f(D)) = CoI_f(f(D))$.*
(2) *If $P \in \mu$ is the intersection $P = \alpha \cap \beta$ of the line α through D and the line $\beta = f(\alpha)$ through E, then the second intersection point Q of μ' with β is the image $Q = f(P)$ of P and the tangent of μ' at Q is the image via f of the tangent of μ at P.*
(3) *The tangent of μ' at E coincides with ED and is the image of the tangent to μ at D.*
(4) *The image point $E' = f(E) = f^2(D)$ is on the tangent to μ at E.*

Proof. *Nr-1* is obvious, since the lines $\{\alpha, \beta\}$ respectively through points $\{D, E = f(D)\}$, generating through their intersection $Q = \alpha \cap \beta$ the conic μ, map via f correspondingly to lines $\{\alpha' = f(\alpha), \beta' = f(\beta)\}$ respectively through $\{f(D) = E, f(E) = E'\}$ also generating through their intersection $Q' = f(Q) = \alpha' \cap \beta'$ the conic $\mu' = CoI_f(E)$.

Nr-2 is also obvious, since for $P \in \alpha \Rightarrow Q = f(P) \in \beta$ and $Q \in \mu' \Rightarrow Q \in \mu' \cap \beta$, and the tangent at P maps to the tangent at Q.

Nr-3 is a consequence of *nr-2*, since, when P tents to D, then line $\alpha = PD$ tends to coincide with the tangent to μ at D, consequently its image $\beta = f(\alpha)$ tends to coincide with the tangent of μ' at E.

Nr-4 is analogous to *nr-3*. As P tends to coincide with E the line EP carrying $Q = f(P)$ tends to the tangent of μ at E and Q tends to E'.

Remark 3.1. *The preceding lemma shows that the projectivity f restricted to the points of μ coincides with a kind of projection $P \mapsto Q$ along lines β through E. Nr-4 of the lemma shows also that this map does not fix E, but maps it to another point E', hence is not a "perspectivity" (*[5, p.242]*).*

It is well known ([23, I, p.213]), that having two conics $\{\mu, \mu'\}$ and selecting three points $\{A, B, C\}$ and $\{A', B', C'\}$ on each, there is a unique projectivity mapping μ to μ' and $\{A, B, C\}$ respectively to $\{A', B', C'\}$. In particular, we have the following corollary.

Corollary 3.2. *Two conics $\{\mu, \mu'\}$ passing through the same three points $\{A, B, C\}$ in general position define a unique projectivity f mapping μ to μ' and fixing these three points.*

Next lemma formulates a method to find this projectivity f and to express the two given conics in the form of $CoI_f(X)$ for appropriate points $X \in \mu$.

Lemma 3.4. *Given two circumconics $\{\mu, \mu'\}$ of the triangle ABC intersecting also at a fourth point E, different from $\{A, B, C\}$, there is a unique projectivity f fixing the vertices of the triangle, mapping μ onto μ'. The point $D = f^{-1}(E) \in \mu$ is the second intersection with μ of the tangent to μ' at E, and the two conics can be represented in the form $\mu = CoI_f(D)$ and $\mu' = CoI_f(E)$.*

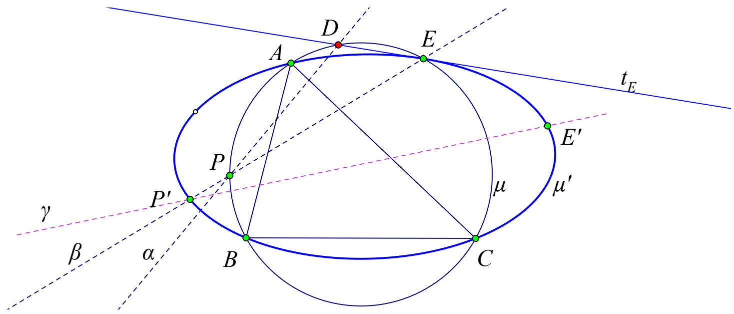

Figure 6. Projectivity which maps μ onto μ'

Proof. We define f by the properties (i) to fix $\{A, B, C\}$ and (ii) to map D to E. The conic μ coincides with $CoI_f(D)$ since both conics pass through the same five points $\{A, B, C, D, E\}$. Also the conic μ' coincides with $CoI_f(E)$ since both pass through the four points $\{A, B, C, E\}$ and, by the preceding lemma, have at E the same tangent.

Theorem 3.2. *Let μ be a circumconic of the triangle ABC. Then, every projectivity f fixing no other point than $\{A, B, C\}$, defines a unique pair $\{D, E\}$ of points of μ, such that $E = f(D)$ and $\mu = CoI_f(D)$.*

Proof. Consider the conic $\mu' = f(\mu)$, and its fourth intersection E with μ (in section 4 we show that E is different from $\{A, B, C\}$, and the conics $\{\mu, \mu'\}$ have no common tangent at E). Let also D be the second intersection point of μ with the tangent t_E of μ' at E (see Figure 6). Define the projectivity f' fixing the points $\{A, B, C\}$ and mapping D to E. As in the proof of the preceding lemma, $f'(\mu)$ and μ' have the points $\{A, B, C, E\}$ in common and their tangents at E coincide, hence the conics coincide and $f'(\mu) = f(\mu) = \mu'$. It follows that $g = f^{-1} \circ f'$ is a projectivity mapping the conic μ onto itself and leaving fixed the three points $\{A, B, C\}$. This implies that g is the identity on μ, hence the identity on the whole plane. This is easily seen by considering again the invariance of the cross ratio on μ by g. For a point $X \in \mu$ and its image $X' = g(X)$ we have for the cross ratio on μ : $(AB; CX) = (AB; CX') \Rightarrow X' = X$. Thus, f and f' coincide everywhere and $f(D) = f'(D) = E$.

4. Digression on homologies

In our discussion in the preceding section we assumed that circumconics of the triangle ABC intersect each other, besides the vertices $\{A, B, C,\}$ also at a fourth point E, different from these three. There are though cases, like the one of figure 7, in which the two circumconics of ABC are tangent at one of the three vertices.

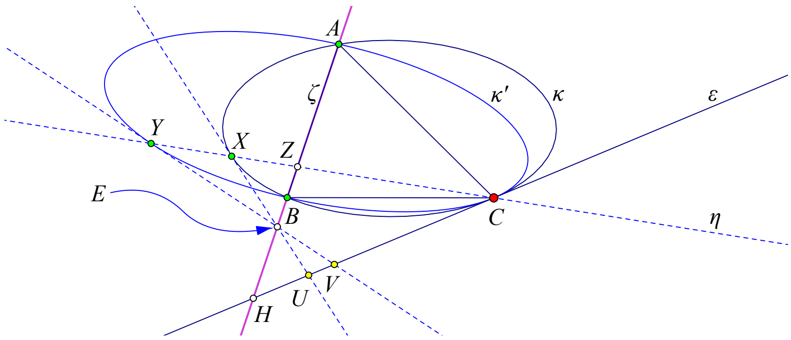

FIGURE 7. Two circumconics of triangle ABC tangent at C

Certainly, two different conics $\{\kappa, \kappa'\}$ cannot pass through $\{A, B, C\}$ and also through a fourth point E, different from these three points, and have at E a common tangent. This because of the fact, that *"through four distinct points in general position and a line through one of them, passes a unique conic tangent to this line"* ([16, § 8.1]), which would imply that the conics coincide. Thus, tangency of circumconics may appear only as in the above configuration, where the contact of the conics occurs at a vertex of the triangle.

In this case we can again prove that there is a projectivity f mapping κ to κ', but this is not *"generic"* with only three distinct fixed points. It has instead infinite many fixed points, in figure 7 their set coinciding with the line $\zeta = AB$ and the isolated point C. This kind of projectivity is called *"homology"* ([7, p.53], [8, p.17]). It is characterized by a line, like ζ, of fixed points, called *"axis of the homology"* and an additional fixed point like the point C, called *"center of the homology"*. In addition hold the properties: (i) every line through the center remains invariant by the homology, and (ii) every point X maps to Y, later lying on the line CX, which intersects the axis at Z and the cross ratio $(XY; ZC) = k$ is constant, independent of the position of X. The constant k is called *"coefficient or parameter"* of the homology ([8, p.63]). Next theorem establishes the kind of projectivity relating the conics κ and κ' of figure 7.

Theorem 4.1. *Two circumconics of the triangle ABC, tangent at the vertex C, are related by a homology $\kappa' = f(\kappa)$ with axis the side-line AB and center the vertex C of the triangle.*

Proof. By corollary 3.2 we have a projectivity f mapping κ onto κ' and fixing the points $\{A, B, C\}$. Since $\{A, B\}$ are fixed the side-line $\zeta = AB$ remains invariant under f. Also the tangent ε, by assumption maps via f onto itself. It follows that the intersection point $H = \varepsilon \cap \zeta$ is also fixed, implying that every point of the line ζ is a fixed point for f.

Let the line η through C intersect the conics $\{\kappa, \kappa'\}$ and the line ζ respectively at the points $\{X, Y, Z\}$. Since $\{C, Z\}$ are fixed by f, the line

$\eta = CZ$ is invariant by f and $Y = f(X)$. To accomplish the proof that f is a homology, we show that $(XY;ZC) = k$ is a constant independent of the position of $X \in \kappa$.

For this we show first, that the tangents at $\{X,Y\}$ intersect at a point E of ζ using the invariance of the cross ratio by projectivities. In fact, $(AB;CX)$ on κ maps to $(AB;CY)$ on κ', hence the two cross ratios are equal. But using the tangent at X and measuring the cross ratio on κ through the pencil $X(AB;CE)$ we see that this is equal to $(AB;DE)$. Analogously using the tangent at Y and measuring the cross ratio $(AB;CY)$ on κ' through the pencil Y we find this equal to $(AB;DE')$ for a point E' on ζ. Since $(AB;CX) = (AB;CY)$ we conclude that $E' = E$, as claimed.

Now, the pencil of lines through E relates the cross ratios $(XY;ZC) = (UV;HC)$. But U is the pole of line η w.r.t. κ and the map $\eta \mapsto U$ and $U = f_1(\eta)$ is a homography of the pencil C^* of lines through C to the line ε ([20, § 9]). Analogously V is the pole of η w.r.t. κ' and the map $V = f_2(\eta)$ is also a homography of C^* onto the line ε. It follows that the composition $V = g(U) = f_2(f_1^{-1}(U))$ is a homography mapping the line ε onto itself. Besides, we see easily that for $U = H$ i.e. when X is the contact of the other than ε tangent from H to κ, then $V = U = H$ and when $X = C$, then $U = V = C$. This means that $\{H,C\}$ are fixed points of the homography $V = g(U)$ and ([20, § 3]) $(UV;HC) = k$ is constant, thereby completing the proof that f is a homology.

Remark 4.1. *We notice that in the case of homologies all the $CoI_f(P)$ for P non fixed by f are generated by intersections of lines α through P and $\beta = f(\alpha)$ through $P' = f(P)$ intersecting at the axis of the the homology, consequently all these conics contain the homology axis and consist of the union of it and the line PP'.*

Taking into account the discussion in this section, lemma 3.4 and theorem 3.2 we deduce the following corollary.

Corollary 4.1. *Fixing a circumconic μ of the triangle ABC, there is a bijective correspondence of the set $\mathcal{C}$ of conics μ' circumscribing the triangle ABC and intersecting μ at a point E different from $\{A,B,C\}$, and the set $\mathcal{P}$ of projectivities fixing no other point than $\{A,B,C\}$. The set $\mathcal{P}$ of these projectivities in turn is in bijective correspondence with the set $\mathcal{S}$ of pairs (D,E) of points of μ different from $\{A,B,C\}$.*

5. The parabola of the generic projectivity

The parabola associated to a given generic projectivity f, as we noticed in the introduction, can be defined by its property to be tangent to the side-lines of the triangle ABC of fixed points and also tangent to the line ε_f sent to infinity by f. In this section we proceed however to an alternative definition, more appropriate for immediate deduction of the properties to be discussed below.

Theorem 5.1. *Given the generic projectivity f, we consider the line ε_f sent to infinity by f. For each point $P \in \varepsilon_f$ let λ_P be the line joining P with $f(P) = [P'] \in \ell_\infty$. The envelope of all these lines $\{\lambda_P\}$ is a parabola*

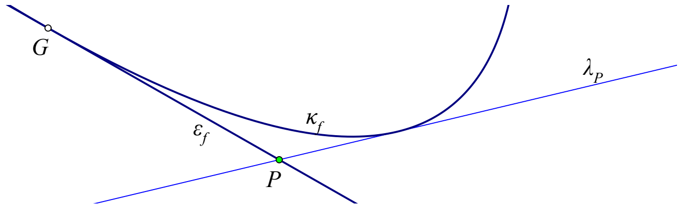

FIGURE 8. Parabola κ_f enveloping λ_P for $P \in \varepsilon_f$

κ_f, tangent to ε_f, tangent to ℓ_∞ and tangent to the side-lines of the triangle of the fixed points ABC.

Proof. The proof is a direct application of the theorem of Chasles-Steiner ([4, p.6]), according to which *"the lines joining points on two fixed lines $\{P \in \alpha, h(P) \in \beta\}$, corresponding under an homography $h : \alpha \to \beta$, envelope a conic tangent to $\{\alpha, \beta\}$"*. The two lines here are ε_f and ℓ_∞. The homography, mapping the first line to the second, is the restriction of f on ε_f. Since a fundamental property of f is that it preserves the cross ratio of points on a line, the restriction of f on ε_f has the same preservation property between the points of ε_f and ℓ_∞. This implies [3, I, p.130] that the restriction of f on ε_f defines a homography $h : \varepsilon_f \to \ell_\infty$ and the theorem of Chasles-Steiner applies, proving that the envelope of lines $\{\lambda_P, P \in \varepsilon_f\}$ is a conic. Since the conic is tangent to the line at infinity ℓ_∞, it is a parabola ([21, p.235]).

To see that this parabola is tangent also to the sides of the triangle ABC of fixed points of the projectivity f, consider one side-line, AB say. This intersects ε_f at a point Q. Since $\{A, B\}$ are fixed points of f, the line maps to itself under f. Thus $Q \in AB$ maps to $[Q'] = f(Q) \in AB$ too, which implies that AB coincides with line $Q[Q']$ and is one of the tangents of the parabola. Analogously is proved the tangency of the other side-lines of the triangle ABC.

Theorem 5.2. *The image-conic $\kappa'_f = f(\kappa_f)$ of the parabola κ_f is also a parabola tangent to the sides of the triangle ABC of the fixed points of f and tangent also to the image-line $\delta_f = f(\ell_\infty)$ of the line at infinity. Further κ'_f coincides with the parabola κ_g of the inverse projectivity $g = f^{-1}$ (see Figure 9).*

Proof. Since projectivities preserve conics, $\kappa'_f = f(\kappa_f)$ is a conic. Since the line ε_f, tangent to κ_f at the point G maps to the line at infinity ℓ_∞, later is tangent to κ'_f at $[G'] = f(G)$, hence it is a parabola. Since ℓ_∞ is tangent to κ_f its image $\delta_f = f(\ell_\infty)$ is tangent to κ'_f. Finally, since the side-lines of the triangle ABC pass, each through two fixed points of f, they are invariant under f hence tangent also to κ'_f.

The coincidence of κ_g with $\kappa'_f = f(\kappa_f)$ follows from the fact that both are tangent to the three side-lines of the triangle and the line δ_f, and the fact that there is a unique parabola with these properties ([16, p.324]).

Corollary 5.1. *With the notation and conventions adopted so far, the following properties are valid.*

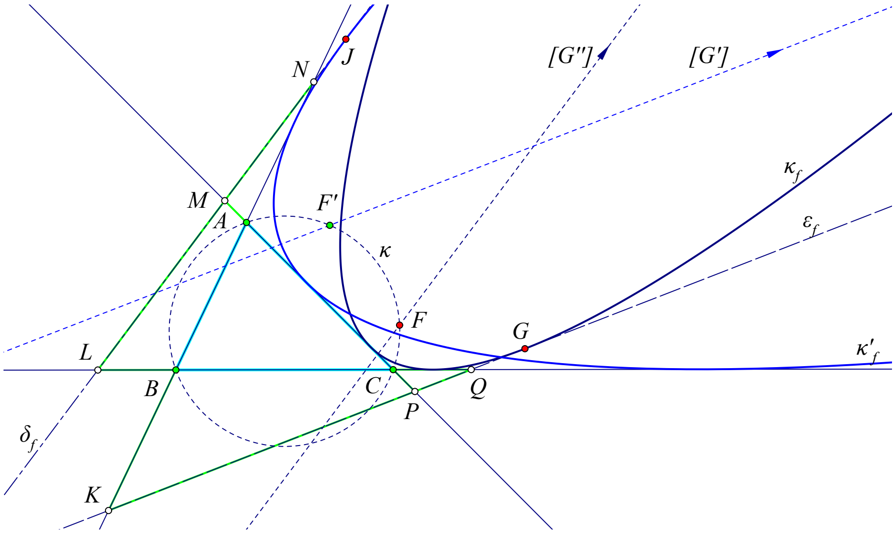

FIGURE 9. The parabolas κ_f and $\kappa'_f = f(\kappa_f)$

(1) *The contact point G with the parabola κ_f, of the line ε_f sent to infinity by f, maps via f to the point at infinity of the line ε_f which is $[G'] = f(G) = \varepsilon_f \cap \ell_\infty$.*
(2) *The image point $[G''] = f([G']) = f(f(G)) = f^2(G)$ is also a point of ℓ_∞.*
(3) *The parallels $\{\lambda_t\}$ to a tangent $\lambda = Qf(Q)$ for $Q \in \varepsilon_f$ of the parabola κ_f map via f to lines $\{\lambda'_t\}$ passing through the fixed point $Q'' = f(f(Q)) = f^2(Q) \in \delta_f$.*
(4) *The parallels $\{\varepsilon_t\}$ to ε_f map via f to lines $\{\varepsilon'_t\}$ which pass through the point $[G''] \in \ell_\infty$, hence are pairwise parallel.*
(5) *The focal points F and F' of the parabolas κ_f and κ'_f are points of the circumcircle κ of the triangle ABC of fixed points of f (see Figure 9).*
(6) *The axis of the parabola κ_f is parallel to δ_f, whose direction is determined by the point at infinity $[G''] = f^2(G)$ and the axis of the parabola κ'_f is parallel to ε_f, whose direction is determined by the point at infinity $[G'] = f(G)$.*

Proof. *Nr-1* follows directly from the definition of the parabola κ_f and the fact that ε_f is tangent to the parabola. The image point $[G'] = f(G) \in \ell_\infty$ must be such that $G[G']$ is tangent κ_f. But the tangent to κ_f at G is already this line ε_f, hence $[G'] \in \varepsilon_f$ and $[G'] = \varepsilon_f \cap \ell_\infty$.

Nr-2 follows from the definition of the line ε_f and the fact that ℓ_∞ is also a tangent to the parabola κ_f. Since $[G'] \in \varepsilon_f$, by the definition of ε_f, its image $[G']' = f([G'])$ will be a point at infinity.

Nr-3 follows from the fact that the parallels $\{\lambda_t\}$ to $\lambda = Qf(Q)$ pass through the point at infinity of λ which is $[Q'] = f(Q)$. Hence their images via f pass all through the point $Q'' = f(f(Q)) \in \delta_f$.

Nr-4 is a direct consequence of *nr-2* and *nr-3*.

Nr-5 is a well-known result of the triangle geometry, according to which ([21, p.130]) *"the circumcircle of a triangle circumscribing a parabola passes through the focus of the parabola"*. For this, related material and references see [16, p.324].

Nr-6 For κ'_f this follows from the fact, mentioned in theorem 5.2, that κ'_f is tangent to the line at infinity ℓ_∞ at $[G'] = f(G)$, which defines the direction of the axis of this parabola and is also the direction of $\varepsilon_f = G[G']$.

For κ_f consider the contact point J of δ_f with the parabola $\kappa'_f = f(\kappa_f)$. Again $f^{-1}(J) = \delta_f \cap \ell_\infty = f(\ell_\infty) \cap f(\varepsilon_f) = f(\ell_\infty \cap \varepsilon_f) = f(f(G)) = [G'']$ is the contact point of $f^{-1}(\kappa'_f) = \kappa_f$ with the line at infinity determining the direction of the axis of κ_f.

Corollary 5.2. *The parallels to ε_f, whose direction is determined by the point $[G'] = f(G)$ map via f to parallels to the axis (diameters) of the parabola κ_f, whose direction is determined by $[G''] = f^2(G)$. The lines through G map via f to parallels to the axis of κ'_f, whose direction is determined by $[G'] = f(G)$.*

Remark 5.1. *Notice that* $J = f(\kappa_f) \cap f(\ell_\infty) = f(\kappa_f \cap \ell_\infty) = f([G'']) = f(f^2(G)) = f^3(G)$.

Lemma 5.1. *With the notation and the conventions adopted so far, a line α through the point P maps via the generic projectivity f to a line $\beta = f(\alpha)$ parallel to α, if and only if P belongs to a tangent of κ_f.*

Proof. The tangents to κ_f are characterized by two points $\{Q \in \varepsilon_f\ ,\ [Q']\} = f(Q) \in \ell_\infty$. If P is in such a line $P \in \alpha = Q[Q']$, then $\beta = f(\alpha) \ni [f(Q)] \in \ell_\infty$ shares the same point at infinity with α hence is parallel to it.

Conversely if $\beta = f(\alpha)$ is parallel to $\alpha \ni [f(Q)]$ for some $Q \in \varepsilon_f$, then β will pass through the same point at infinity $\beta \ni [f(Q)]$ of α. Hence $Q \in f^{-1}(\beta) = \alpha$, and α will contain both Q and $f(Q)$, consequently will coincide with a tangent to ε_f.

Remark 5.2. *Referring to section 4 and figure 7, we notice that the line ε_f sent to infinity by a homology is parallel to its axis. Also, for a homology coefficient $k = (XY; ZC)$ this parallel is at distance $k \cdot d$ from the center of the homology, where d is the distance of the axis from the center. In this case all the lines $\{Xf(X), X \in \varepsilon_f\}$ pass through the center C, their envelope reducing to this point.*

6. A circular conic of intersections

In the preceding section we defined the parabolas κ_f of the generic projectivity f and κ'_f of f^{-1} and showed that they are tangent to the triangle ABC with vertices the fixed points of f. In addition, κ_f is tangent to the line ε_f sent to infinity by f and κ'_f is tangent to the line δ_f sent to infinity by f^{-1}. The aim in this section is to find the relation between the focal points $\{F, F'\}$ of the two parabolas shown in corollary 5.1 to lie on the circumcircle κ of the triangle ABC (see Figure 10), and show that they define the unique circular $CoI_f(P)$ of the projectivity f.

To start with, we consider the *"focal chord"* (passing through the focus) GV of κ_f, which by corollary 5.2 maps to a parallel to the axis of κ'_f. By a well known property of parabolas [1, p.137] *"the tangents at the extremities of a focal chord intersect orthogonally at a point of the directrix"*. Thus, the tangent $t = VW$ at the extremity V of the focal chord is orthogonal to the tangent ε_f at G. From lemma 5.1 we have also that the image $t' = V'W'$ via f of this tangent is parallel to t, hence also orthogonal to ε_f. Thus,

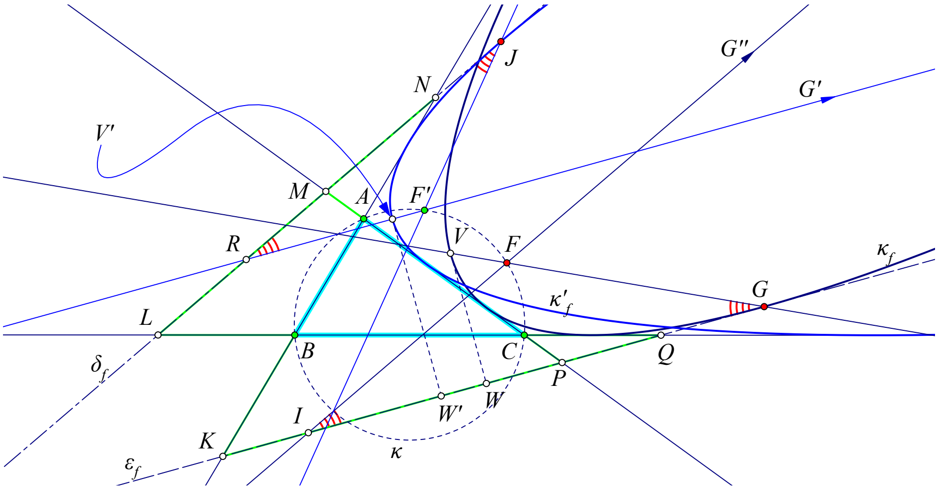

FIGURE 10. Focus F of κ_f maps to the focus F' of κ'_f

at $V' = f(V)$ the two image lines $\{f(VG)\ ,\ V'W'\}$ are, the first parallel to the axis of κ'_f and the second is tangent to κ'_f at V' and orthogonal to the axis of κ'_f. This implies that V' is the vertex of the parabola κ'_f and the line $f(VG)$ coincides with the axis of κ'_f consequently passes through the focus F' of this parabola.

Applying the same arguments to the focal chord of κ'_f contained in the line JF', we see that this maps via f^{-1} to the axis FI of κ_f. Hence, conversely f maps the axis of κ_f to the focal chord along the line $F'J$. In total, the projectivity maps the lines $\{FG, F[G'']\}$ correspondingly to the lines $\{F'[G'], F'J\}$ consequently the intersection F of the first pair of lines to the intersection F' of the second pair. This proves next theorem.

Theorem 6.1. *With the notation and conventions adopted so far, the generic projectivity f maps the focus F of the parabola κ_f to the focus F' of the parabola κ'_f.*

Since two projectivities coinciding at four points coincide everywhere we have also

Corollary 6.1. *The given generic projectivity f coincides with the one fixing the vertices of the triangle ABC and mapping the focus F of the parabola κ_f to the focus F' of the parabola κ'_f.*

Corollary 6.2. *The $CoI_f(F)$ of the generic projectivity f w.r.t. the focus F of the parabola κ_f coincides with the circumcircle κ of the triangle ABC of fixed points of F.*

Proof. Obviously, since both the circumcircle κ of the triangle ABC and $CoI_f(F)$ pass through the five points $\{A, B, C, F, F'\}$.

Corollary 6.3. *The circumcircle κ of the triangle ABC of the fixed points of the generic projectivity f is the unique circular conic of intersections of f and for every point P different from the focus F of the parabola κ_f, the $CoI_f(P)$ is a circumconic of ABC different from κ.*

Proof. This is a direct consequence of lemma 3.2.

7. The shape of the conic of intersections

As we noticed already, the shape of the $CoI_f(P)$, for the generic projectivity f and a non-fixed point $P \neq f(P)$, depends on the location of the point P :

Theorem 7.1. *The shape of the $CoI_f(P)$ depends on the location of P relative to the parabola κ_f of the projectivity f and is:*

(1) *an ellipse/parabola/hyperbola if P is a generic ordinary point respectively inside/on/outside the parabola κ_f.*
(2) *a degenerate conic consisting of two lines, if P is an ordinary generic point on a side-line of the triangle ABC.*
(3) *a hyperbola and in one case a parabola if the point P is at infinity and not on the side-lines of the triangle ABC.*

Proof. *Nr-1* follows directly from lemma 5.1. If the generic ordinary point P of the plane is inside/on/outside the parabola κ_f, then there are respectively none/one/two tangents passing through P. Consequently there are correspondingly none/one/two lines through P that have corresponding parallel image-lines through $P' = f(P)$. This implies that the corresponding $CoI_f(P)$ has respectively none/one/two points at infinity and proves the claim.

Nr-2 comprises two cases. The first concerning ordinary points of the side-lines of ABC and the second concerning a point at infinity: the intersection point of a side-line with the line at infinity.

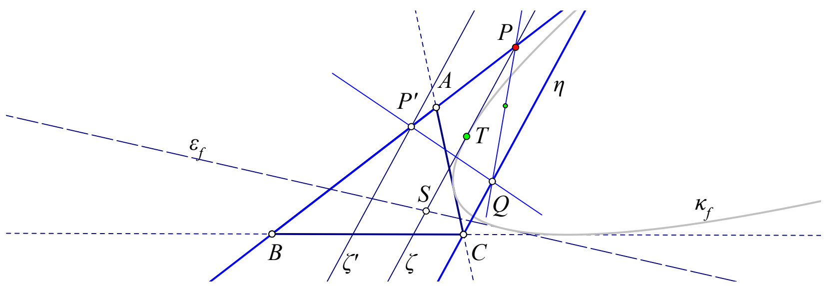

Figure 11. Degenerate conic $CoI_f(P) = AB \cup \eta$ for $P \in AB$

In the first case shown in figure 11 the conic is degenerate. Point $P \in AB$ is outside the parabola κ_f and there is, besides AB a second tangent ζ to κ_f through P. By lemma 5.1 the image line $\zeta' = f(\zeta)$ is then parallel to ζ. The direction of the two parallels is determined by the point at infinity $f(S)$, where $S = \zeta \cap \varepsilon_f$. The conic consists of the union of lines $CoI_f(P) = AB \cup \eta$, where η the parallel to ζ through the third vertex C of the triangle. The line η takes the position of the parallel to AB through C when P coincides with the contact point of AB with the parabola κ_f.

In the second case of *nr-2* $[P]$ is the point at infinity of the side-line AB. Then the lines through $[P]$ are parallels $\{\sigma\}$ to AB and their images $\{\sigma'\}$ are lines through the image point $P' = f([P])$ lying on AB (see Figure 12). If the line σ intersects ε_f at S, then $[\sigma'] = f(\sigma)$ contains $[S'] = f(S) \in \ell_\infty$

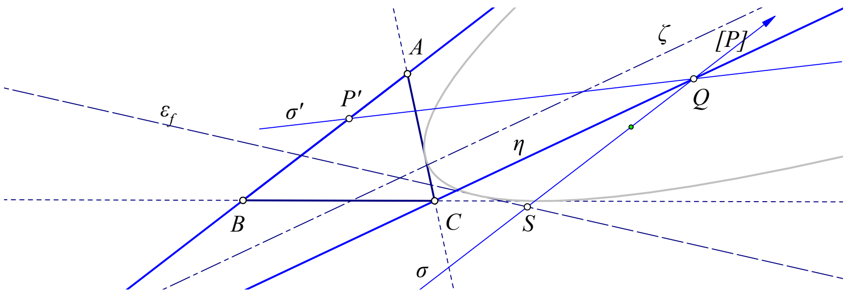

FIGURE 12. Degenerate conic $CoI_f([P]) = AB \cup \eta$ for $[P] = AB \cap \ell_\infty$

hence points in the direction of $[S']$, which is the direction of the other than ε_f tangent to κ_f from S. When S takes the position of $[G'] = \varepsilon_f \cap \ell_\infty$, then $[S'] = f([G']) = [G''] \in \ell_\infty$ is the direction of the axis ζ of κ_f. Thus, the conic $CoI_f([P])$ consists in this case of the union of lines $AB \cup \eta$, where $\eta = CG''$ is the parallel to the axis of the parabola κ_f through C.

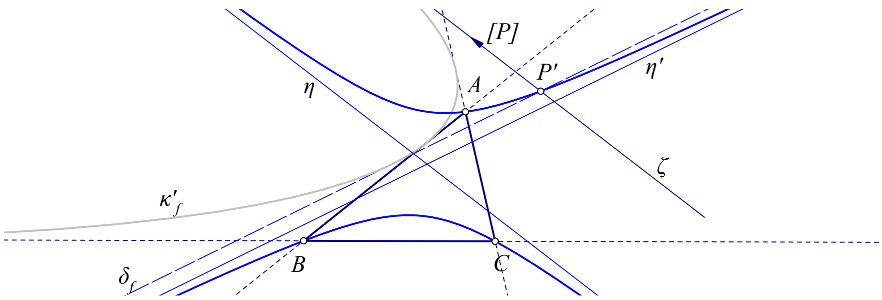

FIGURE 13. Hyperbola $CoI_f(P)$ for a generic point $P \in \ell_\infty$

Nr-3 comprises three cases. Generic points on the line at infinity and the special cases of $[G']$ and $[G'']$.

In the first case $[P]$ is a generic point at infinity and $P' = f([P]) \in \delta_f$. The conic $CoI_f([P])$ passes also through $\delta_f \cap \ell_\infty = [G'']$, which determines the direction of the axis of the parabola κ_f. Thus, the conic has two points at infinity $\{[P], [G'']\}$ and is a hyperbola through the points $\{A, B, C\}$ with asymptotes parallel to the directions determined by its points at infinity $\{[P], [G'']\}$. This conic can be constructed by standard methods described in [16, p.303]. Figure 13 shows such a hyperbola suggesting also the directions of the two asymptotes $\{\eta, \eta',\}$ which are parallel respectively to the tangent $[P]P'$ to the parabola κ'_f and to line δ_f.

In the second case of *nr-3* the conic $CoI_f([G'])$ passes through two points at infinity $[G'] = f(G)$ and $f([G']) = f^2(G) = [G'']$, which are the directions of the axes respectively of the parabolas $\{\kappa'_f, \kappa_f\}$. Their construction can be carried out using the same procedure referred to in the previous case.

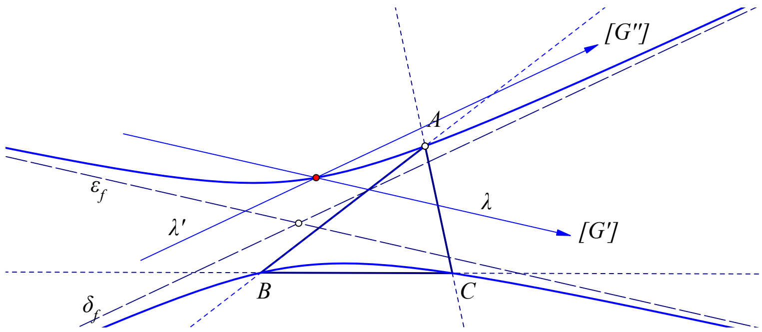

FIGURE 14. $CoI_f(G')$ with asymptotes the lines $\{\varepsilon_f, \delta_f\}$

Figure 14 shows the hyperbola which has asymptotes the lines $\{\delta_f, \varepsilon_f\}$. It is generated through the intersections of lines λ parallel to ε_f and their images $\lambda' = f(\lambda)$ which are parallel to δ_f passing all through $[G''] \in \ell_\infty$.

In the third case of *nr-3* the conic $CoI_f([G''])$ is generated by the intersections of lines λ parallel to δ_f, all of them passing through $[G''] \in \ell_\infty$ and their images $\lambda' = f(\lambda)$, which pass through $J = f([G''])$ (see Figure 15). The fact that the conic is a parabola follows from the special case of $\lambda = \ell_\infty$ and $\lambda' = \delta_f \ni [G'']$, which implies that the conic is tangent to ℓ_∞ at $[G'']$. Alternatively, this can be seen by taking into account

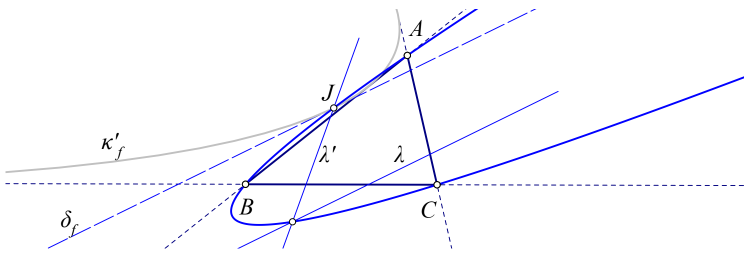

FIGURE 15. $CoI_f([G''])$ is a parabola with axis parallel to δ_f

that $CoI_f([G'']) = f(CoI_f([G']))$. Since $CoI_f([G'])$ is tangent to ε_f at $[G']$ we have that $f(CoI_f([G'])) = CoI_f([G''])$ is tangent to $f(\varepsilon_f) = \ell_\infty$ at $[G''] = f([G'])$, hence a parabola.

8. One fixed point at infinity

Considering a projectivity f with two ordinary fixed points $\{A, B\}$ and one fixed point at infinity $[C] \in \ell_\infty$, we have an *"infinite triangle"* $AB[C]$ of fixed points, consequently all the conics of intersections $CoI_f(P)$ in this case are unbounded, passing through $[C]$. In this section we examine such projectivities and notice the differences from the generic kind.

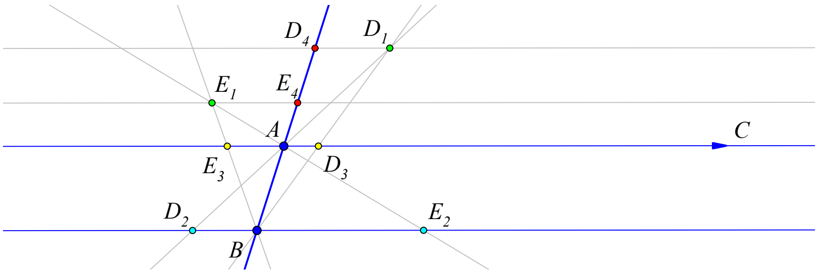

FIGURE 16. $\{D_1, D_2, D_3, D_4\}$ map respectively to $\{E_1, E_2, E_3, E_4\}$

Knowing the direction represented by $[C]$, one ordinary point D_1 and its image $E_1 = f(D_1)$, assumed to be also an ordinary point, we can easily find three other points, non-collinear by three, and their images and through them identify the projectivity with one mapping a quadruple of ordinary points to another quadruple. Figure 16 suggests a simple recipe to find from the known pair $\{D_1, E_1\}$ three additional pairs corresponding under the projectivity f fixing $\{A, B, [C]\}$ and mapping D_1 to E_1. The recipe uses the fact, that lines $\{AB, A[C], B[C]\}$ are invariant under f and a line in the direction of $[C]$ maps via f to a parallel to it. We have the correspondences of lines

$$AD_1 \xmapsto{f} AE_1 \ , \ BD_1 \xmapsto{f} BE_1 \ , \ D_1[C] \xmapsto{f} E_1[C] \ , \quad \Rightarrow$$
$$\text{on } B[C] : D_2 \xmapsto{f} E_2 \ , \text{on } A[C] : D_3 \xmapsto{f} E_3 \ , \text{on } AB : D_4 \xmapsto{f} E_4 \ .$$

Figure 17 shows the invariant lines $\{\varepsilon = AB, A[C], B[C]\}$. It shows also the lines $\{\varepsilon_f, \delta_f\}$, the first sent to infinity by f and the second coming from infinity $\delta_f = f(\ell_f)$. Next lemma is trivially verified.

Lemma 8.1. *The lines $\{\delta_f, \varepsilon_f\}$ are parallel to the direction determined by $[C] \in \ell_\infty$.*

Proof. In fact, $\delta_f = f(\ell_\infty) \ni [C]$ since $[C]$ is fixed by f. Thus, point $[C]$ is the point at infinity of δ_f and determines the direction of δ_f as claimed. From this follows that ε_f is also parallel to the direction determined by $[C]$, since $\varepsilon_f = \delta_{f^{-1}}$ and f^{-1} shares the same fixed points with f.

In this case there is no parabola enveloping the lines $\{Uf(U)\}$ for $U \in \varepsilon_f$, as was the case when all three fixed points were ordinary points of the plane. Next theorem shows what is going on.

Theorem 8.1. *If the projectivity f has two ordinary fixed points $\{A, B\}$ and one fixed point $[C]$ at infinity, then all lines $Uf(U)$ for $U \in \varepsilon_f$ pass through the same point $W \in AB$.*

Proof. Consider for each point $U \in \varepsilon_f$ the line ε_U parallel to $\varepsilon = AB$ and its image $\zeta_U = f(\varepsilon_U)$ (see Figure 17). Since $U \in \varepsilon_U \Rightarrow f(U) \in f(\varepsilon_U) = \zeta_U$, line ζ_U is parallel to $Uf(U)$ and intersects ε_f at a point V. Denoting by $[\varepsilon]$ the point at infinity of ε, we have $[\varepsilon] \in \varepsilon_U$ consequently $f([\varepsilon]) = J \in \zeta_U$. The fact that $J = AB \cap \delta_f$, as seen in the figure, follows immediately from the invariance of AB under f, which implies $f([\varepsilon]) \in \varepsilon$.

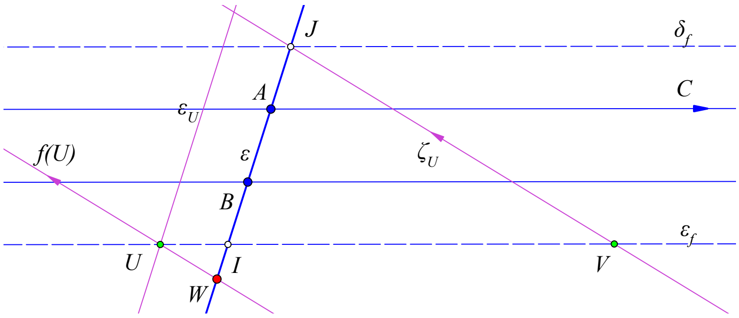

FIGURE 17. Lines $\{Uf(U)\ ,\ U \in \varepsilon_f\}$ pass through W

The correspondence of lines $f^* : \varepsilon_U \longmapsto \zeta_U$ is a homography induced by f between the pencil $[\varepsilon]^*$ of lines parallel to ε, and J^* the pencil of lines passing through J. This induces a homography $g : U \longmapsto V$ on line ε_f, which taking $I = \varepsilon_f \cap AB$ as origin of coordinates on ε_f, is represented by a rational function of the form $y = g(x) = (ax+b)/(cx+d)$ with $ad - bc \neq 0$ ([23, I, p.154]). It is readily seen that $g(\infty) = \infty$ and $g(0) = 0$, which implies that actually g is a similarity $g(x) = kx$ for a constant $k \neq 0$. This implies that $\frac{IV}{IU} = k$ and by the similar triangles $\{IVJ, IUW\}$ we obtain $\frac{IJ}{IW} = k$, independent of the position of $U \in \varepsilon_f$, thereby proving the theorem.

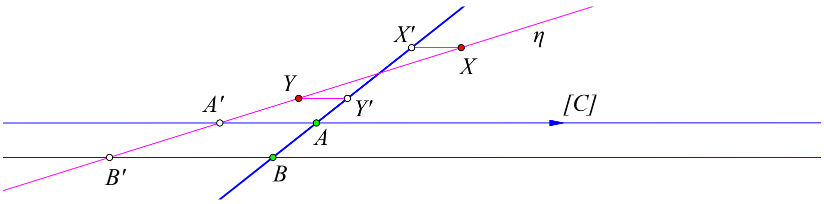

FIGURE 18. Constant cross ratio $(A'B'; XY) = (AB; X'Y') = k$

Lemma 8.2. *If the projectivity f has two ordinary fixed points $\{A, B\}$ and one fixed point $[C]$ at infinity, then for every point X of the plane, its image $Y = f(X)$ and the projections $\{X', Y'\}$ of these points parallel to $[C]$ on the invariant line AB, the cross ratio $(AB; X'Y') = k$ is constant (see Figure 18).*

Proof. The proof follows from the corresponding property for points on the invariant line AB. In fact if $X' \in AB$ then also $Y' = f(X') \in AB$, and the map $X' \mapsto Y'$ between points of the line AB is a line homography with two fixed points $\{A, B\}$. By a well known elementary property of homographies the cross ratio $(AB; X'Y') = k$ is constant ([20]). This implies the proof, since the line XX' maps via f to YY' and the cross ratio is preserved by the parallel projection in the direction of $[C]$.

Next we examine the shape of the conic $CoI_f(D)$ for a projectivity with fixed points $\{A, B, [C]\}$. Figure 19 shows the case of $CoI_f(D)$ for a generic point of the plane with $E = f(D)$. It is a hyperbola passing through $\{A, B, [C]\}$, hence having an asymptote parallel to the direction of $[C]$. The

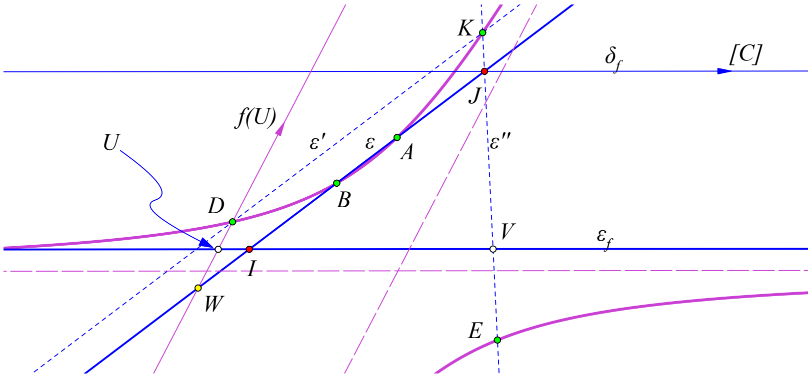

FIGURE 19. The hyperbola $CoI_f(D)$ for a generic point

second point at infinity is determined by the line WD, where W is the point contained in all lines $\{Uf(U)\ ,\ U \in \varepsilon_f\}$ (theorem 8.1). If $U = DW \cap \varepsilon_f$, then $f(DW) \ni f(U)$ and the lines $\{UD = UW, f(U)E\}$ intersect at the point at infinity $f(U)$ defining the direction of the second asymptote.

We notice, that in this case the conic can be identified with the one passing through five points $\{A, B, D, E, K\}$, where the last point is the intersection $K = \varepsilon' \cap \varepsilon''$, line ε' being the parallel to AB through D and $\varepsilon'' = f(\varepsilon')$, later passing through two known points $\{D', J\}$.

The conics $CoI_f(D)$ of non-generic points D occur when later is on one of the remarkable lines or points of the configuration, such as $AB, AC, BC, \varepsilon_f$, $\delta_f, \ell_\infty, \ldots$. I proceed here to short account, leaving the details to the reader. For points D on the side-lines of the (infinite) triangle $AB[C]$ it is easily seen that the conic $CoI_f(D)$ degenerates to a product of lines, one of them being the side containing D. In the other cases the conic is again a hyperbola, with the exception of points $D \in \eta = W[C]$ i.e. points contained in the parallel to $[C]$ from W (see Figure 20). In this case one verifies easily that $[C]$ is a contact point of the conic with the line at infinity, hence $CoI_f(D)$ is a parabola.

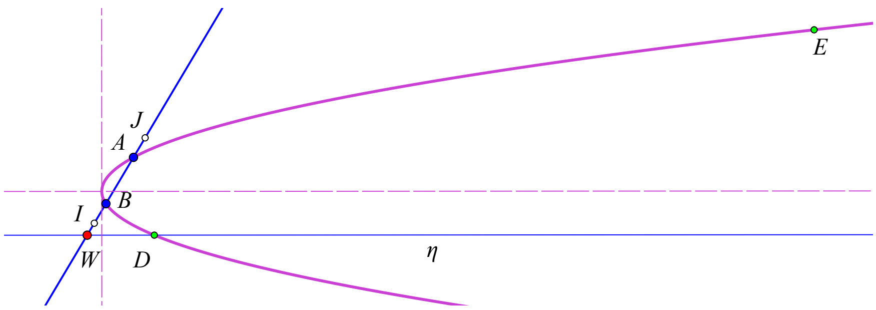

FIGURE 20. For $D \in \eta = W[C]$ the conic $CoI_f(D)$ is a parabola

9. The Steiner ellipse

In this and the subsequent sections we discuss a few of well known examples of circumconics from the view point of the conics of intersections

$\{CoI_f(D)\}$ and apply in each case the results of the preceding discussion. The concepts referred to below belong mostly to the *"triangle geometry"* ([22], [13], [24], [12]) and the *"triangle centers"* [14].

We start with the *"Steiner ellipse"* κ' of the triangle ABC ([24, p.109]) which is well known to be the circumconic with center at the centroid G of the triangle and fourth intersection point E with the circumcircle κ the triangle center (Steiner point of the triangle) $E = X(99)$. It is known also that the tangent to κ' at E intersects a second time the circumcircle at the triangle center $D = X(110)$, known to be the focus of the so called *"Kiepert parabola"*, tangent to the side-lines and the *"Lemoine line"* of ABC ([11]), which is also the *"tripolar"* of the *"Symmedian point"* $K = X(6)$ of the triangle. The directrix of this parabola is the Euler line of the triangle.

According to our discussion the Steiner ellipse is the image $\kappa' = f(\kappa)$ of the circumcircle via the projectivity f fixing the vertices of the triangle and mapping D to E (see Figure 21).

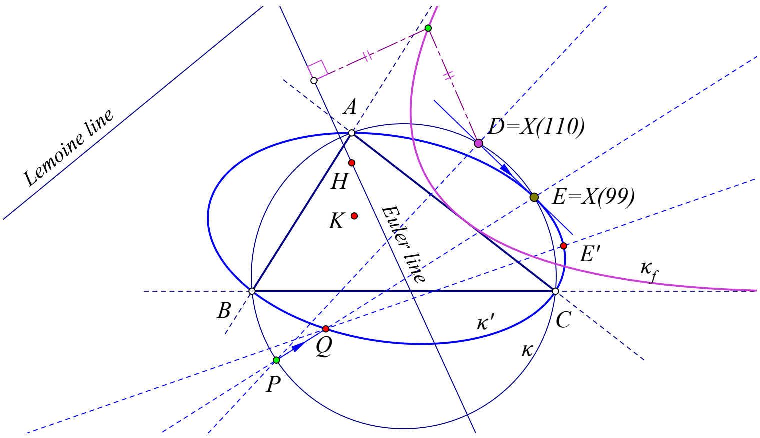

FIGURE 21. The Steiner ellipse in the form of $CoI_f(E)$

Theorem 9.1. *With the notation and conventions adopted so far, the projectivity f fixing the vertices of the triangle ABC and mapping $D = X(110)$ to $E = X(99)$ has the following properties.*

(1) *It represents the Steiner ellipse as a conic of the form $\kappa' = CoI_f(E)$ generated by the intersections $Q = \lambda \cap \lambda'$ of lines $\lambda \ni E$ and $\lambda' = f(\lambda) \ni E' = f(E)$.*
(2) *Restricted on the circumcircle κ the projectivity f coincides with the radial projection $P \mapsto Q$ along the radii EP trough the point E.*
(3) *The line ε_f sent to infinity by f coincides with the Lemoine line of the triangle.*
(4) *The parabola κ_f of f coincides with the Kiepert parabola of the triangle.*

Proof. *Nrs 1-2* result from theorem 3.2 and the known facts identifying $E = X(99)$ and $D = X(110)$.

Nrs 3-4 result by showing that the line ε_f coincides with the Lemoine line and the known fact that the Kiepert parabola is tangent to the line-sides and the Lemoine line of the triangle ABC ([11]). To show this coincidence of lines we work with barycentric coordinates (barycentrics) w.r.t. the triangle ABC ([17]). In these coordinates the work needed for the matrix representation of f can be transferred verbatim from that done for Cartesian coordinates in section 2. The corresponding matrices are $P = Q = I$ the identity matrix, and $M = K$ the diagonal matrix, whose non-zero diagonal entries are up to constant multiple, equal to $F(e_1/d_1 : e_2/d_2 : e_3/d_3)$. In these $\{d_i\}$ and $\{e_i\}$ are correspondingly the barycentrics of D and E ([14]):

$$D\left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right) \quad , \quad E\left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right) \ ,$$

where $\{a = |BC|, b = |CA|, c = |AB|\}$ are the side-lengths of the triangle. Thus, the projectivity f is in these coordinates represented by a diagonal matrix with non-zero diagonal entries $F(1/a^2 : 1/b^2 : 1/c^2)$. and since the line at infinity in these coordinates is represented by $\lambda : x + y + z = 0$, the line ε_f has coefficients satisfying

$$(p : q : r)F^{-1} \ = \ G(1 : 1 : 1) \quad \Rightarrow \quad (p : q : r) \ = \ GF \ = \ \left(\frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2}\right) \ ,$$

which are the coefficients of the Lemoine line.

10. Digression in Barycentrics

Taking the opportunity from the preceding calculations, we should notice that they generalize for arbitrary generic projectivities and their corresponding conics of intersections $\mu_P = CoI_f(P)$. In fact, using the concept of *"barycentric product"* ([24, p.99])

$$X \cdot Y \ = \ (x_1y_1 : x_2y_2 : x_3y_3) \quad \text{for} \quad X(x_1 : x_2 : x_3) \ , \ Y(y_1 : y_2 : y_3) \ ,$$

we see that the expressions of f, the point $E' = f(E)$, and the line ε_f, in terms of the points $\{D, E\}$ take quite simple forms. In fact, the entries $F(f_1 : f_2 : f_3)$ of the diagonal matrix representing f in barycentrics are $F = E/D$. The image point $E' = f(E)$ is expressed then with $E' = E^2/D$ and the coefficients of the line sent to infinity by $\varepsilon_f = G \cdot (E/D) = E/D$, are the same with those expressing f.

Since the parabola κ_f is completely determined by its focus D and the fact that it is tangent to the side-lines of the triangle ABC, the various projectivities $f = f_{DE}$ resulting by fixing D and varying E on the circumcircle κ are represented in barycentrics with diagonal matrices whose non-zero diagonal entries $\{F(f_1 : f_2 : f_3)\}$ coincide with the coefficients of the tangent lines ε_f to κ_f.

Transferring to matrices the relation of the conics $\mu_E = CoI_f(E) = f(\kappa)$ where κ the circumcircle, we have the relations between the matrices M_0, M_1, F correspondingly of $\{\kappa, \mu_E, f\}$:

$$M_0 = \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \quad , \quad M_1 = F^{-1}M_0F^{-1} = \begin{pmatrix} 0 & c^2\frac{d_1}{e_1}\frac{d_2}{e_2} & b^2\frac{d_1}{e_1}\frac{d_3}{e_3} \\ c^2\frac{d_1}{e_1}\frac{d_2}{e_2} & 0 & a^2\frac{d_2}{e_2}\frac{d_3}{e_3} \\ b^2\frac{d_1}{e_1}\frac{d_3}{e_3} & a^2\frac{d_2}{e_2}\frac{d_3}{e_3} & 0 \end{pmatrix} \ .$$

Since the matrix of the projectivity is defined up to a non-zero constant multiple, multiplying the last matrix with $\frac{e_1e_2e_3}{d_1d_2d_3}$ we obtain the matrix of the conic in barycentrics

$$\mu_E = CoI_f(E) \ : \ X^tM_1X = 0 \quad \text{with}$$
$$M_1 = \begin{pmatrix} 0 & c^2e_3/d_3 & b^2e_2/d_2 \\ c^2e_3/d_3 & 0 & a^2e_1/d_1 \\ b^2e_2/d_2 & a^2e_1/d_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c^2f_3 & b^2f_2 \\ c^2f_3 & 0 & a^2f_1 \\ b^2f_2 & a^2f_1 & 0 \end{pmatrix} .$$

11. The Jerabek hyperbola

This well known ([18]) circumconic of the triangle ABC is a rectangular hyperbola passing, besides the vertices of the triangle, also through the orthocenter H, the circumcenter O, the symmedian point K, and many other remarkable triangle centers. It intersects the circumcircle at the triangle center $E = X(74)$ (see Figure 22). Using §10 and adapting to barycentrics some computations done in the aforementioned reference, we obtain the formulas: $f : F\,(S_A(S_B - S_C) : \ldots) \ = \ \varepsilon_f$,

$$E = X(74) = \left(\frac{a^2}{S_A(S_B+S_C)-2S_BS_C} : \ldots\right) ,$$
$$D = E/F = \left(\frac{a^2(S_A-S_B)(S_A-S_C)}{S_A(S_A(S_B+S_C)-2S_BS_C)} : \ldots\right) ,$$
$$E' = f(E) = E^2/D = \left(\frac{a^2S_A(S_C-S_B)}{S_A(S_B+S_C)-2S_BS_C} : \ldots\right) .$$

In these the dots denote the remaining two barycentric coordinates, obtained from the first by cyclic permutations of the letters $\{a, b, c\}$ and $\{A, B, C\}$. The symbols S_A denote the *"Conway triangle symbols"* ([19])

$$S_A = \frac{1}{2}(b^2+c^2-a^2) \ , \ S_B = \frac{1}{2}(c^2+a^2-b^2) \ , \ S_C = \frac{1}{2}(a^2+b^2-c^2) .$$

The first of the preceding formulas shows that the line sent to infinity ε_f

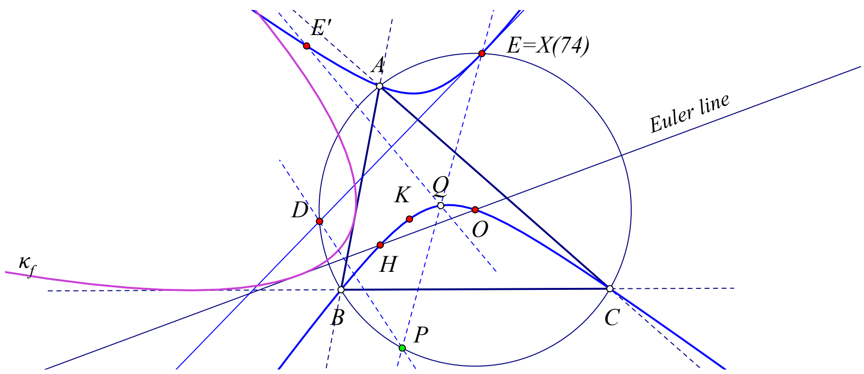

Figure 22. The Jerabek hyperbola represented as a $CoI_f(E)$

coincides in this case with the Euler line. Thus, the parabola κ_f of the projectivity f is characterized by its tangency to the side-lines and the Euler line of the triangle ABC. Using the points $\{E, E' = f(E)\}$ defined above, the Jerabek hyperbola is represented in the form of $CoI_f(E)$, i.e. as locus of intersections $Q = \lambda \cap \lambda'$ for lines $\{\lambda \ni E, \lambda' = f(\lambda) \ni E'\}$.

12. Parabolas represented as conics of intersections

From theorem 7.1 we know that a parabola circumscribing a triangle ABC and represented as a conic of intersections $\mu_E = CoI_f(E)$ for some generic projectivity f, has necessarily point E lying on the parabola κ_f of f (see Figure 23). According to lemma 3.4 the projectivity f is defined by its properties (i) to fix $\{A, B, C\}$ and (ii) to map D to E. Point D is the second intersection with the circumcircle of the tangent to the parabola at E, which in turn is the fourth intersection point of the circumcircle with the parabola.

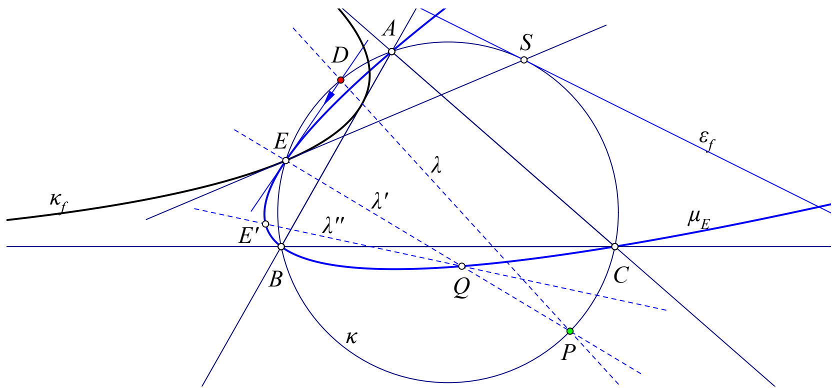

FIGURE 23. $\{Q = \lambda' \cap \lambda''\}$ generating Parabola $\mu_E = CoI_f(E)$

The circumcircle of the triangle ABC is represented as $\kappa = CoI_f(D)$ and described by the intersections $\{P = \lambda \cap \lambda'\}$ for λ revolving about D and $\lambda' = f(\lambda) \ni E$. The parabola is the image $\mu_E = f(CoI_f(D)) = CoI_f(E)$ described by the intersections $\{Q = \lambda' \cap \lambda''\}$ for $\lambda'' = f(\lambda') \ni E' = f(E)$. The following theorem lists a couple of properties immediately following from the preceding discussion.

Theorem 12.1. *With the notation and conventions adopted so far the parabola $\mu_E = CoI_f(E)$ has the following properties (see Figure 23):*

(1) *The line ε_f sent to infinity by f is tangent to the circumcircle κ at a point S.*
(2) *The line ES is parallel to the axis of the parabola μ_E.*

Proof. Referring to figure 23 and lemma 3.3, we have that f maps κ onto μ_E along the lines λ' through E sending $f : P \mapsto Q$. Assume that S is the second intersection with κ of the tangent to κ_f at E. Since ES is tangent to κ_f the projectivity f sends ES to $E'S'$ parallel to ES (lemma 5.1). Thus, $S' = f(S) \in \mu_E$ is on the parabola and also on the line at infinity, since the corresponding lines $\{\lambda', \lambda''\}$ become parallel. Hence $S' = f(S)$ is the contact point of the parabola with the line at infinity $\ell_f = f(\varepsilon_f)$. This implies that ε_f is tangent to the circumcircle κ and also that $E'S'$ as well as ES are parallel to the axis of the parabola, as claimed.

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