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A new geometric inequality in acute triangles and its applications

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Abstract. We find a new acute triangle inequality, which gives a lower bound of a single median for acute triangles. As its applications, we establish three new symmetric acute triangle inequalities with the help of software Maple. We also propose several related conjectures checked by the computer.

1. Introduction

Given a triangle ABC, denote by a, b, c the side lengths, m_a, m_b, m_c the medians, h_a, h_b, h_c the altitudes, r_a, r_b, r_c the radii of excircles, s, R and r the semiperimeter, the inradius and the circumradius, respectively. In addition, we denote $\sum$ and $\prod$ by cyclic sums and products respectively.

In the recent paper [12], Theorem 1.1 gives a upper bound of a single median for any triangle ABC, i.e,

$$m_a \leq h_a + R\left(\frac{b-c}{a}\right)^2, \tag{1.1}$$

with equality if and only if $b = c$ or $A = \pi/2$. There are several equivalent forms of (1.1). For example, the author pointed out in [13] that it is equivalent to

$$4m_a \leq r_b + r_c + 2h_a + \frac{(b-c)^2}{r_b + r_c} \tag{1.2}$$

and

$$m_a \leq h_a + \frac{(r_b - r_c)^2 + (b-c)^2}{4(r_b + r_c)}. \tag{1.3}$$

Many years ago, the author gave the following linear inequality related to (1.2) for the acute triangle ABC in a Chinese paper [5]:

$$4m_a \geq r_b + r_c + 2h_a, \tag{1.4}$$

with equality if and only if $b = c$.

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Very recently, the author [13] considered improvements of (1.4) and obtained the following result:

$$4m_a \geq r_b + r_c + 2h_a + \frac{3}{4} \cdot \frac{(b-c)^2}{r_b + r_c}, \tag{1.5}$$

We also pointed out in [13] that (1.5) is equivalent to

$$m_a \geq h_a + \frac{3(b-c)^2 + 4(r_b - r_c)^2}{16(r_b + r_c)}, \tag{1.6}$$

with equality if and only if $b = c$.

From [12] and [13], we see that both inequalities (1.1) and (1.6) can be used to prove some geometric inequalities involving medians of a triangle.

Inspired by inequality (1.3), the author finds that if $(b-c)^2$ is replaced by $(h_b - h_c)^2$ in (1.3) then the inequality holds for the acute triangle. Specifically, we have the following conclusion:

Theorem 1.1. *Let $\triangle ABC$ be an acute triangle, then*

$$m_a \geq h_a + \frac{(r_b - r_c)^2 + (h_b - h_c)^2}{4(r_b + r_c)}, \tag{1.7}$$

with equality if and only if $b = c$ or $A = \pi/2$.

By the identity given in [13]:

$$r_b + r_c = 2h_a + \frac{(r_b - r_c)^2}{r_b + r_c}, \tag{1.8}$$

we see that inequality (1.7) is equivalent to

$$4m_a \geq r_b + r_c + 2h_a + \frac{(h_b - h_c)^2}{r_b + r_c}, \tag{1.9}$$

which is also an improvement of inequality (1.4). In addition, we have known that the values of the right hand side of (1.6) and (1.7) are not comparable.

The aim of this paper is to prove Theorem 1.1 and establish three new inequalities by making use of it. We also present some related conjectures.

2. Proof of Theorem 1.1

In this section, we give a simple proof of Theorem 1.1 as follows:

Proof. We denote by S the area of $\triangle ABC$. Applying the known formula:

$$r_a = \frac{S}{s-a}, \tag{2.1}$$

we get

$$r_b + r_c = \frac{aS}{(s-b)(s-c)}, \tag{2.2}$$

and

$$r_b - r_c = \frac{(b-c)S}{(s-b)(s-c)}. \tag{2.3}$$

Denoting by v_a the value of the right hand side of (1.7), then using (2.2), (2.3) and $h_a = 2S/a$, we get

$$v_a = \frac{2S}{a} + \frac{(s-b)(s-c)S}{4a}(b-c)^2\left[\frac{1}{(s-b)^2(s-c)^2} + \frac{4}{b^2c^2}\right].$$

Again, using $s = (a+b+c)/2$ and simplifying, we obtain

$$v_a = \frac{N_0 S}{M_0}, \tag{2.4}$$

where

$$\begin{aligned} M_0 =& 4a(c+a-b)(a+b-c)b^2c^2, \\ N_0 =& (b-c)^2a^4 - 2(b^2+c^2)(b^2-4bc+c^2)a^2 \\ &+ (b^2+c^2)(b^2-4bc+c^2)(b-c)^2. \end{aligned}$$

In order to prove inequality (1.7) we need to show that

$$m_a^2 - v_a^2 \geq 0.$$

Applying the following known formula:

$$4m_a^2 = 2b^2 + 2c^2 - a^2 \tag{2.5}$$

and Heron's formula:

$$S = \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}, \tag{2.6}$$

we easily obtain the following identity:

$$m_a^2 - v_a^2 = \frac{(b-c)^2(b^2+c^2-a^2)^2X_0}{256(c+a-b)(a+b-c)a^2b^4c^4}, \tag{2.7}$$

where

$$\begin{aligned} X_0 =& (b-c)^2a^6 - (3b^4 - 12b^3c + 2b^2c^2 - 12bc^3 + 3c^4)a^4 \\ &+ (3b^4 - 12b^3c - 10b^2c^2 - 12bc^3 + 3c^4)(b-c)^2a^2 \\ &- (b-c)^2(b+c)^2(b^2-4bc+c^2)^2. \end{aligned}$$

After analyzing, we find the following identity:

$$X_0 = X_1 + X_2 + X_3, \tag{2.8}$$

where

$$\begin{aligned} X_1 =& 6bc(c+a-b)(a+b-c)(b^2+c^2-a^2)a^2, \\ X_2 =& 2bc(c^2+a^2-b^2)(a^2+b^2-c^2)\left[2bc + 3(c+a-b)(a+b-c)\right], \\ X_3 =& (a+b+c)(b+c-a)(c+a-b)(a+b-c)(b^2+c^2-a^2)(b-c)^2. \end{aligned}$$

In fact, we easily verify identity (2.8) by expanding. Since $\triangle ABC$ is an acute triangle, one has $X_1 \geq 0, X_2 \geq 0$ and $X_3 \geq 0$. Note that X_1 and X_2 are not zero at the same time. Thus inequality $X_1 + X_2 + X_3 > 0$ holds strictly, so that $X_0 > 0$. Clearly, we have $m_a^2 - v_a^2 \geq 0$ from identity (2.7) and it is easily seen that the equality in (1.7) holds if and only if $b = c$ or $b^2 + c^2 - a^2 = 0$, i.e. $A = \pi/2$. This completes the proof of Theorem 1.1.

Remark 2.1. *It is well known that for any triangle ABC it holds that*

$$m_a \geq \frac{b^2+c^2}{4R}. \tag{2.9}$$

Here, we wish to point out that for the acute triangle ABC, the inequality of Theorem 1.1 actually improves (2.9). *In fact, it is easy to prove that inequality* (1.7) *is equivalent to*

$$m_a \geq \frac{b^2+c^2}{4R} + \frac{S(b-c)^2(b^2+c^2-a^2)^2}{a(c+a-b)(a+b-c)b^2c^2}. \tag{2.10}$$

Hence, the above statement is true.

3. Applications of Theorem 1.1 (I)

In this section and next two sections, we shall apply Theorem 1.1 and other well-known triangle inequalities to establish three new acute triangle inequalities. As done as in [12], we shall omit the details of deducing some identities in a triangle. For the deductions of some complex identities in triangles we refer the reader to [7], [8] and [10].

Next, we prove the following inequality involving medians and radii of excircles of an acute triangle.

Theorem 3.1. *Let $\triangle ABC$ be an acute triangle, then*

$$\sum \frac{1}{m_b+m_c} \geq \sum \frac{1}{r_a+m_a}, \tag{3.1}$$

with equality if and only if $\triangle ABC$ is equilateral.

Proof. We write

$$v_a = h_a + \frac{(r_b-r_c)^2+(h_b-h_c)^2}{4(r_b+r_c)}, \tag{3.2}$$

$$v_b = h_b + \frac{(r_c-r_a)^2+(h_c-h_a)^2}{4(r_c+r_a)}, \tag{3.3}$$

$$v_c = h_c + \frac{(r_a-r_b)^2+(h_a-h_b)^2}{4(r_a+r_b)}. \tag{3.4}$$

By Theorem 1.1, we only need to show that

$$\sum \frac{1}{m_b+m_c} \geq \sum \frac{1}{r_a+v_a}. \tag{3.5}$$

Again, we set

$$q_a = \frac{1}{4}(r_b+r_c) + \frac{m_a^2}{r_b+r_c}, \tag{3.6}$$

$$q_b = \frac{1}{4}(r_c+r_a) + \frac{m_b^2}{r_c+r_a}, \tag{3.7}$$

$$q_c = \frac{1}{4}(r_a+r_b) + \frac{m_c^2}{r_a+r_b}. \tag{3.8}$$

By the simplest arithmetic-geometric mean inequality, we have

$$m_a \leq q_a. \tag{3.9}$$

(This actually is equivalent to (1.1), cf. [12]). Two similar relations $m_b \le q_b$ and $m_c \le q_c$ are also valid. Thus, for proving (3.5) we need to show

$$\sum \frac{1}{q_b + q_c} \ge \sum \frac{1}{r_a + v_a}. \tag{3.10}$$

After computing, we obtain

$$\prod (q_b + q_c) = \frac{M_1}{8R^2 s^4}, \tag{3.11}$$

$$\sum (q_c + q_a)(q_a + q_b) = \frac{N_1}{16R^2 s^4}, \tag{3.12}$$

where

$$\begin{aligned} M_1 =& (R + 2r)s^8 + (16R^3 + 32R^2 r + 12Rr^2 - 2r^3)s^6 \\ & + 2(R + r)(32R^4 - 32R^3 r - 56R^2 r^2 - 20Rr^3 \\ & - r^4)s^4 - 2(8R^3 + 8R^2 r - 2Rr^2 - r^3)(4R + r)^3 rs^2 \\ & + (4R + r)^6 Rr^2, \\ N_1 =& s^8 + 8(4R + 5r)Rs^6 + (192R^4 + 192R^3 r + 16R^2 r^2 \\ & - 40Rr^3 - 2r^4)s^4 - 8(4R + 3r)(4R + r)^3 Rrs^2 \\ & + (4R + r)^6 r^2. \end{aligned}$$

It follows from (3.11) and (3.12) that

$$\sum \frac{1}{q_b + q_c} = \frac{N_1}{2M_1}. \tag{3.13}$$

In addition, after computing we obtain

$$\prod (r_a + v_a) = \frac{M_2}{1024R^5}, \tag{3.14}$$

$$\sum (r_b + v_b)(r_c + v_c) = \frac{N_2}{256R^4}, \tag{3.15}$$

where

$$\begin{aligned} M_2 =& s^8 + 4(7R - r)(R - r)s^6 + (304R^4 - 448R^3 r + 352R^2 r^2 \\ & - 48Rr^3 + 2r^4)s^4 + (1856R^6 - 1280R^5 r + 1216R^4 r^2 \\ & - 192R^3 r^3 - 100R^2 r^4 + 80Rr^5 - 4r^6)s^2 \\ & + (4R - 3r)(4R^2 + r^2)^2 (4R + r)^3, \\ N_2 =& s^6 + (40R^2 - 24Rr + 3r^2)s^4 + (592R^4 - 384R^3 r \\ & + 304R^2 r^2 - 8Rr^3 + 3r^4)s^2 + (4R^2 + r^2)(36R^2 \\ & - 24Rr + r^2)(4R + r)^2. \end{aligned}$$

It follows from (3.14) and (3.15) that

$$\sum \frac{1}{r_a + v_a} = \frac{4RN_2}{M_2}. \tag{3.16}$$

From (3.13) and (3.16), one sees that inequality (3.10) is equivalent to

$$\frac{N_1}{2M_1} \ge \frac{4RN_2}{M_2}.$$

Note that $M_1 > 0$ and $M_2 > 0$, we have to prove that

$$Q_0 \equiv M_2N_1 - 8RN_2M_1 \geq 0. \tag{3.17}$$

With the help of Maple, we easily obtain

$$\begin{aligned}Q_0 =& s^{16} + (52R^2 - 8Rr + 4r^2)s^{14} + 8(118R^3 - 108R^2r\\&- 65Rr^2 + 5r^3)Rs^{12} + (6592R^6 - 19840R^5r - 6400R^4r^2\\&+ 8544R^3r^3 - 1220R^2r^4 + 56Rr^5 - 12r^6)s^{10} + (7168R^8\\&- 167424R^7r + 35968R^6r^2 + 125312R^5r^3 - 8016R^4r^4\\&- 8192R^3r^5 + 4160R^2r^6 - 184Rr^7 - 6r^8)s^8 - (110592R^{10}\\&+ 364544R^9r - 1029120R^8r^2 - 465408R^7r^3 + 441984R^6r^4\\&+ 128768R^5r^5 + 66944R^4r^6 + 7424R^3r^7 - 3476R^2r^8\\&- 88Rr^9 + 12r^{10})s^6 - 8(3072R^{10} - 17408R^9r - 35200R^8r^2\\&+ 19136R^7r^3 - 3712R^6r^4 - 7088R^5r^5 + 1758R^4r^6\\&+ 2124R^3r^7 + 197R^2r^8 - 23Rr^9 - r^{10})(4R + r)^2s^4\\&+ 4(2560R^8 - 1856R^7r - 1744R^6r^2 - 32R^5r^3 - 656R^4r^4\\&+ 20R^3r^5 + 209R^2r^6 + 30Rr^7 - r^8)(4R + r)^5rs^2\\&- (4R^2 + r^2)(224R^4 - 160R^3r + 4R^2r^2 + 8Rr^3\\&+ 3r^4)(4R + r)^8r^2.\end{aligned} \tag{3.18}$$

We recall that for the acute $\triangle ABC$ the following inequality holds (cf.[9]):

$$v_0 \equiv s^2 - 4R^2 + Rr - 13r^2 \geq 0. \tag{3.19}$$

According to this inequality, we can rewrite Q_0 as follows:

$$\begin{aligned}Q_0 =& v_0^8 + m_7v_0^7 + m_6v_0^6 + m_5v_0^5 + m_4v_0^4 + m_3v_0^3\\&+ m_2v_0^2 + m_1v_0 + m_0,\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}m_7 =& 84R^2 - 16Rr + 108r^2,\\m_6 =& 2848R^4 - 1676R^3r + 7320R^2r^2 - 1444Rr^3 + 5096r^4,\\m_5 =& 50304R^6 - 60352R^5r + 212900R^4r^2 - 119000R^3r^3\\&+ 271468R^2r^4 - 55792Rr^5 + 137216r^6,\\m_4 =& 499968R^8 - 1041024R^7r + 3320928R^6r^2\\&- 3548668R^5r^3 + 6533114R^4r^4 - 3486852R^3r^5\\&+ 5552040R^2r^6 - 1196384Rr^7 + 2306064r^8,\end{aligned}$$

$$\begin{aligned}m_3 =&2790400R^{10} - 9395200R^9r + 29610240R^8r^2\\&- 49902720R^7r^3 + 85535228R^6r^4 - 82554208R^5r^5\\&+ 105209060R^4r^6 - 53860384R^3r^7 + 67593664R^2r^8\\&- 15378752Rr^9 + 24770368r^{10},\\m_2 =&8044544R^{12} - 42974208R^{11}r + 149402112R^{10}r^2\\&- 348326144R^9r^3 + 628071904R^8r^4 - 887623460R^7r^5\\&+ 1075542272R^6r^6 - 947878924R^5r^7 + 935654840R^4r^8\\&- 461311488R^3r^9 + 489499744R^2r^{10}\\&- 118511168Rr^{11} + 166070016r^{12},\\m_1 =&9240576R^{14} - 80642048R^{13}r + 384334848R^{12}r^2\\&- 1117246464R^{11}r^3 + 2385879680R^{10}r^4 - 4274776000R^9r^5\\&+ 5711920908R^8r^6 - 6932730040R^7r^7 + 6598564932R^6r^8\\&- 5356135280R^5r^9 + 4340759104R^4r^{10} - 2069231936R^3r^{11}\\&+ 1950372544R^2r^{12} - 506989056Rr^{13} + 635378688r^{14},\\m_0 =&-(R - 2r)(24969216R^{14} - 262062080R^{13}r + 705221632R^{12}r^2\\&- 2072245504R^{11}r^3 + 3082592896R^{10}r^4 - 5729865120R^9r^5\\&+ 5890241044R^8r^6 - 7111394629R^7r^7 + 5795953986R^6r^8\\&- 4171641268R^5r^9 + 3521715512R^4r^{10} - 1114222496R^3r^{11}\\&+ 1547364480R^2r^{12} - 198922752Rr^{13} + 531062784r^{14})r.\end{aligned}$$

It is easily seen that $m_7 > 0$ and $m_6 > 0$ by Euler's inequality (in any $\triangle ABC$):

$$e \equiv R - 2r \geq 0. \tag{3.21}$$

Substituting $R = 2r + e$ into the expression of m_5 and expanding gives

$$\begin{aligned}m_5 =&50304e^6 + 543296e^5r + 2627620e^4r^2 + 7218760e^3r^3\\&+ 11911868e^2r^4 + 11245088er^5 + 4854096r^6,\end{aligned} \tag{3.22}$$

so $m_5 > 0$ holds strictly. Similarly, we can easily show that $m_4 > 0, m_3 > 0$ and $m_2 > 0$.

Therefore, according to identity (3.20) and inequality (3.19), it remains to prove that

$$Q_1 \equiv m_1v_0 + m_0 \geq 0. \tag{3.23}$$

We now recall that for any $\triangle ABC$ the following Gerretsen's inequality holds:

$$g_2 \equiv 4R^2 + 4Rr + 3r^2 - s^2 \geq 0. \tag{3.24}$$

And, for the acute triangle ABC we have the following equivalent form of Ciamberlini's inequality $s \geq 2R + r$ (see [2] and [15]):

$$u_0 \equiv s^2 - (2R + r)^2 \geq 0. \tag{3.25}$$

Based on inequalities (3.21), (3.24) and (3.25), after analysis we obtain the following identity:

$$Q_1 = (k_1 g_2 + k_2 u_0)e^3 + erk_3 + k_4 g_2, \tag{3.26}$$

where

$$\begin{aligned}
k_1 =&64r(393728R^{10} + 115632R^8r^2 + 272503209Rr^9 \\
&+ 1409244696r^{10}), \\
k_2 =&4R^2(2310144R^9 + 30564096R^7r^2 + 168202912R^5r^4 \\
&+ 207237584R^4r^5 + 638169891R^3r^6 + 954609124R^2r^7 \\
&+ 1377157957r^8R + 573963562r^9), \\
k_3 =&21233664R^{14} - 159629312R^{13}r + 1290377216R^{12}r^2 \\
&- 3832424192R^{11}r^3 + 9824856576R^{10}r^4 - 17967689248R^9r^5 \\
&+ 26393956008R^8r^6 - 31408505195R^7r^7 + 33349831506R^6r^8 \\
&- 23500243332R^5r^9 + 27111934168R^4r^{10} - 34934157824R^3r^{11} \\
&- 10162335744R^2r^{12} - 2336022528Rr^{13} + 2645830656r^{14}, \\
k_4 =&1679616(185489R^2 - 561004Rr + 429204r^2)r^{12}.
\end{aligned}$$

Using the above method to prove $m_5 > 0$, we can easily show $k_3 > 0$ and $k_4 > 0$. Therefore, by Euler's inequality (3.21), Ciamberlini's (3.25) and Gerretsen's inequality (3.24) we deduce that $Q_1 \geq 0$ holds for the acute $\triangle ABC$. Hence, inequality (3.1) is proved, and it is easy to determine its equality condition. This completes the proof of Theorem 3.1.

Remark 3.1. *By Theorem 3.1, it is easy to obtain the following consequence:*

$$\sum \frac{1}{m_a + r_a} \leq \frac{1}{2} \sum \frac{1}{m_a}, \tag{3.27}$$

which holds for the acute triangle ABC.

4. Applications of Theorem 1.1 (II)

In [12], the author established an inequality involving medians and sides of the acute triangle ABC, i.e.

$$\sum \frac{a}{m_b + m_c} \leq \sqrt{3}. \tag{4.1}$$

We here give an extension of this inequality:

Theorem 4.1. *Let $\triangle ABC$ be an acute triangle, then*

$$\sum \frac{a}{r_a + m_a} \leq \sum \frac{a}{m_b + m_c}, \tag{4.2}$$

with equality if and only if $\triangle ABC$ is equilateral.

Proof. According to Theorem 1.1 and the previous inequality (3.9), we only need to prove

$$\sum \frac{a}{q_b + q_c} \geq \sum \frac{a}{r_a + v_a}. \tag{4.3}$$

Computing gives

$$\sum a(q_c+q_a)(q_a+q_b)=\frac{N_3}{4Rs^3}, \tag{4.4}$$

where

$$\begin{aligned}N_3 =&(4R+9r)s^6+(4R+3r)(8R^2+8Rr-3r^2)s^4\\&-(24R^2+28Rr+r^2)(4R+r)^2rs^2+(4R+r)^5r^2.\end{aligned}$$

Thus by the previous identity (3.11) we get

$$\sum\frac{a}{q_b+q_c}=\frac{2sRN_3}{M_1}. \tag{4.5}$$

On the other hand, we can obtain the following identity after computing:

$$\sum a(r_b+v_b)(r_c+v_c)=\frac{sN_4}{64R^3}, \tag{4.6}$$

where v_a, v_b, v_c are given by (3.2)-(3.4) respectively, and

$$\begin{aligned}N_4 =&(4R+5r)s^4+(96R^3+24R^2r-68Rr^2+10r^3)s^2\\&+(4R+r)(80R^4-48R^3r-40R^2r^2+52Rr^3-11r^4).\end{aligned}$$

Thus, by the previous identity (3.14), we get

$$\sum\frac{a}{r_a+v_a}=\frac{16sR^2N_4}{M_2}, \tag{4.7}$$

From (4.5) and (4.7), one sees that inequality (4.3) is equivalent to

$$\frac{2sRN_3}{M_1}\ge\frac{16sR^2N_4}{M_2}.$$

Note that $M_1>0$ and $M_2>0$, we require the following inequality to be proved:

$$P_0\equiv N_3M_2-8RM_1N_4\ge 0. \tag{4.8}$$

With the help of Maple, we easily get

$$\begin{aligned}P_0 =&(4R+9r)s^{14}+(112R^3+76R^2r-340Rr^2+27r^3)s^{12}\\&+(832R^5-2288R^4r-6096R^3r^2+2892R^2r^3-204Rr^4\\&-19r^5)s^{10}+(256R^7-30912R^6r-19008R^5r^2\\&+62320R^4r^3+18048R^3r^4-4328R^2r^5+1484Rr^6\\&-57r^7)s^8+(-14336R^9-64000R^8r+290560R^7r^2\end{aligned}$$

$$
\begin{aligned}
&+ 394048R^6r^3 - 148736R^5r^4 - 168064R^4r^5 - 12320R^3r^6 \\
&+ 3576R^2r^7 - 1140Rr^8 + 11r^9)s^6 - (4R+r)(8192R^{10} \\
&- 92160R^9r - 396800R^8r^2 + 129024R^7r^3 + 475328R^6r^4 \\
&+ 31104R^5r^5 - 77184R^4r^6 - 2432R^3r^7 + 5604R^2r^8 \\
&+ 160Rr^9 - 33r^{10})s^4 + (4096R^8 - 4864R^7r - 24896R^6r^2 \\
&+ 11200R^5r^3 + 12528R^4r^4 - 2640R^3r^5 - 36R^2r^6 \\
&+ 332Rr^7 - r^8)(4R+r)^4rs^2 - (384R^6 - 256R^5r \\
&- 400R^4r^2 + 480R^3r^3 - 80R^2r^4 + 8Rr^5 \\
&+ 3r^6)(4R+r)^7r^2. \qquad (4.9)
\end{aligned}
$$

Putting $v_0 = s^2 - 4R^2 + Rr - 13r^2$, then we can obtain the following identity (which is easily verified by expanding):

$$P_0 = n_7v_0^7 + n_6v_0^6 + n_5v_0^5 + n_4v_0^4 + n_3v_0^3 + n_2v_0^2 + n_1v_0 + n_0, \qquad (4.10)$$

where

$$
\begin{aligned}
n_7 =& 4R + 9r, \\
n_6 =& 224R^3 + 300R^2r - 39Rr^2 + 846r^3, \\
n_5 =& 4864R^5 + 1216R^4r + 1332R^3r^2 + 29169R^2r^3 \\
&- 17604Rr^4 + 34028r^5, \\
n_4 =& 52736R^7 - 62592R^6r + 84192R^5r^2 + 281740R^4r^3 \\
&- 490407R^3r^4 + 1132202R^2r^5 - 736236Rr^6 + 759208r^7, \\
n_3 =& 302080R^9 - 957696R^8r + 1519872R^7r^2 - 455104R^6r^3 \\
&- 4390324R^5r^4 + 12341731R^4r^5 - 17882008R^3r^6 \\
&+ 22757432R^2r^7 - 14246624Rr^8 + 10148032r^9, \\
n_2 =& 868352R^{11} - 5260288R^{10}r + 12472064R^9r^2 \\
&- 20809472R^8r^3 - 3435168R^7r^4 + 43260132R^6r^5 \\
&- 148694813R^5r^6 + 222422946R^4r^7 - 266859096R^3r^8 \\
&+ 251802416R^2r^9 - 147938176Rr^{10} + 81266816r^{11}, \\
n_1 =& 983040R^{13} - 10993664R^{12}r + 44618752R^{11}r^2 \\
&- 104931072R^10r^3 + 119000576R^9r^4 - 54887936R^8r^5 \\
&- 385401860R^7r^6 + 805665315R^6r^7 - 1595272484R^5r^8 \\
&+ 1828090652R^4r^9 - 1873016416R^3r^{10} + 1461489856R^2r^{11} \\
&- 801489920Rr^{12} + 361030656r^{13},
\end{aligned}
$$

$$
\begin{aligned}
n_0 = & -(R-2r)(4915200R^{13} - 38961152R^{12}r + 75459584R^11r^2 \\
& - 203163392R^{10}r^3 + 33298560R^9r^4 - 56791776R^8r^5 \\
& - 804721572R^7r^6 + 990417869R^6r^7 - 1969598540R^5r^8 \\
& + 1914132836R^4r^9 - 1866891168R^3r^{10} + 1383155904R^2r^{11} \\
& - 721906176Rr^{12} + 343203840r^{13})r.
\end{aligned}
$$

It is easily seen that $n_6 > 0, n_5 > 0$ and $n_4 > 0$ hold by Euler's inequality $R \geq 2r$. Let $e = R - 2r$, then it is easy to get

$$
\begin{aligned}
n_3 = & 302080e^9 + 4479744e^8r + 29696256e^7r^2 + 116558912e^6r^3 \\
& + 297763148e^5r^4 + 512063451e^4r^5 + 591341408e^3r^6 \\
& + 448363440e^2r^7 + 214742784er^8 + 61517520r^9,
\end{aligned}
\tag{4.11}
$$

so that $n_3 > 0$(since $e \geq 0$). Thus, by identity (4.10) and inequality (3.19), to prove $P_0 \geq 0$ it remains to prove that

$$
P_1 \equiv n_2v_0^2 + n_1v_0 + n_0 \geq 0. \tag{4.12}
$$

Simplifying gives

$$
\begin{aligned}
P_1 = & (868352R^{11} - 5260288R^{10}r + 12472064R^9r^2 \\
& - 20809472R^8r^3 - 3435168R^7r^4 + 43260132R^6r^5 \\
& - 148694813R^5r^6 + 222422946R^4r^7 - 266859096R^3r^8 \\
& + 251802416R^2r^9 - 147938176Rr^{10} + 81266816r^{11})s^4 \\
& + (-5963776R^{13} + 32825344R^{12}r - 88255488R^{11}r^2 \\
& + 223256320R^{10}r^3 - 219410688R^9r^4 + 133206944R^8r^5 \\
& + 979991276R^7r^6 - 2395871311R^6r^7 + 4850511314R^5r^8 \\
& - 6503043464R^4r^9 + 6752430320R^3r^{10} - 6031383840R^2r^{11} \\
& + 3207436288Rr^{12} - 1751906560r^{13})s^2 + 9961472R^{15} \\
& - 51068928R^{14}r + 179355648R^{13}r^2 - 553756672R^{12}r^3 \\
& + 897704192R^{11}r^4 - 1415582528R^{10}r^5 - 373964704R^9r^6 \\
& + 3659771168R^8r^7 - 13770272875R^7r^8 + 25264298291R^6r^9 \\
& - 39108686525R^5r^{10} + 47661845038R^4r^{11} - 43929968344R^3r^{12} \\
& + 37177253872R^2r^{13} - 18121105536Rr^{14} + 9727101056r^{15}.
\end{aligned}
\tag{4.13}
$$

We now recall that for any triangle ABC we have the fundamental inequality (see [1] and [14]):

$$
t_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - (4R + r)^3r \geq 0, \tag{4.14}
$$

Gerretsen's inequality (see [4] and [14]):

$$
g_1 \equiv s^2 - 16Rr + 5r^2 \geq 0, \tag{4.15}
$$

and the following Yang's inequality (cf. [9]):

$$
g_4 \equiv 4R^3 - 2Rr^2 - r^3 - (R - r)s^2 \geq 0. \tag{4.16}
$$

Based on these three inequalities, after analysis we obtain the following identity:

$$(R-r)P_1 = (R-r)(f_1t_0 + f_2u_0g_1) + f_3g_4 + f_4(R-2r)r^2, \tag{4.17}$$

where

$$\begin{aligned}
f_1 =& Rr(5260288R^9 + 20809472R^7r^2 + 3435168R^6r^3\\
&+ 148694813R^4r^5 + 266859096R^2r^7 + 147938176r^9),\\
f_2 =& 868352R^{11} + 12472064R^9r^2 + 43260132R^6r^5\\
&+ 222422946R^4r^7 + 251802416R^2r^9 + 81266816r^{11},\\
f_3 =& 2490368R^{13} - 29151232R^{12}r + 147046400R^{11}r^2\\
&- 399980288R^{10}r^3 + 699229056R^9r^4\\
&- 279163056R^8r^5 - 1257285000R^7r^6\\
&+ 4653116315R^6r^7 - 8528923476R^5r^8\\
&+ 11722707504R^4r^9 - 11730444128R^3r^{10}\\
&+ 9672289760R^2r^{11} - 5128648960Rr^{12}\\
&+ 2076973824r^{13},\\
f_4 =& 1245184R^{13} - 27208704R^{12}r + 257814272R^{11}r^2\\
&- 910566784R^{10}r^3 + 1682561728R^9r^4\\
&- 1105962876R^8r^5 - 2916304483R^7r^6\\
&+ 10574332574R^6r^7 - 20268016148R^5r^8\\
&+ 27432724020R^4r^9 - 27479086112R^3r^{10}\\
&+ 21739795392R^2r^{11} - 11389301632Rr^{12}\\
&+ 4028230656r^{13}.
\end{aligned}$$

By using Euler's inequality (3.21), we easily show that $f_3 > 0$ and $f_4 > 0$. Therefore, by identity (4.17), Euler's inequality, Ciamberlini's acute inequality (3.25), the fundamental inequality (4.14), Gereetsen's inequality (4.15) and Yang's inequality (4.16), we conclude that $P_1 \geq 0$ holds for the acute triangle ABC. And, inequality (4.2) is proved. Also, it is easy to know that the equality in (4.2) holds if and only if $\triangle ABC$ is equilateral. Theorem 4.1 is proved.

5. Applications of Theorem 1.1 (III)

In this section, we apply Theorem 1.1 to prove the following inequality:

Theorem 5.1. *Let $\triangle ABC$ be an acute triangle, then*

$$\sum \frac{1}{r_a + 2m_a} \leq \frac{3}{\sum m_a}, \tag{5.1}$$

with equality if and only if $\triangle ABC$ is equilateral.

Proof. By Theorem 1.1, we only need to prove

$$\sum \frac{1}{r_a + 2v_a} \le \frac{3}{\sum m_a}, \tag{5.2}$$

where v_a, v_b, v_c are given by (3.2), (3.3) and (3.4), respectively.

Computing gives

$$\sum \frac{1}{r_a + 2v_a} = \frac{2RF_0}{E_0}, \tag{5.3}$$

$$\begin{aligned} E_0 =& s^8 + (12R^2 - 34Rr + 4r^2)s^6 + 2(14R - r)(4R^3 \\ & - 6R^2r + 13Rr^2 - r^3)s^4 + 2(104R^4 + 52R^3r \\ & + 114R^2r^2 + 29Rr^3 - 2r^4)(2R - r)^2s^2 \\ & + (2R - 3r)(4R^2 + r^2)^2(4R + r)^3, \\ F_0 =& s^6 + (24R^2 - 24Rr + 3r^2)s^4 + (336R^4 - 128R^3r \\ & + 272R^2r^2 - 8Rr^3 + 3r^4)s^2 + (4R^2 + r^2)(20R^2 \\ & - 24Rr + r^2)(4R + r)^2. \end{aligned}$$

Now, we recall that for any triangle ABC it holds that

$$\left(\sum m_a\right)^2 \le 4s^2 - 16Rr + 5r^2, \tag{5.4}$$

which was established by Chu X.G and Yang X.Z in [3] (for an improvement of (5.4), see [6]). Thus, in order to prove inequality (5.2) we needs to show that the following inequality holds for acute triangle ABC:

$$G_0 \equiv 9E_0^2 - 4R^2(4s^2 - 16Rr + 5r^2)F_0^2 \ge 0. \tag{5.5}$$

With the help of Maple, it is easy to obtain

$$\begin{aligned} G_0 =& 9s^{16} + (200R^2 - 612Rr + 72r^2)s^{14} + (2544R^4 - 9680R^3r \\ & + 17920R^2r^2 - 3420Rr^3 + 180r^4)s^{12} + (19200R^6 \\ & - 88448R^5r + 185408R^4r^2 - 260912R^3r^3 + 59832R^2r^4 \\ & - 3780Rr^5 + 72r^6)s^{10} + (30464R^8 - 516608R^7r \\ & + 1042432R^6r^2 - 1563264R^5r^3 + 1765312R^4r^4 \\ & - 383936R^3r^5 - 11024R^2r^6 + 5220Rr^7 - 306r^8)s^8 \\ & - (669696R^{10} + 816128R^9r - 3584000R^8r^2 \\ & + 3787264R^7r^3 - 4849408R^6r^4 + 3600896R^5r^5 \\ & - 261056R^4r^6 - 575328R^3r^7 + 90568R^2r^8 - 5940Rr^9 \end{aligned}$$

$$+360r^{10})s^6-(3403776R^{12}-10219520R^{11}r$$
$$-1802240R^{10}r^2-2753536R^9r^3+693248R^8r^4$$
$$+3106304R^7r^5+2192128R^6r^6-121088R^5r^7$$
$$+1403728R^4r^8+72464R^3r^9-57888R^2r^{10}$$
$$+3636Rr^{11}-36r^{12})s^4+4(4R^2+r^2)(17408R^{10}$$
$$+199936R^9r-312704R^8r^2+147200R^7r^3$$
$$-225248R^6r^4+71328R^5r^5-5928R^4r^6$$
$$+9120R^3r^7-3022R^2r^8-819Rr^9+54r^{10})(4R+r)^2s^2$$
$$+(9216R^8+2560R^7r-57344R^6r^2+55744R^5r^3$$
$$-9776R^4r^4+3904R^3r^5+1096R^2r^6+540Rr^7$$
(5.6) $$+81r^8)(4R^2+r^2)^2(4R+r)^4.$$

Putting

$$x_3=488R^2-684Rr+1008r^2,$$
$$x_2=12176R^4-30232R^3r+68880R^2r^2-66168Rr^3+49320r^4,$$
$$x_1=179712R^6-599456R^5r+1760552R^4r^2-2834060R^3r^3+3726912R^2r^4-2738448Rr^5+1376928r^6,$$
$$x_0=1634304R^8-6878208R^7r+23708592R^6r^2-50184664R^5r^3+88610922R^4r^4-108409896R^3r^5+105175256R^2r^6-62854560Rr^7+23990544r^8,$$

then it is easy to show that $x_3>0, x_2>0, x_1>0$ and $x_0>0$ in the same way to prove the previous inequality $m_5>0$. So, by the previous inequality (3.19), for the acute triangle ABC we have the following inequality:

(5.7) $$G_1\equiv(9v_0^4+x_3v_0^3+x_2v_0^2+x_1v_0+x_0)v_0^4\geq 0.$$

Putting

$$y_1=8454144R^{10}-47644672R^9r+184210432R^8r^2-474948032R^7r^3+1009390840R^6r^4-1622741732R^5r^5+2170095280R^4r^6-2175491488R^3r^7+1709281728R^2r^8-864124416Rr^9+267120000r^{10},$$
$$y_2=18874368R^{12}-139198464R^{11}r+457072640R^{10}r^2-996445184R^9r^3+1493202224R^8r^4-1068850736R^7r^5-899648316R^6r^6+4757436632R^5r^7-8677926856R^4r^8+10423632896R^3r^9-8641648992R^2r^{10}+4589170560Rr^{11}-1349316864r^{12},$$

$$
\begin{aligned}
y_3 = {} & 100663296R^{14} - 488636416R^{13}r + 4406902784R^{12}r^2 \\
& - 19865489408R^{11}r^3 + 60769593856R^{10}r^4 \\
& - 147857566800R^9r^5 + 291870859632R^8r^6 \\
& - 481805172895R^7r^7 + 672580302642R^6r^8 \\
& - 780632760708R^5r^9 + 754989195288R^4r^{10} \\
& - 577124774080R^3r^{11} + 340673039872R^2r^{12} \\
& - 133932201984Rr^{13} + 30422310912r^{14}, \\
y_4 = {} & 264241152R^{13} - 1578041344R^{12}r \\
& + 6416596992R^{11}r^2 - 18119720448R^{10}r^3 \\
& + 41669273248R^9r^4 - 79326701048R^8r^5 \\
& + 126855932388R^7r^6 - 173394751264R^6r^7 \\
& + 197394940496R^5r^8 - 188441627808R^4r^9 \\
& + 142309372800R^3r^{10} - 83361408896R^2r^{11} \\
& + 32500242432Rr^{12} - 7359012864r^{13}.
\end{aligned}
$$

Then we can also prove $y_1 > 0, y_2 > 0, y_3 > 0$ and $y_4 > 0$ in the same way to prove $m_5 > 0$. Thus, in view of the previous inequalities (3.21), (3.24) and (3.25), we deduce that the following inequality:

$$G_2 \equiv (u_0y_1 + y_2)v_0^2 + (R - 2r)ry_3 + rg_2y_4 \geq 0 \tag{5.8}$$

holds for the acute triangle ABC.

Finally, it is easy to verify the following identity:

$$G_0 = G_1 + G_2. \tag{5.9}$$

Therefore, the two inequalities (5.7) and (5.8) show that $G_0 \geq 0$ holds for the acute triangle ABC. We thus finish the proof of inequality (5.1). Also, we easily know that equality in (5.1) holds if and only if $\triangle ABC$ is equilateral. This completes the proof of Theorem 5.1.

Remark 5.1. *The author has known that the previous inequality* (1.6) *can not be used to prove Theorem 3.1, Theorem 4.1 and Theorem 5.1.*

6. Some conjectures

In the last section, we give some interesting related conjectures.

Considering exponential generalizations of Theorem 3.1, we propose the following conjecture:

Conjecture 6.1. *If $k > 1$, then for the acute triangle ABC the following inequality holds:*

$$\sum \frac{1}{m_b^k + m_c^k} \geq \sum \frac{1}{r_a^k + m_a^k}. \tag{6.1}$$

Conjecture 6.2. *Let p and q be real numbers such that $p \le 1$ and $q > 0$, then for the acute triangle ABC the following inequality holds:*

$$\sum \frac{a^p}{(m_b+m_c)^q} \ge \sum \frac{a^p}{(r_a+m_a)^q}, \tag{6.2}$$

If $p \ge 1$ and $q < 0$, then the inequality holds reversely for any triangle ABC.

For the inequality of Theorem 5.1, we have the following conjecture:

Conjecture 6.3. *Inequality* (5.1) *holds for any triangle ABC.*

We have known that in the acute triangle ABC the following inequality holds (cf. [11]):

$$\sum m_a \ge \frac{3}{2}\sqrt{\sum bc}, \tag{6.3}$$

which together with Theorem 1.1 yields

$$\sum \frac{1}{r_a+2m_a} \le \frac{2}{\sqrt{\sum bc}}. \tag{6.4}$$

This inequality seems to be true for any triangle ABC. We present the following general inequality with one parameter:

Conjecture 6.4. *If $1 \le k \le 6.12$, then for any triangle ABC the following inequality holds:*

$$\sum \frac{1}{r_a+km_a} \le \frac{6}{(k+1)\sqrt{\sum bc}}. \tag{6.5}$$

If $0 < k \le 0.37$, then the inequality holds reversely for any triangle ABC.

Inequality (6.5) also motivates the author to propose the following inequality with two parameters:

Conjecture 6.5. *Let p and q be real numbers such that $p \ge q \ge 1$, then for any triangle ABC the following inequality holds:*

$$\sum \frac{1}{pr_a+m_a} \ge \frac{q+1}{p+1}\sum \frac{1}{qm_a+r_a}. \tag{6.6}$$

If $0 < q \le p \le 1$, then the inequality holds reversely.

Finally, we give a generalization of the previous acute triangle inequality (3.27) after checking by the computer, that is

Conjecture 6.6. *If $k \ge 1.45$, then for any triangle ABC the following inequality holds:*

$$\sum \frac{1}{r_a+km_a} \le \frac{1}{k+1}\sum \frac{1}{m_a}. \tag{6.7}$$

If $0.65 \le k \le \sqrt{2}$, then the inequality holds reversely. If $0 < k \le 0.24$, then the inequality holds reversely for the acute triangle ABC.

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