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Tango-Quadrilaterals

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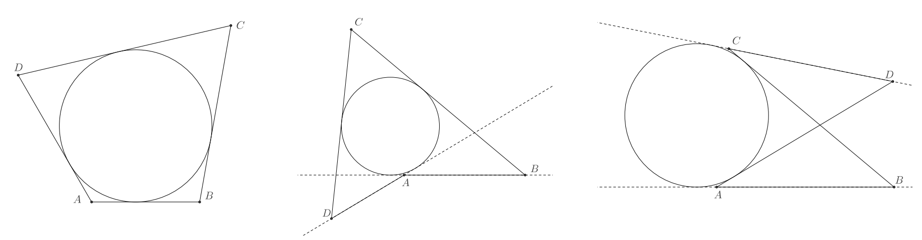
Abstract. In this note, we represent and characterize a tango-quadrilateral as a quadrilateral having a circle that is tangent to the lines supporting its sides.

1. Introduction

The problem of tangential and extangential quadrilaterals characterizations is a well known problem; see [3] and [4] respectively. By definition an extangential quadrilateral is a convex quadrilateral with an external circle tangent to the extensions of all four sides. Similarly a tangential quadrilateral is a convex quadrilateral with an incircle, that is a circle tangent at the inside of the quadrilateral to all four sides.

Our purpose here is to give a unified presentation for both tangential and extangential quadrilaterals. Throughout this paper we assume quadrilaterals without three aligned vertices.

Definition 1.1. *A tango-quadrilateral is any quadrilateral having a circle tangent to the lines supporting its four sides, (see Figure* 1*).*



Convex **Concave** **Crossed**

Figure 1.

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We start by some classic properties of tangential and extangential quadrilaterals:

Lemma 1.1 (Pitot's theorem)**.** [3] *A convex quadrilateral is tangential if and only if the sums of opposite sides are equal.*

Lemma 1.2. [4] *A convex quadrilateral is extangential if and only if the sum of two adjacent sides is equal to the sum of the other two.*

The reader can see [3, 4] and the references therein for further equivalent properties of such quadrilaterals.

Lemma 1.3 (Newton's line)**.** [4, 2] *Suppose $ABCD$ is a tangential or extangential quadrilateral. If M and N are the midpoints of its diagonals and O the center of the tangent circle, then M,N and O are collinear.*

See also [1] for a proof of Lemma 1.3 with a historical view. We reprove previous lemmas analytically in a slightly more general setting, together with a trigonometric characterization for tango-quadrilaterals.

Given a quadrilateral $ABCD$ with corners A, B, C, D and sides $[AB]$, $[BC]$, $[CD]$ and $[DA]$, we call an inside (internal) bisector of a corner, the bisector of the angle formed by the segments of adjacent sides. The other bisector is called the outside (external) bisector (the one of the supplementary angle). The following is a basic formulation of tango-quadrilaterals:

Proposition 1.1. *A quadrilateral $ABCD$ is a tango-quadrilateral if and only if there exist four (equivalently 3) angle bisectors, one from each corner such that they intersect at a single point denoted by O.*

2. MAIN RESULT

Before stating the main results we consider which angle bisectors are like to intersect.

Proposition 2.1. *For a tango-quadrilateral $ABCD$, if 4 corner bisectors intersect at O, then no consecutive two are both outside bisectors.*

Proof. This is straightforward, for suppose the converse; the two outside consecutive bisectors intersect at point O outside of $ABCD$ and it is easy to see that, the fourth side (line) can not be tangent to that circle (using that from a point T on circle (C), there is a single line tangent to (C) and from a point T' outside of (C), there are only two tangents to (C)).

Similarly we have:

Proposition 2.2 (Figure 2)**.** *For a crossed tango-quadrilateral $ABCD$, if 4 corner bisectors intersect at O, no consecutive two are both inside bisectors.*

Hereafter consider a line segment $[AB]$ (B on the right from A) with two angle corners $2a$ and $2b$ at A and B, respectively, a and b in $]0; 90°[$. The points A' and B' verify $(\overrightarrow{AB}; \overrightarrow{AA'}) = 2a$ and $(\overrightarrow{BB'}; \overrightarrow{BA}) = 2b$, respectively, (all directed angles are assumed to have an anticlockwise positive orientation).

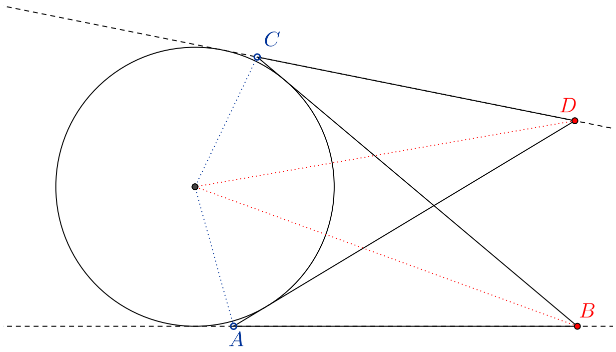

FIGURE 2.

Assume that the angle bisectors of $2a$ and $2b$ intersect at O and that without loss of generality O is in the upper half plane divided by (AB). Notice here that $a+b<180°$. For $C \in (BB')$, $(C \neq B)$ let $(\overrightarrow{CO};\overrightarrow{CB}) = d$ and for $D \in (AA')$, $(D \neq A)$ let $(\overrightarrow{DA};\overrightarrow{DO}) = x$, see Figure 3. The quadrilateral $ABCD$ is a tango-quadrilateral if and only if $x = 180° - a - b - d$ or $x = 540° - a - b - d$. In particular if $x+a+b+d = 180°$, $ABCD$ has an inside tangent circle and for $x+a+b+d = 540°$, $ABCD$ has an external tangent circle. The proof of this main characterization is straightforward in term of angle values by drawing a line (CC') with $(\overrightarrow{CC'};\overrightarrow{CB}) = 2d$ and $(CC') \cap (AA') = D$.

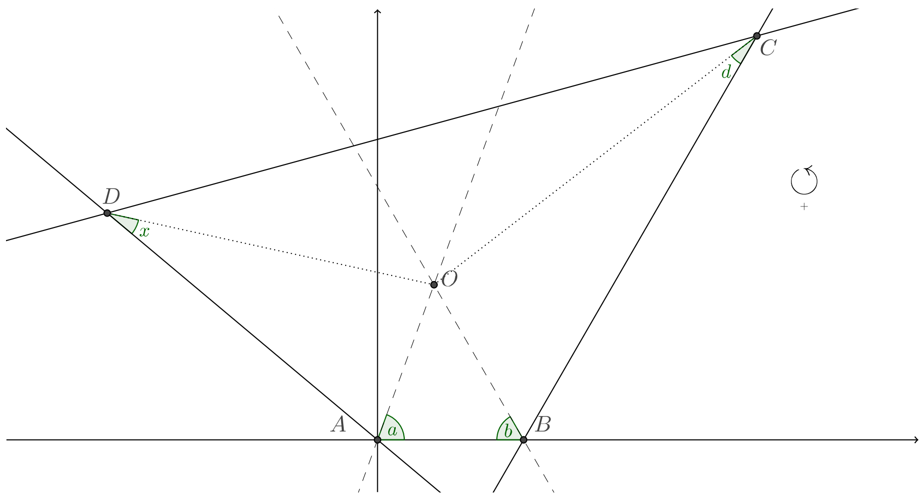

FIGURE 3. The main configuration

Remark 2.1. *The quadrilateral $ABCD$ is uniquely determined given the directed angles x and d. Notice also that $x+a+b+d \equiv 0 \pmod{180°}$ is* *equivalent to* $x+a+b+d = 180°$ *or* $x+b+d+a = 540°$*; as one can see that* $x+a+b+d = 360°$ *and* $x+a+b+d = 720°$ *are not possible; for example if* $a+b+d < 180°$ *then* $x > 180°$ *and for such point* D*,* $x > 360° - a$*, a contradiction. Similarly for* $x+a+b+d = 540°$*,* $x+d > 360°$ *with* x *or* d $< 180°$*.*

Given an orthonormal axis $(O; \overrightarrow{i}; \overrightarrow{j})$, the equation of a line (l) is $\mathtt{y} = \alpha\mathtt{x} + \beta$ for some reals α and β, the real α is called the slope of (l). In the sequel we take α (without loss of generality) to be finite and well defined; the case of a vertical line is taken as a tending limit for $\alpha \to \pm\infty$ and $\alpha = \dfrac{\mathtt{y} - y_A}{\mathtt{x} - x_A}$ for some point $A \in (l)$.

Lemma 2.1. *Let* I *and* J *be the midpoints of the diagonals* $[DB]$ *and* $[AC]$*, respectively. For* $a+b \neq 90°$*, the points* I*,* J *and* O *are collinear if and only if* $ABCD$ *is a tango-quadrilateral.*

Proof. We take here an orthonormal axis centered at A, up to a scalar multiple let $d(O; (AB)) = 1$, so the circle is of radius one. We find the slope of (OJ); the slope of (OI) can be deduced by replacing $a \leftrightarrow b$, $d \leftrightarrow x$ and adding a minus sign to the corresponding slopes (vertical reflection). From the angle definitions we have: $A(0;0)$, $O(\frac{\cos(a)}{\sin(a)};1)$, $B(\frac{\cos(a)}{\sin(a)} + \frac{\cos(b)}{\sin(b)};0)$, $C(\frac{\cos(a)}{\sin(a)} - \frac{\cos(2b+d)}{\sin(d)};1+\frac{\sin(2b+d)}{\sin(d)})$. If m is the slope of (OJ), it can be verified that

$$m = \frac{\sin(a)(\sin(d) - \sin(2b+d))}{\sin(a)\cos(2b+d) + \sin(d)\cos(a)}$$

consequently the slope n of (OI) is

$$n = \frac{-\sin(b)(\sin(x) - \sin(2a+x))}{\sin(b)\cos(2a+x) + \sin(x)\cos(b)}.$$

Solving $n = m$ in x gives:

$$\frac{\sin(a)(\sin(d+b-b) - \sin(b+b+d))}{\sin(a)\cos(b+b+d) + \sin(d+b-b)\cos(a)} = \frac{-\sin(b)(\sin(x+a-a) - \sin(a+a+x))}{\sin(b)\cos(a+a+x) + \sin(x+a-a)\cos(b)},$$

after expanding and simplifying we get:

$$\frac{2\sin(a)\sin(b)}{\sin(b-a) + \tan(a+x)\cos(a+b)} = \frac{2\sin(a)\sin(b)}{\sin(b-a) - \tan(b+d)\cos(a+b)},$$

equivalently $(\sin(a)\sin(b) \neq 0)$

$$\tan(a+x)\cos(a+b) = \tan(-b-d)\cos(a+b)$$

which implies $x+a+b+d \equiv 0 \pmod{180°}$ as $\cos(a+b) \neq 0$.

A similar characterization holds if we consider two angle bisectors of two opposite corners for the quadrilateral $ABCD$.

Lemma 2.2. *In an orthonormal system* $(O; \overrightarrow{i}; \overrightarrow{j})$*, let* $A(r\cos(\theta); r\sin(\theta))$*,* $C(c;0)$ *for some positives* r, c *and* $\theta \in \mathbb{R}$*. If a point* $B(t\cos(\alpha); t\sin(\alpha))$ *is taken such that* (OB) *is an angle bisector of the lines* (BA) *and* (BC) *at* B*, then*

$$t = \frac{rc\sin(2\alpha - \theta)}{c\sin(\alpha) - r\sin(\theta - \alpha)}.$$

Proof. We use the characterization of an angle bisector so:

$$\underbrace{d(O;(BA))}_{k} = \underbrace{d(O;(BC))}_{h}.$$

By the law of cosines:

$$4Area(OCB)^2 = h^2(t^2 + c^2 - 2tc\cos(\alpha)) = t^2c^2\sin(\alpha)^2.$$
$$4Area(OAB)^2 = k^2(t^2 + r^2 - 2tr\cos(\theta - \alpha)) = t^2r^2\sin(\theta - \alpha)^2.$$

The equation $h^2 = k^2$ gives the quadratic

$$(t(c\sin(\alpha)+r\sin(\theta-\alpha))-rc\sin(\theta))(t(r\sin(\alpha-\theta)+c\sin(\alpha))-rc\sin(2\alpha-\theta)) = 0.$$

The roots are $t_1 = \dfrac{rc\sin(\theta)}{c\sin(\alpha) + r\sin(\theta - \alpha)}$ and $t_2 = \dfrac{rc\sin(2\alpha - \theta)}{c\sin(\alpha) - r\sin(\theta - \alpha)}$. The first root gives A, B and C aligned which we have already excluded.

Theorem 2.1. *Given a quadrilateral $ABCD$ with no three aligned vertices; let the angle bisectors of two opposite corners intersect at O with an angle θ. If $\theta \neq 90°$, the midpoints of the diagonals (I and J) are collinear with O if and only if $ABCD$ is a tango-quadrilateral.*

Proof. Take as in Lemma 2.2 the points $O(0;0)$, $A(r\cos(\theta); r\sin(\theta))$, $C(c;0)$ and $B(t\cos(\alpha); t\sin(\alpha))$ where r and c are positives reals with $\theta \in [0;180°]$. We start by the following figure, let B_1 and B_2 be the symmetric of B with respect to (OA) and (OC), respectively. Assume that $(B_1A) \cap (B_2C) = D$. It can be verified that (B_2C) is the line $\mathtt{y} = \dfrac{y_B\mathtt{x}}{c - x_B} + \dfrac{y_Bc}{x_B - c}$ and (B_1A) is

$$\mathtt{y} = \frac{x_B\sin(2\theta) - y_B\cos(2\theta) - r\sin(\theta)}{x_B\cos(2\theta) + y_B\sin(2\theta) - r\cos(\theta)}\mathtt{x} + \frac{ry_B\cos(\theta) - rx_B\sin(\theta)}{x_B\cos(2\theta) + y_B\sin(2\theta) - r\cos(\theta)}.$$

This gives point $D(x_D; y_D)$ where

$$x_D = \frac{rx_B\sin(\theta)(c - x_B) + y_B(rx_B\cos(\theta) - cx_B\cos(2\theta) - cy_B\sin(2\theta))}{(cx_B - t^2)\sin(2\theta) + y_B(r\cos(\theta) - c\cos(2\theta)) + r\sin(\theta)(x_B - c)}.$$

Simplify $\dfrac{y_D + y_B}{x_D + x_B}$ to get

$$\textit{slope}\,(OI) = \frac{\sin(\alpha)\left(t - r\frac{\cos(\alpha)}{\cos(\theta)}\right)}{\cos(\alpha)\left(t - \dfrac{c\sin(2\theta - 2\alpha) + r\sin(2\alpha)\cos(\theta)}{\sin(2\theta)\cos(\alpha)}\right)},\ (ID = IB),$$

$$\textit{slope}\,(OJ) = \frac{r\sin(\theta)}{r\cos(\theta) + c} = m,\ (JA = JC),$$

with $n = \textit{slope}\,(OI) = \dfrac{2\sin(\theta)\sin(\alpha)(r\cos(\alpha) - t\cos(\theta))}{c\sin(2(\theta - \alpha)) + r\sin(2\alpha)\cos(\theta) - t\sin(2\theta)\cos(\alpha)}$. Solving $n = m$ gives

$$\sin(\theta)\cos(\theta)(t(c\sin(\alpha) + r\sin(\alpha - \theta)) + rc\sin(\theta - 2\alpha)) = 0,$$

this is true for $\theta \equiv 0 \pmod{90°}$ and for $t = \dfrac{rc\sin(2\alpha - \theta)}{c\sin(\alpha) + r\sin(\alpha - \theta)}$. The case $\theta \equiv 0 \pmod{180°}$ gives $ABCD$ a kite, otherwise the given solution is the one for $ABCD$ a tango-quadrilateral by Lemma 2.2.

Now assume $\theta \not\equiv 0 \pmod{90°}$, the particular case when

$$r\frac{\cos(\alpha)}{\cos(\theta)} = \frac{c\sin(2\theta-2\alpha)+r\sin(2\alpha)\cos(\theta)}{\sin(2\theta)\cos(\alpha)}$$

gives $c = r\dfrac{\cos(\alpha)}{\cos(\theta-\alpha)} = c_0$. For $c = c_0$, from $t = r\frac{\cos(\alpha)}{\cos(\theta)}$, $(B_1A) \equiv (B_2C)$ (excluded) and from *slope* $(OI) = \frac{\sin(\alpha)}{\cos(\alpha)} = m$ we get $c = r\frac{\sin(\theta-\alpha)}{\sin(\alpha)} = c_1$. The equation $c_0 = c_1$ is equivalent to $\sin(2\alpha) = \sin(2\theta - 2\alpha)$ so $\alpha \equiv \frac{\theta}{2} \pmod{90°}$, $c = r$ and $ABCD$ is a kite, or $\theta = 90°$. This completes the proof.

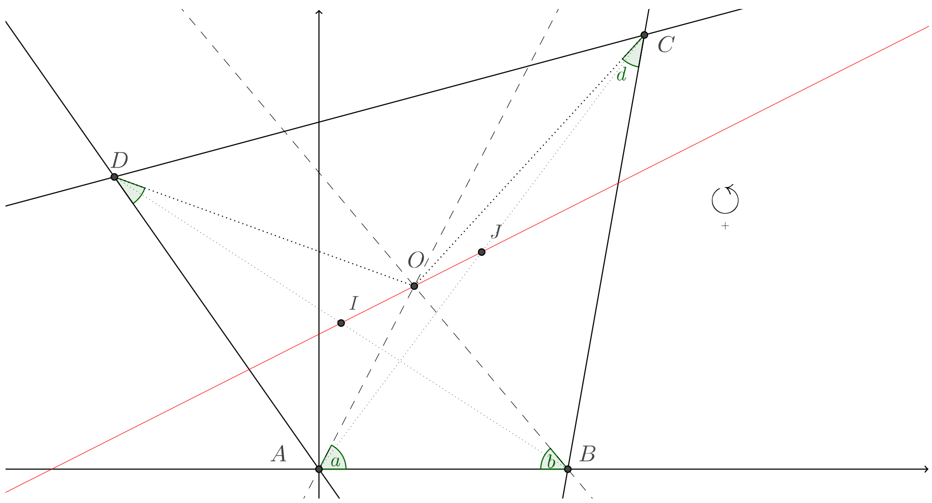

FIGURE 4. The Newton line

A particular case of Theorem 2.1 is proved in [4] (Theorem 5.2). A rhombus is a tangential quadrilateral and if only one diagonal is the perpendicular bisector of the other one, then such quadrilateral (a kite) has an incircle and an external circle tangent to the lines of its sides.

Now consider an orthonormal system centered at O with $d(O;(AB)) = 1$, we have $A(-\frac{\cos(a)}{\sin(a)};-1)$, $B(\frac{\cos(b)}{\sin(b)};-1)$, $C(-\frac{\cos(2b+d)}{\sin(d)};\frac{\sin(2b+d)}{\sin(d)})$. The midpoints I and J are:

$$I\left(\frac{\cos(b)}{2\sin(b)}+\frac{\cos(2a+x)}{2\sin(x)};\frac{\sin(2a+x)}{2\sin(x)}-\frac{1}{2}\right),$$

$$J\left(-\frac{\cos(a)}{2\sin(a)}-\frac{\cos(2b+d)}{2\sin(d)};\frac{\sin(2b+d)}{2\sin(d)}-\frac{1}{2}\right),$$

and $y_J = 0$ if and only if $\sin(2b+d) = \sin(d)$ thus $b + d \equiv 90° \pmod{180°}$. Equivalently $b+d = 90°$ for b acute and $b+d = 450°$ for b obtuse (d verifies $d < 180° - b$ or $d > 360° - b$); this gives $J(-\frac{\cos(a)}{2\sin(a)} + \frac{\sin(b)}{2\cos(b)};0)$. When $a + x = 90°$ or $a + x = 450°$ we get $I(\frac{\cos(b)}{2\sin(b)} - \frac{\sin(a)}{2\cos(a)};0)$. It is easy to verify that $x_I x_J > 0$ for a or b obtuse and $x_I x_J < 0$ for a and b acute; with $x_I = x_J = 0$ if and only if $a + b = 90°$ ($ABCD$ is a rhombus). With the same previous notation:

Proposition 2.3. *For $x+a+b+d=180^\circ$, $\overrightarrow{OJ}=\alpha\overrightarrow{IO}$ with $\alpha\in\mathbb{R}^+$ and α is nul if and only if $ABCD$ is a rhombus.*

Proof. The case $y_J=0$ is already discussed so assume $y_J\neq 0$ and so is y_I, we need to prove $y_Iy_J<0$, equivalently

$$(\sin(2b+d)-\sin(d))(\sin(b+d-a)-\sin(b+d+a))<0. \tag{1}$$

- Say $90^\circ<b+d\Leftrightarrow 2b+d>180^\circ-d\Leftrightarrow b+d-a>180^\circ-b-d-a$. Since $b+d<180^\circ$ and a acute this implies that $\sin(2b+d)<\sin(d)$ and $\sin(b+d-a)>\sin(b+d+a)$.
- When $2b+d<180^\circ-d$ so $0<b+d<90^\circ$; $\sin(2b+d)>\sin(d)$ and $\sin(b+d-a)<\sin(b+d+a)$.

Proposition 2.4. *For $x+a+b+d=540^\circ$, $\overrightarrow{OJ}=\alpha\overrightarrow{IO}$ with $\alpha\in\mathbb{R}^{*-}$.*

Proof. We may assume y_I and y_J are not zero, by Remark 2.1 we know that $\sin(x)\sin(d)<0$. Let without loss of generality $d<180^\circ-b<180^\circ$ so $180^\circ<b+d+a<360^\circ$ and (1) is proved as follows:

- If $180^\circ<b+d+a<270^\circ$, for $b+d>90^\circ$, $\sin(b+d-a)-\sin(b+d+a)>0$ and similarly to the last proof $\sin(2b+d)<\sin(d)$. When $b+d-a<180^\circ-a-b-d$, $b+d<90^\circ$, $a>90^\circ$ with $\sin(b+d-a)-\sin(b+d+a)<0$ and $\sin(2b+d)>\sin(d)$.
- If $270^\circ\le b+d+a<360^\circ$, from $180^\circ>b+d>90^\circ$, $a>90^\circ$ and $b<90^\circ$: $\sin(b+d-a)-\sin(b+d+a)>0$ and $\sin(2b+d)<\sin(d)$.

Proposition 2.5. *Under the previous notation:*

$$\left(\frac{\cos(x)}{\sin(x)}+\frac{\cos(d)}{\sin(d)}\right)^2=\frac{1}{\sin(x)^2}+\frac{1}{\sin(d)^2}+\frac{2}{\sin(x)\sin(d)}\cos(2b+2a+d+x)$$

if and only if $x+a+b+d\equiv 0 \pmod{180^\circ}$.

Proof. Expanding and simplifying gives $\cos(x+d)=\cos(2a+2b+d+x)$ and thus $x+a+b+d\equiv 0 \pmod{180^\circ}$.

It can be verified that:

$$DC^2=\frac{1}{\sin(x)^2}+\frac{1}{\sin(d)^2}+\frac{2}{\sin(x)\sin(d)}\cos(2b+2a+d+x). \tag{2}$$

Notice that $(\frac{\cos(x)}{\sin(x)}+\frac{\cos(d)}{\sin(d)})$ has the same sign as $\frac{\sin(x+d)}{\sin(x)\sin(d)}$. From Remark 2.1, given a tango-quadrilateral $\sin(x+d)>0$. For $x+a+b+d=180^\circ$: $\sin(x)\sin(d)>0$ and for $x+a+b+d=540^\circ$: $\sin(x)\sin(d)<0$.

For $d<180^\circ$ (resp. $x<180^\circ$), $BC=\frac{\cos(b)}{\sin(b)}+\frac{\cos(d)}{\sin(d)}$ with $b+d<180^\circ$ (resp. $AD=\frac{\cos(x)}{\sin(x)}+\frac{\cos(a)}{\sin(a)}$ with $a+x<180^\circ$). For $d>180^\circ$ (resp. $x>180^\circ$), $BC=\frac{\cos(180^\circ-b)}{\sin(180^\circ-b)}+\frac{\cos(360^\circ-d)}{\sin(360^\circ-d)}=-(\frac{\cos(b)}{\sin(b)}+\frac{\cos(d)}{\sin(d)})$ with $b+d>360^\circ$ (resp. $AD=-(\frac{\cos(a)}{\sin(a)}+\frac{\cos(x)}{\sin(x)})$ with $a+x>360^\circ$). Consequently we have the following:

- If $x<180^\circ$ and $d<180^\circ$, then $ABCD$ is a tango-quadrilateral with an incircle if and only if $AB+DC=AD+BC$.
- If $x>180^\circ$ and $d<180^\circ$, then $ABCD$ is a tango-quadrilateral with an excircle if and only if $AD+AB=DC+BC$.

- If $x < 180°$ and $d > 180°$, then $ABCD$ is a tango-quadrilateral with an excircle if and only if $AB + BC = DC + AD$.
- If $x > 180°$ and $d > 180°$, $ABCD$ is not a tango-quadrilateral.

Acknowledgments

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