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FIFTEEN COLLINEAR POINTS IN BICENTRIC QUADRILATERALS

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Abstract. We prove there are another thirteen significant points on the line defined by the incenter and the circumcenter in a bicentric quadrilateral.

1. The line of centers

A *bicentric quadrilateral* is a convex quadrilateral that has the capacity of having both an incircle (it is tangential) and a circumcircle (it is cyclic), meaning that its sides are tangent to the former circle and its vertices are concyclic on the latter. The centers of these two circles are in general two different points, and they coincide only in a square according to Theorem 6.2 in [1]. Hence except in a square, the incenter I and the circumcenter O in a bicentric quadrilateral define a line, which has been called the *line of centers* in [19]. There are more than a dozen other interesting points located on the line OI that will be studied in this paper. Most of these are known to lie on this line (although certainly not well-known), but these facts are scattered in various books, papers, and online math forums. Thus the main merit of this paper is to collect for the first time in one place that there are at least fifteen significant points located on the line of centers.

The origin for this paper was in the spring of 2019 when I discovered five collinear points in a bicentric quadrilateral when making investigations in GeoGebra (whey will be denoted O, I, P, V, and Q). At that time I was unable to prove this result, so this study was shelved for one and a half years until the author of [11] contacted me and we had some correspondence about a few of these points. As time went by I found references for several additional points on the line of centers and thought eventually that it was such a fascinating subject that it was worthy of another paper (besides [11]) even if I had not contributed so much to the original proofs of the collinearity for most of the points. I shall present proofs for why these fifteen points are collinear in my own words, providing references were such are known.

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2. The diagonal point

The intersection of the diagonals in a quadrilateral, which has been called the *diagonal point* in [19], is probably the most well-known point to belong to the line OI in a bicentric quadrilateral. We will denote this point by P. That O, I, and P are collinear was proved almost a century ago in [20]. This result resurfaced again in 1989, when Shailesh Shirali from India (see [18]) proposed as a problem for the International Mathematical Olympiad to prove that the line through the two centers of a bicentric quadrilateral also contains the diagonal point. This problem was not chosen as one of the contest problems, but has nonetheless attracted much attention and several different (mostly long) solutions have been given thereafter in [2], [3], [7, p. 521], and [10, pp. 100–105]. The method we will use here was suggested at [21], which we consider to be the most attractive in regards to being short and applying elementary methods.

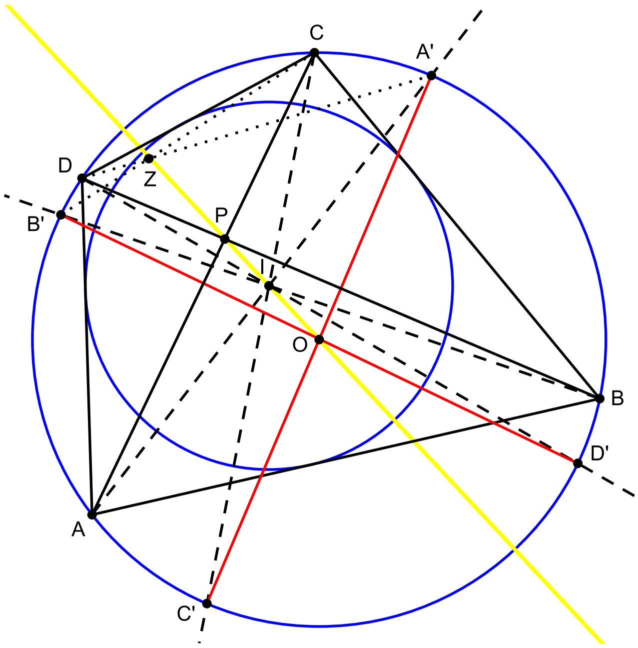

Figure 1. P, I, and O are collinear

In a bicentric quadrilateral $ABCD$ with circumcenter O, incenter I, and diagonal point P, let AI intersect the circumcircle at A' and define the points B', C', and D' in the same way (see Figure 1). Then $A'C'$ and $B'D'$ are diameters of the circumcircle. To see this, we have that

$$\angle C'OB + \angle A'OB = 2(\angle C'CB + \angle A'AB) = \angle C + \angle A = \pi$$

according to the inscribed angle theorem and applying that the dashed lines are angle bisectors to the vertex angles. This proves that $A'C'$ is a diameter, the argument for $B'D'$ is similar. Now let Z be the intersection of CB' and DA'. Then the points Z, P, and I are collinear according to Pascal's theorem applied in hexagram $B'CAA'DB$. (Pascal's theorem is proved using Menelaus' theorem, which in turn can be proved by similarity, see for instance [15, pp. 110–112, 119–121].) But we also have that Z, I, and O are collinear by applying Pascal's theorem in hexagram $B'CC'A'DD'$, since diameters $A'C'$ and $B'D'$ intersect at O. It follows that the lines ZPI and ZIO coincide (since there is a unique line through Z and I), concluding the proof that O, I, and P are collinear.

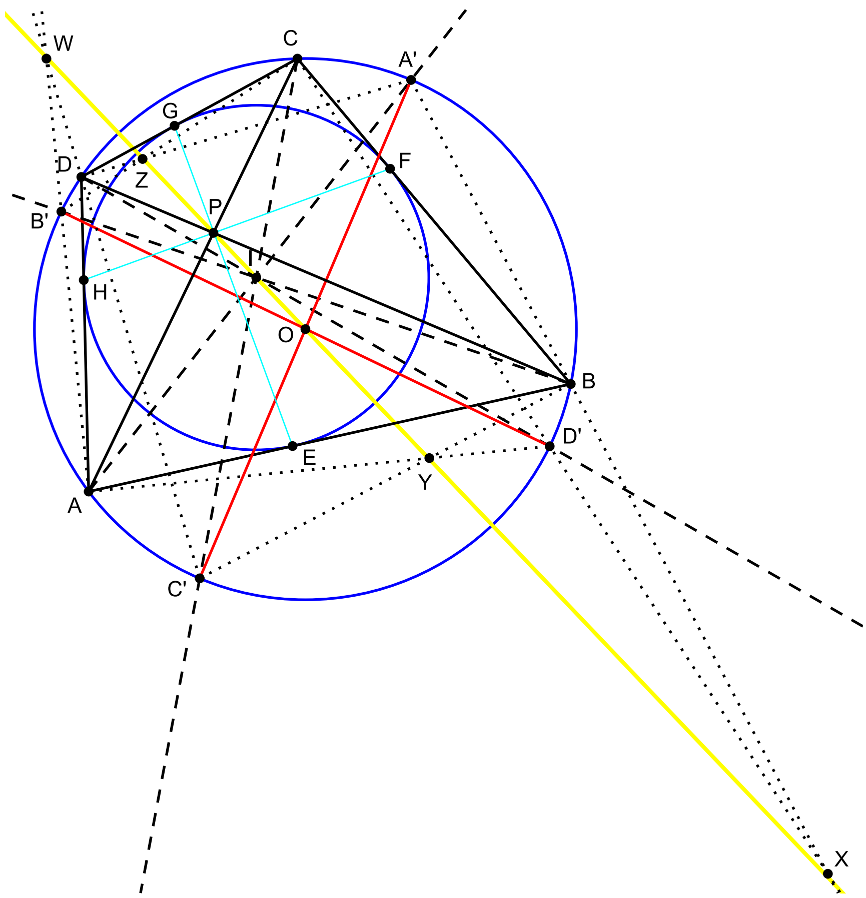

FIGURE 2. W, Z, P, I, O, Y, and X are collinear

The point Z used in the proof was an auxiliary point and can perhaps not be considered as significant as other points we will study in this paper. It was nonetheless located on the line OI, and for this reason it is noteworthy that there are three additional similar points. Let Y be the intersection of AD' and BC', let X be the intersection of $A'B$ and CD', and let W be the intersection of AB' and $C'D$. Applying Pascal's theorem in hexagrams $AD'DBC'C$, $A'BDD'CA$, and $AB'BDC'C$, we get that Y, X, and W are also located on the line of centers (see Figure 2). We don't know of any reference where these three points are discussed.

Let the incircle be tangent to the sides AB, BC, CD, DA at E, F, G, and H respectively. Then it is well-known that the diagonals EG and FH of the contact quadrilateral $EFGH$ intersect at the diagonal point P of the quadrilateral $ABCD$ (this holds for all tangential quadrilaterals). Proofs of this concurrency can be found in many places; our favorite, that applies

Menelaus' theorem, appears in [14, pp. 416–417]. This does not add another point to the line OI, but rather duplicates the importance of the main point studied in this section (see Figure 2).

3. The circumcenter of the excenter quadrilateral

Let $H'E'$, $E'F'$, $F'G'$, $G'H'$ be the exterior angle bisectors to a bicentric quadrilateral $ABCD$. Then E', F', G', H' are the centers of the four escribed circles that are each tangent to a side of the quadrilateral and the extensions of the two adjacent sides (these circles are not drawn in Figure 3), so we call $E'F'G'H'$ the *excenter quadrilateral*, as was done in [11]. We will prove that $E'F'G'H'$ is cyclic (which is well-known) and that its circumcenter V lies on the line of centers in such a way that it is the reflection of I in O. This collinearity was proved as Theorem 159 already in 1912 (see [9]), but from a different starting point (given the cyclic orthodiagonal quadrilateral $E'F'G'H'$ instead of bicentric quadrilateral $ABCD$, see Figure 3). We note that the point V was called the Bevan point in [11] since Benjamin Bevan considered the similar point in a triangle two centuries ago.

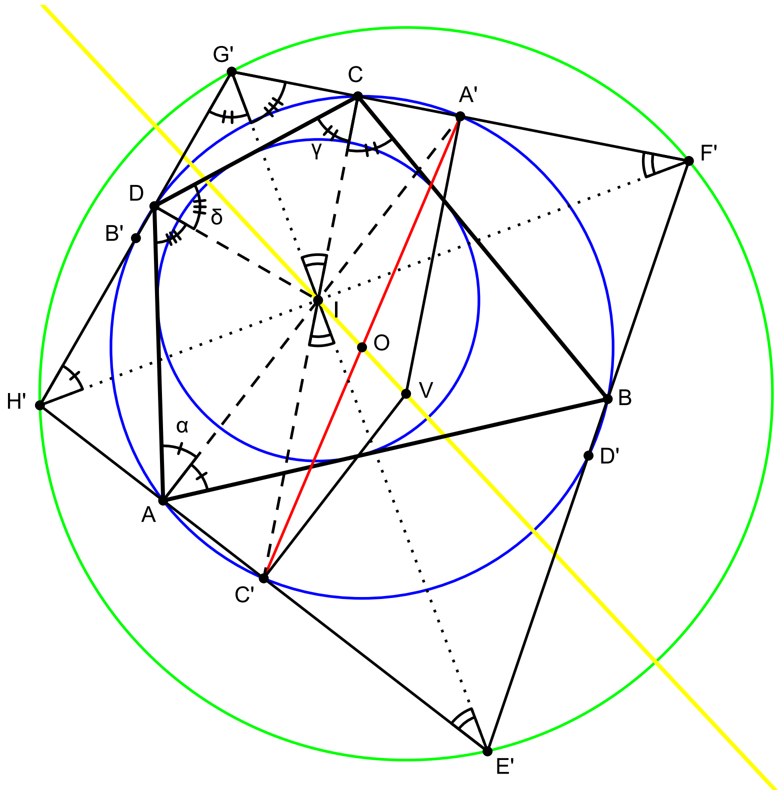

Figure 3. I, O, and V are collinear

Let the vertex angles of $ABCD$ be 2α, 2β, 2γ and 2δ. Quadrilateral $CIDG'$ is cyclic since it has a pair of opposite right angles, so

$$\angle H'G'F' = \angle DG'I + \angle CG'I = \angle DCI + \angle CDI = \gamma + \delta.$$

In the same way $\angle H'E'F' = \alpha+\beta$, so two opposite angles in $E'F'G'H'$ have the sum $\alpha+\beta+\gamma+\delta = \pi$, which proves that the excenter quadrilateral is cyclic. It also holds that its diagonals are perpendicular. To see this, we have that triangle $H'IG'$ is a right angled triangle at I since

$$\angle H'IG' = \pi - (\angle DH'I + \angle DG'I) = \pi - (\alpha+\gamma) = \pi - \tfrac{\pi}{2} = \tfrac{\pi}{2}.$$

In the same way triangles $F'IG'$, $F'IE'$ and $E'IH'$ are all right angled at I, which proves that the diagonals $E'G'$ and $F'H'$ in the excenter quadrilateral are perpendicular, meeting at the incenter I of $ABCD$.

Next we prove that A', B', C', D' are the midpoints of the sides in the excenter quadrilateral. We have

$$\angle C'IE' = \angle CIG' = \angle IF'G' = \angle H'F'G' = \angle H'E'G' = \angle C'E'I$$

which implies that triangle $C'E'I$ is isosceles with $C'E' = C'I$. A similar argument shows that $C'H' = C'I$, so C' is indeed the midpoint of $E'H'$. In the same way the other three points A', B', D' are midpoints of sides in quadrilateral $E'F'G'H'$.

Consider finally the normals to the sides $E'H'$ and $F'G'$ at C' and A'. They intersect at the circumcenter V of $E'F'G'H'$. Further, we have that VC' and $A'A$ are parallel since they are both perpendicular to $E'H'$. In the same way VA' and $C'C$ are parallel, so $C'VA'I$ is a parallelogram. One of its diagonals, $A'C'$, is bisected by the circumcenter O of $ABCD$ (this was proved in Section 2). Hence I, O, V are collinear with $IO = OV$ since the diagonals of a parallelogram bisect each other (see Figure 3).

4. Four concurrent lines

In a bicentric quadrilateral $ABCD$, the four lines $E'E$, $F'F$, $G'G$, and $H'H$ through the excenters we studied in the previous section and the points where the incircle is tangent to the sides, are concurrent at a point Q that lies on the line of centers. In order to prove this, we will show that the quadrilaterals $EFGH$ and $E'F'G'H'$ are homothetic with center Q. To prove the concurrency of these four lines was Problem 3 for Grade 10 on the All-Russian Mathematical Olympiad in 2004 (see [17]; our proof is inspired by post 4 at that link).

First note that triangle AEH is isosceles with $AE = AH$, so $\angle AEH = \frac{\pi-\angle A}{2}$, and that $\angle EAE' = \frac{\pi}{2} - \frac{\angle A}{2}$ since $E'H'$ is an external angle bisector (see Figure 4). Thus $\angle AEH = \angle EAE'$, and we have that $EH \parallel E'H'$. In the same way it holds that $EF \parallel E'F'$, $FG \parallel F'G'$, and $GH \parallel G'H'$.

Next we must prove that the diagonals of $EFGH$ and $E'F'G'H'$ are also parallel. Let us consider one pair of diagonals, the second is proved in the same way. Extend AD and BC to meet at Q_2, and let $H'F'$ intersect AQ_2 and BQ_2 at K_1 and K_2 respectively. Triangle Q_2HF is isosceles since two sides are tangent to the incircle. If we can prove that $Q_2K_1K_2$ is also isosceles, then triangles Q_2HF and $Q_2K_1K_2$ would be similar and thus $HF \parallel H'F'$. We have that

$$\angle Q_2K_1I = \angle K_1AI + \angle K_1IA = \frac{\angle A}{2} + \angle H'DA = \frac{\angle A}{2} + \frac{\pi}{2} - \frac{\angle D}{2}$$

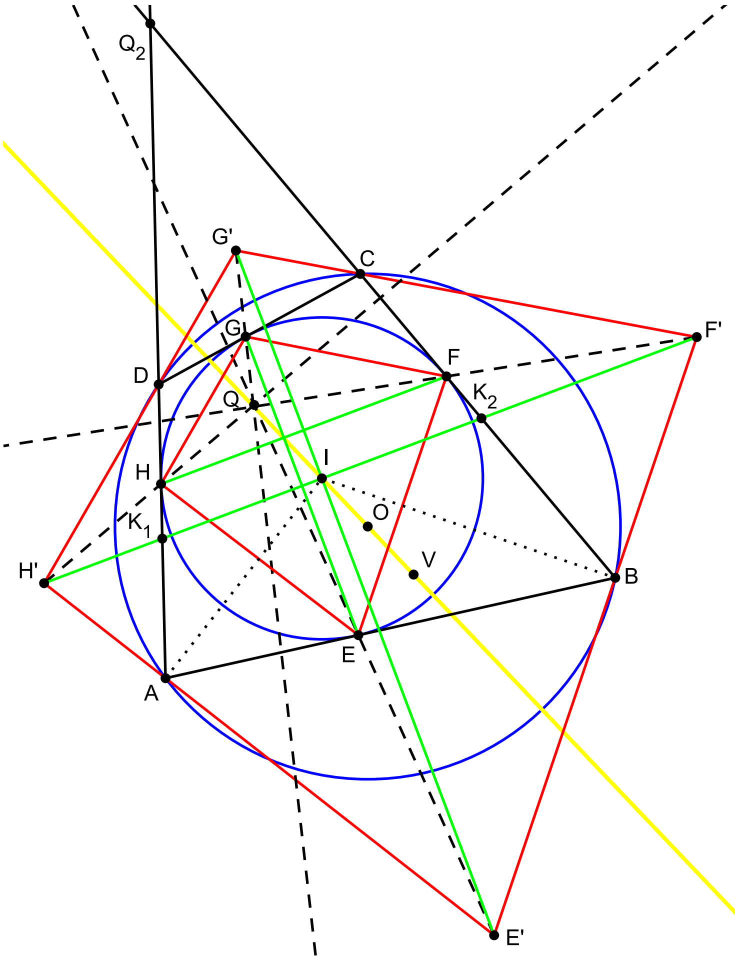

Figure 4. Q, I, O, and V are collinear

where we used the exterior angle theorem and that $H'AID$ is a cyclic quadrilateral due to two opposite right angles. In the same way

$$\angle Q_2K_2I = \frac{\angle B}{2} + \frac{\pi}{2} - \frac{\angle C}{2}$$

and using that $\angle A + \angle C = \angle B + \angle D$ (since $ABCD$ is cyclic), which is equivalent to $\angle A - \angle D = \angle B - \angle C$, we get that $\angle Q_2K_1I = \angle Q_2K_2I$. Hence triangle $Q_2K_1K_2$ is also isosceles and so $HF \parallel H'F'$. Together with the similar result that $EG \parallel E'G'$, this proves that quadrilaterals $EFGH$ and $E'F'G'H'$ are homothetic with center Q. Since both of these quadrilaterals are cyclic, their circumcenters I and V are collinear with the center of homothety Q, proving that Q lies on the line of centers.

5. The Miquel point

If opposite sides AB, CD and AD, BC of a convex quadrilateral $ABCD$ intersect at Q_1 and Q_2 respectively, then it is well-known that the circumcircles of triangles ABQ_2, DCQ_2, BCQ_1, and ADQ_1 are concurrent at a point M, often called the *Miquel point* of the quadrilateral. We include the short proof here. Suppose the circumcircles of triangles ABQ_2 and ADQ_1 intersect at a point, which we call M (see Figure 5 and note that we don't use the fact that $ABCD$ is drawn cyclic). It holds that

$$\angle CQ_1M = \angle DQ_1M = \angle DAM = \angle Q_2AM = \angle Q_2BM = \angle CBM$$

proving that $BCMQ_1$ is cyclic, so the circumcircle of BCQ_1 goes through M. Then we have

$$\angle MQ_2D = \angle MQ_2A = \angle MBA = \angle MBQ_1 = \angle MCQ_1 = \angle MCD$$

which shows that the circumcircle to DCQ_2 also passes through M. The main result of this section is to prove that the Miquel point lies on the line of centers in a bicentric quadrilateral. Our proof is inspired by [16, p. 134].

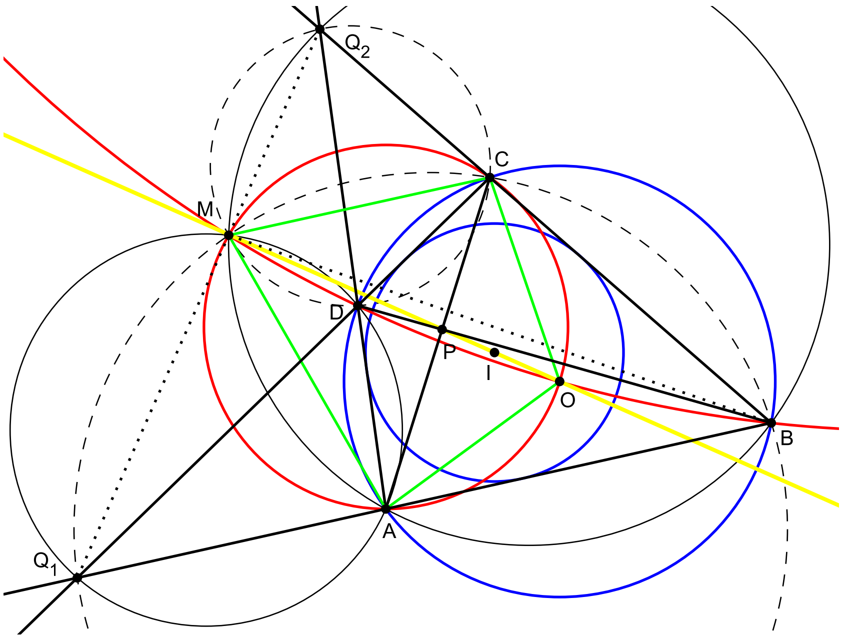

FIGURE 5. M, P, I, and O are collinear

We start by proving that when $ABCD$ is cyclic with circumcenter O, then the circumcircle of triangle AOC pass through M, that is, quadrilateral $AOCM$ is cyclic. We have for a pair of opposite angles that

$$\begin{aligned}\angle AMC + \angle AOC &= \angle AMD + \angle CMD + \angle AOC \\ &= \angle AQ_1D + \angle CQ_2D + 2\angle ABC \\ &= \angle BQ_1C + \angle BQ_2A + \angle Q_1BC + \angle Q_2BA \\ &= (\angle BQ_1C + \angle Q_1BC) + (\angle BQ_2A + \angle Q_2BA) \\ &= (\pi - \angle C) + (\pi - \angle A) \\ &= \pi\end{aligned}$$

confirming that $AOCM$ is cyclic, where we applied the inscribed angle theorem, the angle sum of triangles BCQ_1 and ABQ_2, and that $\angle A + \angle C = \pi$. In the same way $BODM$ is proved to be a cyclic quadrilateral.

The radical axis of the circumcircles to $ABCD$ and $AOCM$ is AC, the radical axis of the circumcircles to $ABCD$ and $BODM$ is BD, and the radical axis of the circumcircles to $AOCM$ and $BODM$ is OM. These radical axes are concurrent at the radical center of the three circles, which means that AC, BD, and OM are concurrent. But AC and BD intersect at the diagonal point P, which proves that P lies on the line OM, or in other words, that the Miquel point M lies on the line of centers since we know that this line coincides with the line OP (Section 2). Note that we did not use any properties of the incircle in this section, so the result that M lies on the line OP is valid in all cyclic quadrilaterals.

6. Four circumcircles through P

The points we study in this and the next two sections were found in [5, pp. 206–207, 210–211]. In a cyclic quadrilateral $ABCD$ (and thus also in a bicentric one) with diagonal point P and circumcenter O, let the circumcenters of triangles ABP, BCP, CDP, and DAP be O_1, O_2, O_3, and O_4 respectively. We shall prove that the line segments O_1O_3 and O_2O_4 intersect at a point, which we call S, that is located on the line OP.

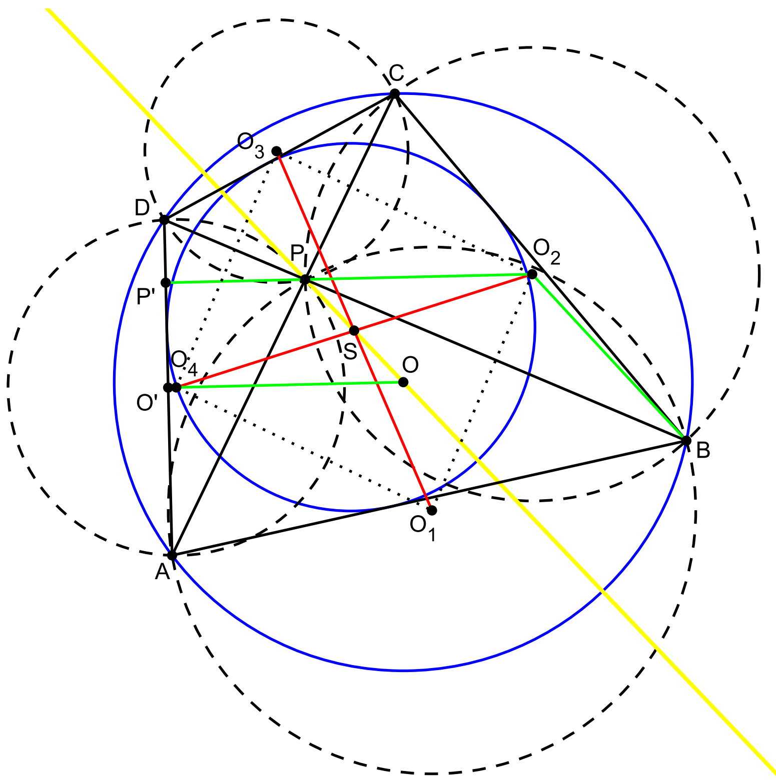

Figure 6. P, S, and O are collinear

First we conclude that $O_1O_2O_3O_4$ is a parallelogram since its opposite sides O_1O_2 and O_3O_4 are perpendicular to diagonal BD, and O_2O_3 and O_1O_4 are perpendicular to diagonal AC due to the fact that O_1O_2 is a perpendicular bisector to BP and so on (see Figure 6). Thus S is the midpoint of O_2O_4.

Next we prove that OO_2PO_4 is also a parallelogram. Let OO_4 intersect AD at O'. Then $OO' \perp AD$ due to the fact that O and O_4 are circumcenters in $ABCD$ and ADP which share the side AD. Let O_2P intersect AD at P'. In order to prove that $O_2P' \perp AD$, we consider a few angles:

$$\angle P'PD + \angle P'DP = \angle O_2PB + \angle ACB = \tfrac{\angle O_2PB + \angle O_2BP}{2} + \tfrac{\angle PO_2B}{2} = \tfrac{\pi}{2}$$

where we used that $\angle ADB = \angle ACB$ in cyclic quadrilateral $ABCD$ and the angle sum of triangle PO_2B in the last step. Thus O_4O and PO_2 are both perpendicular to AB and hence parallel. In the same way OO_2 and O_4P are parallel which shows that OO_2PO_4 is a parallelogram. Together with the fact that S is the midpoint of O_2O_4 from before, this shows that S is the midpoint on OP, so it lies on the line of centers.

7. Four circumcircles through I

In a tangential quadrilateral $ABCD$ (and thus also in a bicentric quadrilateral) with diagonal point P, incenter I, and where the incircle is tangent to AB, BC, CD, and DA at E, F, G, and H respectively, let the circumcenters of triangles AHE, BEF, CFG, and DGH be O_5, O_6, O_7, and O_8 respectively. We shall prove that the line segments O_5O_7 and O_6O_8 intersect at a point, which we call T, that is located on the line IP.

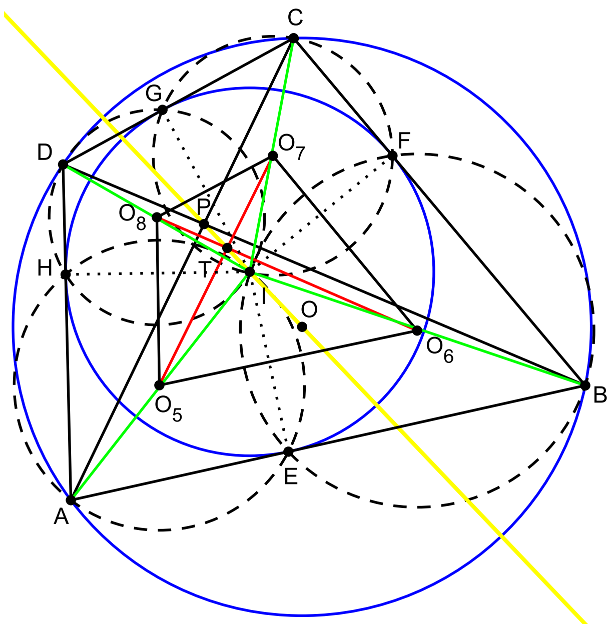

Figure 7. P, T, I, and O are collinear

Note that O_5, O_6, O_7, and O_8 are also the circumcenters of triangles EIH, EIF, FIG, and GIH since $AEIH$, $BEIF$, $CFIG$, and $DGIH$ are cyclic quadrilaterals, all with a pair of opposite right angles (see Figure 7). This means that O_5, O_6, O_7, and O_8 are the midpoints of AI, BI, CI, and DI respectively, which in turn imply that quadrilaterals $O_5O_6O_7O_8$ and $ABCD$ are homothetic, with the former half the size of the latter, and that they have the same incenter I. Hence the diagonal intersection T of $O_5O_6O_7O_8$ is the midpoint of IP, so it lies on the the line of centers.

8. Four concyclic incenters

In a bicentric quadrilateral $ABCD$ with circumcenter O, incenter I, and diagonal point P, let I_1, I_2, I_3, and I_4 be the incenters of triangles ABP, BCP, CDP, and DAP respectively. We shall prove that $I_1I_2I_3I_4$ is a cyclic quadrilateral and that its circumcenter, which we call K, lies on the line of centers.

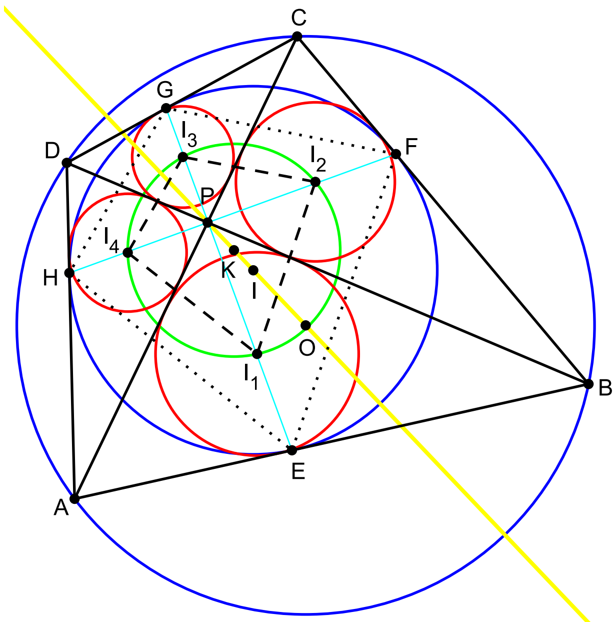

Figure 8. P, K, I, and O are collinear

First let us prove that EG is the angle bisector to one of the angles between the diagonals of $ABCD$. If EA and GD extended meet at a point Q_1, then EQ_1G is an isosceles triangle and so $\angle AEP = \angle DGP$ (see Figure 8). Also, $\angle BAC = \angle BDC$ since $ABCD$ is cyclic, so $\angle APE = \angle DPG$. But vertical angles $\angle BPE = \angle DPG$ confirming that $\angle APE = \angle BPE$, that is, EG is an angle bisector to angle APB. In the same way FH is an angle bisector to angle BPC. Hence all four incenters I_1, I_2, I_3, and I_4 lie on EG or FH.

Next we apply the angle bisector theorem to get

$$\frac{HI_4}{I_4P} = \frac{AH}{AP} = \frac{AE}{AP} = \frac{EI_1}{I_1P}$$

and by the converse of the intercept theorem, it follows that I_1I_4 is parallel to EH. In the same way all the sides of $I_1I_2I_3I_4$ and $ABCD$ are parallel, and since EI_1P, FI_2P, GI_3P, and HI_4P are straight line segments, these two quadrilaterals are homothetic with center P. Since $ABCD$ is cyclic, then so is $I_1I_2I_3I_4$. The homothety implies that their circumcenters K and I are collinear with P, which proves that K lies on the line of centers.

9. Four concyclic excenters

In a bicentric quadrilateral $ABCD$ with circumcenter O, incenter I, and diagonal point P, let J_1, J_2, J_3, and J_4 be the excenters of triangles ABP, BCP, CDP, and DAP opposite AB, BC, CD, and DA respectively. We shall prove that $J_1J_2J_3J_4$ is a cyclic quadrilateral and that its circumcenter, which we call J, lies on the line of centers. We have not found any reference for this collinearity, but the fact that the excenters are concyclic (which holds for all tangential quadrilaterals) was proved in [13]. There ought to be a close connection between the results in this section and the previous one, but we have been unable to provide a nice short proof of the present collinearity. Instead this will be a computational proof applying a few formulas for triangles and quadrilaterals to prove that $J_1J_2J_3J_4$ and $EFGH$ are homothetic. It is well-known that the excenters lie on the angle bisectors to the angles between the diagonals, and from the previous section we also know that E, F, G, and H lie on these. We only need to prove that J_1J_4 and EH are parallel, since then in similar ways the same is true for all corresponding pairs of sides in $J_1J_2J_3J_4$ and $EFGH$.

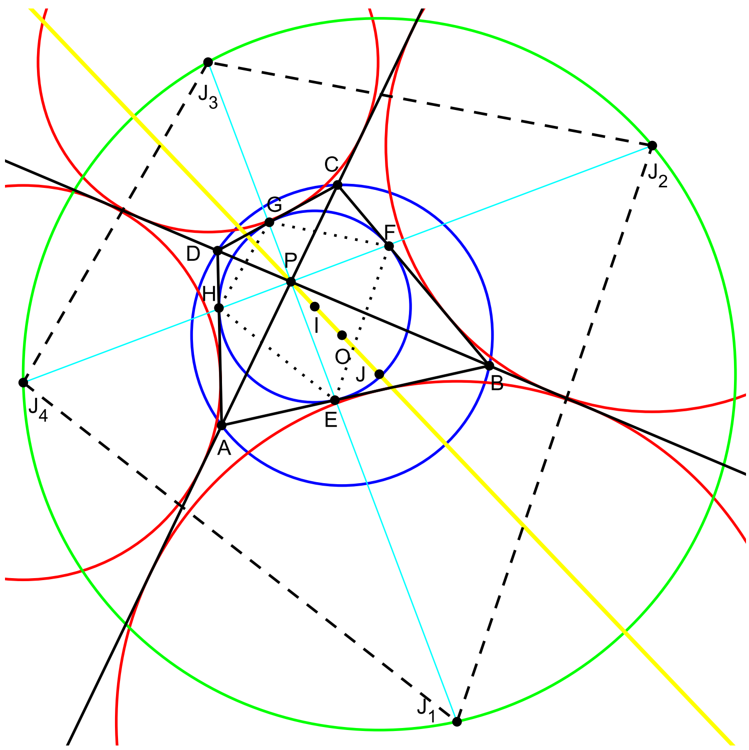

FIGURE 9. P, I, O, and J are collinear

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AP = w$, $BP = x$, and $DP = z$. What we shall prove is (see Figure 9)

$$\frac{PH}{PJ_4} = \frac{PE}{PJ_1}.$$

There are formulas for the length of an angle bisector PH and the distance PJ_4 between a vertex and the opposite excenter in a triangle. (The former is well-known and the latter can easily be derived using the radius of an excircle, the half-angle formula for sine, and Heron's formula. The details are left to the reader.) Applying these formulas, we get

$$\left(\frac{PH}{PJ_4}\right)^2 = \frac{wz(w+z-d)(w+z+d)}{(w+z)^2} \cdot \frac{(w+z-d)}{wz(w+z+d)} = \frac{(w+z-d)^2}{(w+z)^2}$$

so

$$\frac{PH}{PJ_4} = \frac{w+z-d}{w+z}$$

and similarly

$$\frac{PE}{PJ_1} = \frac{w+x-a}{w+x}.$$

Thus we need to prove that

$$\frac{w+z-d}{w+z} = \frac{w+x-a}{w+x}$$

which is simplified to

$$\frac{d}{w+z} = \frac{a}{w+x}.$$

The diagonal parts in cyclic quadrilateral $ABCD$ satisfy

$$\frac{ad}{w} = \frac{ab}{x} = \frac{cd}{z}$$

according to [8, p. 26], and if we call this common quotient u, then the equality we shall prove is

$$\frac{d}{\frac{ad}{u} + \frac{cd}{u}} = \frac{a}{\frac{ad}{u} + \frac{ab}{u}}$$

or

$$\frac{d}{d(a+c)} = \frac{a}{a(d+b)}$$

that is

$$a + c = b + d.$$

But this is the well-known Pitot theorem that holds in all tangential quadrilaterals, and reversing all steps proves that $EH \parallel J_1J_4$. Since J_1EP, J_2FP, J_3GP, and J_4HP are straight line segments, this proves that quadrilaterals $J_1J_2J_3J_4$ and $EFGH$ are homothetic with center P. Hence they are both cyclic and their circumcenters J and I are collinear with P, so J lies on the line of centers.

10. Four concyclic Lemoine points

In a bicentric quadrilateral $ABCD$ with circumcenter O, incenter I, and diagonal point P, let the incircle be tangent to the sides AB, BC, CD, and DA at E, F, G, and H respectively. We know that EG and FH intersect at P. Let L_1, L_2, L_3, and L_4 be the Lemoine points of triangles HEP, EFP, FGP, and GHP respectively. We shall prove that $L_1L_2L_3L_4$ is a cyclic quadrilateral and that its circumcenter, which we call L, lies on the line of centers. This problem was studied at [6]. We present our own solution.

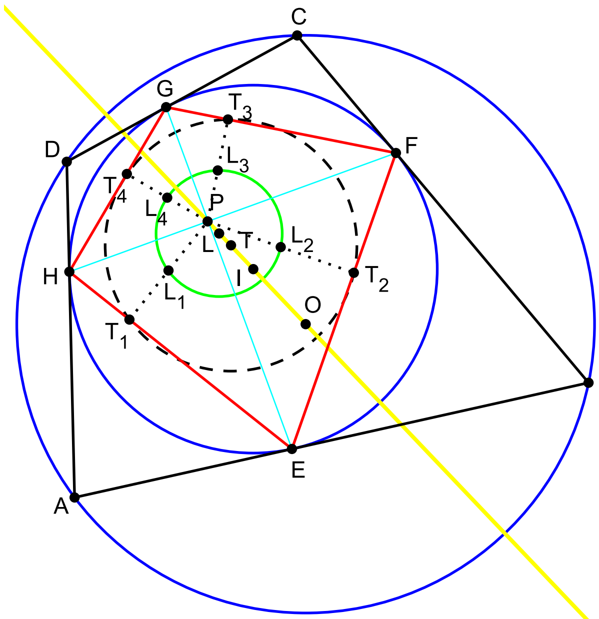

Figure 10. P, L, T, I, and O are collinear

Let T_1, T_2, T_3, and T_4 be the projections of P on the sides of $HEFG$, see Figure 10. Then it is known that L_1, L_2, L_3, and L_4 are the midpoints of the heights PT_1, PT_2, PT_3, and PT_4 in right triangles HEP, EFP, FGP, and GHP (see [4]). The fact that $EG \perp FH$ is a well-known property of bicentric quadrilaterals, proved in many places (for instance [12, pp. 123–124]).

Applying results from Section 3, we have that $T_1T_2T_3T_4$ is a cyclic quadrilateral with center the midpoint of PI. This point was called T is Section 7, so we keep that label. Since T_1L_1, T_2L_2, T_3L_3, and T_4L_4 intersect at P, this means that $T_1T_2T_3T_4$ and $L_1L_2L_3L_4$ are homothetic with center P. Thus the latter quadrilateral is cyclic (with center L). And since the Lemoine points are the midpoints on the heights PT_1, PT_2, PT_3, and PT_4, then L is the midpoint of PT, proving that it lies on the line of centers.

11. Summary

In Figure 11 all fifteen collinear points in a non-square bicentric quadrilateral we have studied in this paper are shown.

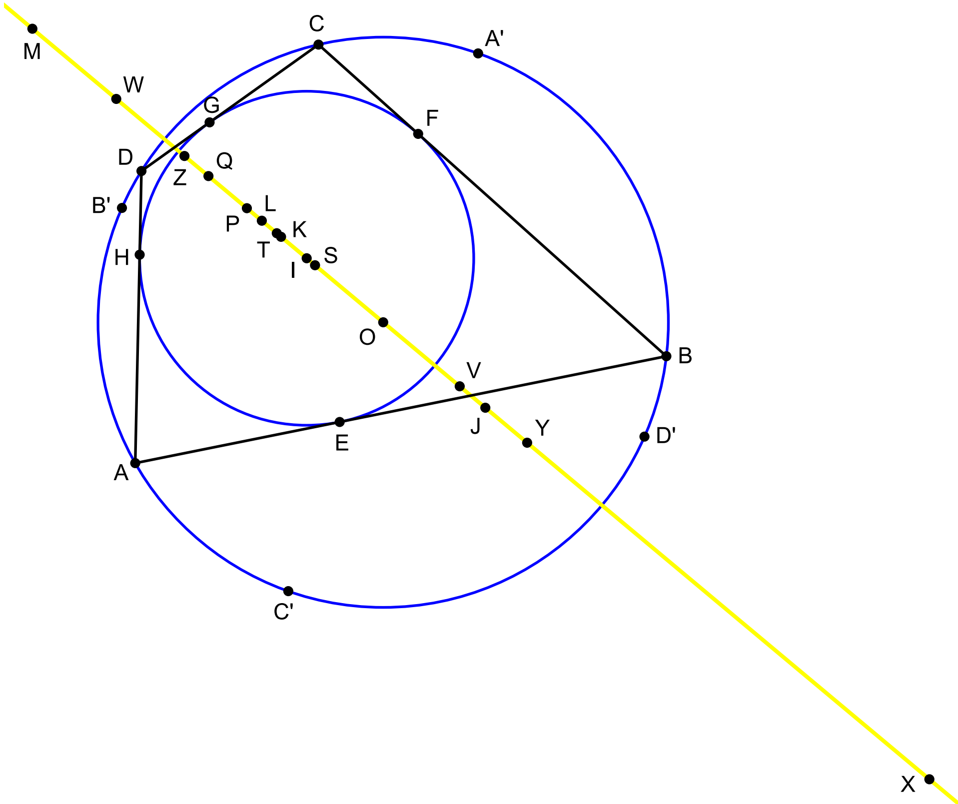

FIGURE 11. M, W, Z, Q, P, L, T, K, I, S, O, V, J, Y, X are collinear

Point	**Description**
I	Incenter
J	Circumcenter of $J_1J_2J_3J_4$
K	Circumcenter of $I_1I_2I_3I_4$
L	Circumcenter of $L_1L_2L_3L_4$
M	Miquel point
O	Circumcenter
P	Diagonal point
Q	$E'E \cap F'F \cap G'G \cap H'H$
S	Diagonal point of $O_1O_2O_3O_4$
T	Diagonal point of $O_5O_6O_7O_8$
V	Circumcenter of $E'F'G'H'$
W	$AB' \cap C'D$
X	$A'B \cap CD'$
Y	$AD' \cap BC'$
Z	$A'D \cap B'C$

TABLE 1. The fifteen collinear points

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