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THE REDUCTION OF DARBOUX'S THEOREM ON WEIL BUNDLES

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Abstract. Let M be a paracompact and connected smooth manifold, $\mathbb{A}$ a Weil algebra and $M^{\mathbb{A}}$ the associated Weil bundle. We construct a Darboux's theorem on Weil bundles, we also give the reduction of this theorem and we revisited the hamiltonian vector field on Weil bundles.

1. Introduction

The most abstract reduction principe states that every coisotropic submanifold of symplectic manifold is foliated by isotropic leaves and if the leaf space is a manifold it carries again a symplectic structure.

In (see [4]), it has been formalized the fact if a n-dimensional symmetry group acts on a hamiltonian system then the number of degrees of freedom can be reduced by $2n$.

In what follows, M denotes a paracompact differentiable manifold, $C^\infty(M)$ the algebra of smooth functions on $M, \mathbb{A}$ a local algebra (in the sense of André Weil), i.e., a real commutative algebra with unit, of finite dimension, and with a unique maximal ideal $\mathfrak{m}$ of codimension 1 over $\mathbb{R}$. In this case, there exists an integer h such that $\mathfrak{m}^{h+1} = (0)$ and $\mathfrak{m}^h \neq (0)$. The integer h is the height of $\mathbb{A}$. Also we have

(1) $$\mathbb{A} = \mathbb{R} \oplus \mathfrak{m}.$$

We recall that a near point of $x \in M$ of kind $\mathbb{A}$ is a morphism of algebras

$$\xi : C^\infty(M) \longrightarrow \mathbb{A}$$

such that

$$[\xi(f) - f(x)] \in \mathfrak{m}$$

for any $f \in C^\infty(M)$. We denote $M_x^{\mathbb{A}}$ the set of near points of $x \in M$ of kind $\mathbb{A}$ and

(2) $$M^{\mathbb{A}} = \bigcup_{x \in M} M_x^{\mathbb{A}}$$

the manifold of infinitely near points on M of kind $\mathbb{A}$.

We have $\mathbb{R}^{\mathbb{A}} = \mathbb{A}, M^{\mathbb{D}} = TM$ where TM is the tangent bundle of M.

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When the dimension of M is m, then the dimension of $M^{\mathbb{A}}$ is $m\times\dim(\mathbb{A})$ (see [7], [3] and the references given there). Let (U,φ) be a local chart with local coordinates $(x_1,x_2,...,x_m)$. The application

$$U^{\mathbb{A}}\longrightarrow\mathbb{A}^m,\xi\longmapsto(\xi(x_1),\xi(x_2),...,\xi(x_m)),$$

is a bijection from $U^{\mathbb{A}}$ to an open of $\mathbb{A}^m$. Thus $M^{\mathbb{A}}$ is an $\mathbb{A}$-manifold of dimension m [2].

The set, $C^\infty(M^{\mathbb{A}},\mathbb{A})$, of smooth functions on $M^{\mathbb{A}}$ with values in $\mathbb{A}$ is a commutative algebra with unit over $\mathbb{A}$.

For any $f\in C^\infty(M)$, the application

$$f^{\mathbb{A}}:M^{\mathbb{A}}\longrightarrow\mathbb{A},\xi\longmapsto\xi(f),$$

is smooth and the application

$$C^\infty(M)\longrightarrow C^\infty(M^{\mathbb{A}},\mathbb{A}),f\longmapsto f^{\mathbb{A}},$$

is a monomorphism of algebras.

The following assertions are equivalent:

(1) X is a derivation of $C^\infty(M^{\mathbb{A}})$, i.e., X is a vector field on $M^{\mathbb{A}}$;
(2) $X:C^\infty(M)\longrightarrow C^\infty(M^{\mathbb{A}},\mathbb{A})$ is a $\mathbb{R}$-linear application such that, for any $f,g\in C^\infty(M)$,

$$X(fg)=X(f)\cdot g^{\mathbb{A}}+f^{\mathbb{A}}\cdot X(g)$$

that is, X is a derivation from $C^\infty(M)$ into $C^\infty(M^{\mathbb{A}},\mathbb{A})$ with respect the module structure

$$C^\infty(M^{\mathbb{A}},\mathbb{A})\times C^\infty(M)\longrightarrow C^\infty(M^{\mathbb{A}},\mathbb{A}),(F,f)\longmapsto F\cdot f^{\mathbb{A}};$$

(3) X is a differentiable section of the tangent bundle $\left(TM^{\mathbb{A}},\pi_{M^{\mathbb{A}}},M^{\mathbb{A}}\right)$;
(4) X is a derivation of $C^\infty(M^{\mathbb{A}},\mathbb{A})$ which $\mathbb{A}$-linear.

Thus the set, $\mathfrak{X}(M^{\mathbb{A}})$, of vector fields on $M^{\mathbb{A}}$ considered as derivations of $C^\infty(M)$ into $C^\infty(M^{\mathbb{A}},\mathbb{A})$ is a module over $C^\infty(M^{\mathbb{A}},\mathbb{A})$.

When

$$\theta:C^\infty(M)\longrightarrow C^\infty(M)$$

is a vector field on M, then the map

$$\theta^{\mathbb{A}}:C^\infty(M)\longrightarrow C^\infty(M^{\mathbb{A}},\mathbb{A}),f\longmapsto[\theta(f)]^{\mathbb{A}},$$

is a vector field on $M^{\mathbb{A}}$. We say that the vector field $\theta^{\mathbb{A}}$ is the prolongation to $M^{\mathbb{A}}$ of the vector field θ.

If X is a vector field on $M^{\mathbb{A}}$, considered as a derivation of $C^\infty(M)$ into $C^\infty(M^{\mathbb{A}},\mathbb{A})$, then there exists, [2], a unique derivation

$$\widetilde{X}:C^\infty(M^{\mathbb{A}},\mathbb{A})\longrightarrow C^\infty(M^{\mathbb{A}},\mathbb{A})$$

such that

(1) $\widetilde{X}$ is $\mathbb{A}$-linear;
(2) $\widetilde{X}\left[C^\infty(M^{\mathbb{A}})\right]\subset C^\infty(M^{\mathbb{A}})$;
(3) $\widetilde{X}(f^{\mathbb{A}})=X(f)$ for any $f\in C^\infty(M)$.

The application

$$[,]:\mathfrak{X}(M^{\mathbb{A}})\times\mathfrak{X}(M^{\mathbb{A}})\longrightarrow\mathfrak{X}(M^{\mathbb{A}}),(X,Y)\longmapsto\widetilde{X}\circ Y-\widetilde{Y}\circ X,$$

is $\mathbb{A}$-bilinear and defines a structure of $\mathbb{A}$-Lie algebra on $\mathfrak{X}(M^{\mathbb{A}})$ (see [2]).

If we denote $Der\left[C^{\infty}(M^{\mathbb{A}},\mathbb{A})\right]$, the $C^{\infty}(M^{\mathbb{A}},\mathbb{A})$-module of derivations of $C^{\infty}(M^{\mathbb{A}},\mathbb{A})$, then the map

$$\mathfrak{X}(M^{\mathbb{A}})\longrightarrow Der\left[C^{\infty}(M^{\mathbb{A}},\mathbb{A})\right],X\longmapsto\widetilde{X},$$

is a morphism of $\mathbb{A}$-Lie algebras [2], [3].

For any $p\in\mathbb{N}$,

$$\Lambda^{p}(M^{\mathbb{A}},\mathbb{A})=\mathcal{L}^{p}_{sks}\left[\mathfrak{X}(M^{\mathbb{A}}),C^{\infty}(M^{\mathbb{A}},\mathbb{A})\right] \tag{3}$$

denotes the $C^{\infty}(M^{\mathbb{A}},\mathbb{A})$-module of skew-symmetric multilinear forms of degree p on $\mathfrak{X}(M^{\mathbb{A}})$. We say that $\Lambda^{p}(M^{\mathbb{A}},\mathbb{A})$ is the $C^{\infty}(M^{\mathbb{A}},\mathbb{A})$-module of differential $\mathbb{A}$-forms of degree p on $M^{\mathbb{A}}$. We have

$$\Lambda^{0}(M^{\mathbb{A}},\mathbb{A})=C^{\infty}(M^{\mathbb{A}},\mathbb{A}). \tag{4}$$

We denote

$$\Lambda(M^{\mathbb{A}},\mathbb{A})=\bigoplus_{p=0}^{n}\Lambda^{p}(M^{\mathbb{A}},\mathbb{A}). \tag{5}$$

If ω is a differential form of degree p on M, then there exists a unique differential $\mathbb{A}$-form of degree p on $M^{\mathbb{A}}$ such that

$$\omega^{\mathbb{A}}(\theta_1^{\mathbb{A}},\theta_2^{\mathbb{A}},...,\theta_p^{\mathbb{A}})=\left[\omega(\theta_1,\theta_2,...,\theta_p)\right]^{\mathbb{A}} \tag{6}$$

for any vector fields $\theta_1,\theta_2,...,\theta_p\in\mathfrak{X}(M)$. We say that the differential $\mathbb{A}$-form $\omega^{\mathbb{A}}$ is the prolongation to $M^{\mathbb{A}}$ of the differential form ω [5], [6], [3].

When

$$d:\Lambda(M)\longrightarrow\Lambda(M)$$

is the exterior differentiation operator, we denote

$$d^{\mathbb{A}}:\Lambda(M^{\mathbb{A}},\mathbb{A})\longrightarrow\Lambda(M^{\mathbb{A}},\mathbb{A})$$

the cohomology operator associated to the representation

$$\mathfrak{X}(M^{\mathbb{A}})\longrightarrow Der\left[C^{\infty}(M^{\mathbb{A}},\mathbb{A})\right],X\longmapsto\widetilde{X}.$$

We recall that for $\eta\in\Lambda^{p}(M^{\mathbb{A}},\mathbb{A})$, we have

$$\begin{aligned}&(d^{\mathbb{A}}\eta)(X_1,X_2,...,X_{p+1})\\&=\sum_{i=1}^{p+1}(-1)^{i-1}\widetilde{X_i}\left[\eta(X_1,X_2,...,\widehat{X_i},...,X_{p+1})\right]\\&+\sum_{i<j}(-1)^{i+j}\ \eta([X_i,X_j],X_1,...,\widehat{X_i},\ ...,\widehat{X_j},...,X_{p+1})\end{aligned}$$

for any vector fields $X_1,X_2,...,X_{p+1}\in\mathfrak{X}(M^{\mathbb{A}})$.

The map

$$d^{\mathbb{A}}:\Lambda(M^{\mathbb{A}},\mathbb{A})\longrightarrow\Lambda(M^{\mathbb{A}},\mathbb{A})$$

is $\mathbb{A}$-linear and

$$d^{\mathbb{A}}(\omega^{\mathbb{A}}) = (d\omega)^{\mathbb{A}} \tag{7}$$

for any $\omega \in \Lambda(M)$ [2]. It is easy to see that if

$$d\omega = 0,$$

then

$$(d\omega)^{\mathbb{A}} = 0. \tag{8}$$

Let (M, ω) be a symplectic manifold. Then the manifold M is a Poisson manifold, i.e., the algebra $C^{\infty}(M)$ carries a structure of Poisson algebra. For any linear form

$$\psi : \mathbb{A} \longrightarrow \mathbb{R},$$

the differential form $\psi \circ \omega^{\mathbb{A}}$ is not necessary a symplectic form on $M^{\mathbb{A}}$. That is the prolongation $\omega^{\mathbb{A}}$ does not always induce a structure of Poisson on $M^{\mathbb{A}}$. Let $\mathfrak{m}$ be the maximal ideal of a local algebra $\mathbb{A}$,

$$ann(\mathfrak{m}) = \{a \in \mathbb{A} \,/ a \cdot x = 0 \text{ for any } x \in \mathfrak{m}\}$$

and

$$\mu_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}, (a, b) \longmapsto a \cdot b,$$

the multiplication on $\mathbb{A}$. Then there exists a linear form $\psi : \mathbb{A} \longrightarrow \mathbb{R}$ such that the bilinear symmetric form

$$\psi \circ \mu_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{R}$$

is nondegenerated if and only if $\dim [ann(\mathfrak{m})] = 1$ [6].

When (M, ω) is a symplectic manifold and $\psi \in \mathbb{A}^*$ a linear form on $\mathbb{A}$, then the scalar 2-form $\psi \circ \omega^{\mathbb{A}}$ is a symplectic form on $M^{\mathbb{A}}$ if and only if $\dim [ann(\mathfrak{m})] = 1$ and $\psi [ann(\mathfrak{m})] \neq 0$: it is the case when $\mathbb{A} = \mathbb{R}[T_1, ..., T_s] / [T_1^{k_1}, ..., T_s^{k_s}]$.

Thus, when (M, ω) is a symplectic manifold, we cannot obtain a Poisson structure on $M^{\mathbb{A}}$ which comes from the prolongation of ω when $\dim [ann(\mathfrak{m})] \neq 1$. For example, it is the case when $\mathbb{A} = \mathbb{R}[T_1, T_2] / (T_1, T_2)^2$.

The main goal of this paper is:

Theorem 2.1. Let $(M^{\mathbb{A}}, \omega^{\mathbb{A}})$ be a $2n$-dimensional symplectic $\mathbb{A}$-manifold, and let ξ be any point in $M^{\mathbb{A}}$. Then there is a coordinate chart $(\mathbb{U}^{\mathbb{A}}, x_1^{\mathbb{A}}, ..., x_{2n}^{\mathbb{A}})$ centered at ξ such that on $\mathbb{U}^{\mathbb{A}}$

$$\omega^{\mathbb{A}} = \sum_{i=1}^{n} d^{\mathbb{A}}(x_i^{\mathbb{A}}) \wedge d^{\mathbb{A}}(x_{n+i}^{\mathbb{A}}).$$

Theorem 3.1. Suppose that $\omega^{\mathbb{A}}$ is a $d^{\mathbb{A}}$-closed 2-form of constant half-rank n on an $\mathbb{A}$-manifold $(M^{\mathbb{A}})^{2n+k}$. Then the null bundle

$$N_{\omega^{\mathbb{A}}}^{\mathbb{A}} = \left\{ \delta \in TM^{\mathbb{A}} \Big/ \omega^{\mathbb{A}}(\delta, v) = 0, \forall v \in T_{\pi_{M^{\mathbb{A}}}(\delta)} M^{\mathbb{A}} \right\}$$

is integrable and of constant rank k. Moreover, any point of $M^{\mathbb{A}}$ has a neighborhood $\mathbb{U}^{\mathbb{A}}$ on which there exist local coordinates $\left(x_1^{\mathbb{A}}, ..., x_{2n}^{\mathbb{A}}, y_1^{\mathbb{A}}, ..., y_k^{\mathbb{A}}\right)$ in which

$$\omega^{\mathbb{A}}_{|_{\mathbb{U}^{\mathbb{A}}}} = \sum_{i=1}^{n} d^{\mathbb{A}}(x_i^{\mathbb{A}}) \wedge d^{\mathbb{A}}(x_{n+i}^{\mathbb{A}}).$$

2. Darboux's theorem on $M^{\mathbb{A}}$

Proposition 2.1. *If ω is a differential form on M and if θ is a vector field on M, then*

$$(i_\theta \omega)^{\mathbb{A}} = i_{\theta^{\mathbb{A}}}(\omega^{\mathbb{A}}). \tag{9}$$

Proof. If the degree of ω is p, then $(i_\theta \omega)^{\mathbb{A}}$ is a unique differential $\mathbb{A}$-form of degree $p-1$ such that

$$\begin{aligned}(i_\theta \omega)^{\mathbb{A}}(\theta_1^{\mathbb{A}}, ..., \theta_{p-1}^{\mathbb{A}}) &= [(i_\theta \omega)(\theta_1, ..., \theta_{p-1})]^{\mathbb{A}} \\ &= [\omega(\theta, \theta_1, ..., \theta_{p-1})]^{\mathbb{A}}\end{aligned}$$

for any $\theta_1, \theta_2, ..., \theta_{p-1} \in \mathfrak{X}(M)$. As $i_{\theta^{\mathbb{A}}}(\omega^{\mathbb{A}})$ is of degree $p-1$ and is such that

$$\begin{aligned}i_{\theta^{\mathbb{A}}}(\omega^{\mathbb{A}}) \left[\theta_1^{\mathbb{A}}, ..., \theta_{p-1}^{\mathbb{A}}\right] &= \omega^{\mathbb{A}}(\theta^{\mathbb{A}}, \theta_1^{\mathbb{A}}, ..., \theta_{p-1}^{\mathbb{A}}) \\ &= [\omega(\theta, \theta_1, ..., \theta_{p-1})]^{\mathbb{A}}\end{aligned}$$

for any $\theta_1, \theta_2, ..., \theta_{p-1} \in \mathfrak{X}(M)$, we conclude that $(i_\theta \omega)^{\mathbb{A}} = i_{\theta^{\mathbb{A}}}(\omega^{\mathbb{A}})$.

Lemma 2.1. *If X is a differentiable section of the tangent bundle $\left(TM^{\mathbb{A}}, \pi_{M^{\mathbb{A}}}, M^{\mathbb{A}}\right)$ and if $(x_1, ..., x_{2n})$ is a system of local coordinates on an open $\mathbb{U}$ of M, then there exist some functions $f_i \in C^\infty(\mathbb{U}^{\mathbb{A}}, \mathbb{A})$ for $i = 1, ..., 2n$ such that*

$$X_{/\mathbb{U}^{\mathbb{A}}} = \sum_{i=1}^{n} f_i \left(\frac{\partial}{\partial x_i}\right)^{\mathbb{A}} + \sum_{i=1}^{n} f_{n+i} \left(\frac{\partial}{\partial x_{i+n}}\right)^{\mathbb{A}}. \tag{10}$$

Theorem 2.1. *Let $(M^{\mathbb{A}}, \omega^{\mathbb{A}})$ be a $2n$-dimensional symplectic $\mathbb{A}$-manifold, and let ξ be any point in $M^{\mathbb{A}}$. Then there is a coordinate chart $(\mathbb{U}^{\mathbb{A}}, x_1^{\mathbb{A}}, ..., x_{2n}^{\mathbb{A}})$ centered at ξ such that on $\mathbb{U}^{\mathbb{A}}$*

$$\omega^{\mathbb{A}} = \sum_{i=1}^{n} d^{\mathbb{A}}(x_i^{\mathbb{A}}) \wedge d^{\mathbb{A}}(x_{n+i}^{\mathbb{A}}). \tag{11}$$

Corollary 2.1. *When (M, ω) is a symplectic manifold, then $(M^{\mathbb{A}}, \omega^{\mathbb{A}})$ is a symplectic $\mathbb{A}$-manifold.*

When (M, ω) is a symplectic manifold, for any $f \in C^\infty(M)$, we denote X_f a unique vector field on M such that

$$i_{X_f}\omega = df \tag{12}$$

and for any $\varphi \in C^\infty(M^{\mathbb{A}}, \mathbb{A})$, we denote X_φ a unique vector field on $M^{\mathbb{A}}$, considered as a derivation of $C^\infty(M)$ into $C^\infty(M^{\mathbb{A}}, \mathbb{A})$, such that

$$i_{X_\varphi}\omega^{\mathbb{A}} = d^{\mathbb{A}}(\varphi). \tag{13}$$

We easily verify that the bracket

$$\{\varphi, \psi\}_{\omega^{\mathbb{A}}} = -\omega^{\mathbb{A}}(X_\varphi, X_\psi) \tag{14}$$

defines a structure of $\mathbb{A}$-Poisson manifold on $M^{\mathbb{A}}$. For more details see [1].

Proposition 2.2. [6] *If (M,ω) is a symplectic manifold, for any $f \in C^{\infty}(M)$ then*

$$X_{f^{\mathbb{A}}} = (X_f)^{\mathbb{A}}. \tag{15}$$

We state the following theorem:

Theorem 2.2. *If (M,ω) is a symplectic manifold, the structure of $\mathbb{A}$-Poisson manifold on $M^{\mathbb{A}}$ defined by $\omega^{\mathbb{A}}$ coincide with the prolongation on $M^{\mathbb{A}}$ of the Poisson structure on M defined by the symplectic form ω.*

3. The reduction of Darboux's theorem on Weil bundles

Theorem 3.1. *Suppose that $\omega^{\mathbb{A}}$ is a $d^{\mathbb{A}}$-closed 2-form of constant half-rank n on an $\mathbb{A}$-manifold $(M^{\mathbb{A}})^{2n+k}$. Then the null bundle*

$$N^{\mathbb{A}}_{\omega^{\mathbb{A}}} = \left\{ \delta \in TM^{\mathbb{A}} \Big/ \omega^{\mathbb{A}}(\delta, v) = 0, \forall v \in T_{\pi_{M^{\mathbb{A}}}(\delta)} M^{\mathbb{A}} \right\} \tag{16}$$

is integrable and of constant rank k. Moreover, any point of $M^{\mathbb{A}}$ has a neighborhood $\mathbb{U}^{\mathbb{A}}$ on which there exist local coordinates $\left(x^{\mathbb{A}}_1, ..., x^{\mathbb{A}}_{2n}, y^{\mathbb{A}}_1, ..., y^{\mathbb{A}}_k\right)$ in which

$$\omega^{\mathbb{A}}_{|_{\mathbb{U}^{\mathbb{A}}}} = \sum_{i=1}^{n} d^{\mathbb{A}}(x^{\mathbb{A}}_i) \wedge d^{\mathbb{A}}(x^{\mathbb{A}}_{n+i}). \tag{17}$$

Proof. Note that $X \in \mathfrak{X}(M^{\mathbb{A}})$ is a section of $N^{\mathbb{A}}_{\omega^{\mathbb{A}}}$ if and only if $i_X\omega^{\mathbb{A}} = 0$. We deduce that $\mathcal{L}_X\omega^{\mathbb{A}} = 0$. If X and Y are two sections of $N^{\mathbb{A}}_{\omega^{\mathbb{A}}}$, then $i_{[X,Y]}\omega^{\mathbb{A}} = 0$, so it follows that $[X,Y]$ is a section of $N^{\mathbb{A}}_{\omega^{\mathbb{A}}}$ as well. Thus, $N^{\mathbb{A}}_{\omega^{\mathbb{A}}}$ is integrable.

For any point $\xi \in M^{\mathbb{A}}$, there exists a neighborhood $\mathbb{U}^{\mathbb{A}}$ on which there exist local coordinates $y^{\mathbb{A}}_1, ..., y^{\mathbb{A}}_{2n+k}$ so that $N^{\mathbb{A}}_{\omega^{\mathbb{A}}}$ restricted to $\mathbb{U}^{\mathbb{A}}$ is spanned by the vector fields $Y_i = \frac{\partial}{\partial y_i}$ for $1 \leq i \leq k$. Since $i_{Y_i}\omega^{\mathbb{A}} = \mathcal{L}_{Y_i}\omega^{\mathbb{A}} = 0$ for $1 \leq i \leq k$, it follows that $\omega^{\mathbb{A}}$ can be expressed on $\mathbb{U}^{\mathbb{A}}$ in terms of the variables $y^{\mathbb{A}}_1, ..., y^{\mathbb{A}}_{2n+k}$ alone. In particular, $\omega^{\mathbb{A}}$ restricted to $\mathbb{U}^{\mathbb{A}}$ may be regarded as a nondegenerate $d^{\mathbb{A}}$-closed 2-form on an open set in $\mathbb{A}^{2n}$. The result now follows from theorem 2.1.

4. Hamiltonian vector fields

We now want to examine some of the special vector fields which are defined on symplectic $\mathbb{A}$-manifolds. Consider a symplectic $\mathbb{A}$-manifold $\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$, let $Sp\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right) \subset Diff\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$ denote the subgroup of symplectomorphisms of $\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$. We would like to follow Lie in regarding $Sp\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$ as an infinite dimensional Lie group. In this case, the Lie algebra of $Sp\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$ should be the space of vector fields whose flows preserve $\omega^{\mathbb{A}}$. Of course, $\omega^{\mathbb{A}}$ will be invariant under the flow of a vector field X if and only if $\mathcal{L}_X\omega^{\mathbb{A}} = 0$. This motivates the following definition:

A vector field X on $M^{\mathbb{A}}$ is said to be symplectic if $\mathcal{L}_X\omega^{\mathbb{A}} = 0$. The space of symplectic vector fields on $M^{\mathbb{A}}$ will be denoted $Sp\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$.

Since $d^{\mathbb{A}}\omega^{\mathbb{A}} = 0$, for any vector field X on $M^{\mathbb{A}}$, we have $\mathcal{L}_X\omega^{\mathbb{A}} = d^{\mathbb{A}}\left(i_X\omega^{\mathbb{A}}\right)$. It is a very simple characterization of the symplectic vector fields on $M^{\mathbb{A}}$. Thus we have the following statements:

(1) X is a symplectic vector field on $M^{\mathbb{A}}$;
(2) $i_X\omega^{\mathbb{A}}$ is $d^{\mathbb{A}}$-closed.

Since $TM^{\mathbb{A}}$ and $T^*M^{\mathbb{A}}$ have the same rank, it follows that the map

$$\tau : \mathfrak{X}\left(M^{\mathbb{A}}\right) \longrightarrow \Lambda^1\left(M^{\mathbb{A}}\right)$$

is an isomorphism of $C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$-modules. Let

$$\varkappa : \Lambda^1\left(M^{\mathbb{A}}\right) \longrightarrow \mathfrak{X}\left(M^{\mathbb{A}}\right)$$

an inverse of τ.

Remark that the 1-form $i_X\omega^{\mathbb{A}}$ vanishes only where X does.

With this notation, we shown that

$$sp\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right) = \varkappa\left(\mathcal{Z}^1\left(M^{\mathbb{A}}\right)\right) \tag{18}$$

where $\mathcal{Z}^1\left(M^{\mathbb{A}}\right)$ denotes the vector space of closed 1-forms on $M^{\mathbb{A}}$. Since $d^{\mathbb{A}} \circ d^{\mathbb{A}} = 0, \mathcal{Z}^1\left(M^{\mathbb{A}}\right)$ contains a subspace,

$$\mathcal{B}^1\left(M^{\mathbb{A}}\right) = d^{\mathbb{A}}\left(C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)\right) \tag{19}$$

which is the space of exact 1-forms on $M^{\mathbb{A}}$.

For each $f^{\mathbb{A}} \in C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$, the vector field

$$X_{f^{\mathbb{A}}} = \varkappa\left(d^{\mathbb{A}} f^{\mathbb{A}}\right) \tag{20}$$

is called the hamiltonian vector field associated to $f^{\mathbb{A}}$.

The set of all hamiltonian vector fields on $M^{\mathbb{A}}$ is denoted $\mathcal{H}\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$. We obtain

$$\mathcal{H}\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right) = \varkappa\left(\mathcal{B}^1\left(M^{\mathbb{A}}\right)\right). \tag{21}$$

For this reason, hamiltonian fields are often called exact.

Corollary 4.1. *For $X, Y \in Sp\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$, we have*

$$[X, Y] = Z_{\omega^{\mathbb{A}}(X,Y)}. \tag{22}$$

In particular,

$$\left[X_{f^{\mathbb{A}}}, X_{g^{\mathbb{A}}}\right] = X_{\{f^{\mathbb{A}}, g^{\mathbb{A}}\}} \tag{23}$$

where, by definition $\left\{f^{\mathbb{A}}, g^{\mathbb{A}}\right\} = -\omega^{\mathbb{A}}\left(X_{f^{\mathbb{A}}}, X_{g^{\mathbb{A}}}\right)$.

If $M^{\mathbb{A}}$ is connected, then we get an exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{A} \longrightarrow C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right) \longrightarrow \mathcal{H}\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right) \longrightarrow 0,$$

which is not, in general split. Since

$$\left\{f^{\mathbb{A}}, 1_{C^\infty(M^{\mathbb{A}}, \mathbb{A})}\right\} = 0 \tag{24}$$

for all functions $f^{\mathbb{A}}$ on $M^{\mathbb{A}}$, it follows that the $\mathbb{A}$-Poisson bracket on $C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$ makes it into a central extension plays an important role of the algebra of hamiltonian vector fields. The geometry of this central extension plays an important role in quantization theories on symplectic $\mathbb{A}$-manifolds.

Also of great interest is exact sequence

$$0 \longrightarrow \mathcal{H}\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right) \longrightarrow sp\left(M^{\mathbb{A}}, \mathbb{A}\right) \longrightarrow \mathcal{H}^1_{dR}\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right) \longrightarrow 0,$$

where the right hand arrow is just the map described by $X \longmapsto \left[i_X \omega^{\mathbb{A}}\right]$. Since the bracket of two elements in $sp\left(M^{\mathbb{A}}, \mathbb{A}\right)$ lies in $\mathcal{H}\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$, it follows that this $\mathbb{A}$-linear map is actually a Lie $\mathbb{A}$-algebra homomorphism when $\mathcal{H}^1_{dR}\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$ is given the abelian Lie $\mathbb{A}$-algebra structure. This sequence also may or may not split, and the properties of this extension have a great deal to do with the study of groups of symplectomorphisms of $M^{\mathbb{A}}$.

5. Conclusion

The construction discussed in this paper permits to produce new symplectic $\mathbb{A}$-manifolds from a given one, $\left(M^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$, by combining two operations: restriction to a submanifold $N^{\mathbb{A}}$ of $M^{\mathbb{A}}$ upon which $\omega^{\mathbb{A}}$ induces a 2-form $\omega^{\mathbb{A}}_{N^{\mathbb{A}}}$ of constant rank, and than taking the quotient of $N^{\mathbb{A}}$ by the characteristic distribution of $\omega^{\mathbb{A}}_{N^{\mathbb{A}}}$. This construction, frequently used in mechanics. It is particularly important when the submanifold $N^{\mathbb{A}}$ is the preimage of a point under the momentum map associated with the hamiltonian action of the Lie group.

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