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Infinite Points and Isogonal Conjugate

FRANCISCO JAVIER GARCÍA CAPITÁN

Abstract. The point of view of a point on the circumcircle as the isogonal conjugate of an infinte point can lead to interesting questions and answers.

1. Introduction

Let ABC be a triangle. We draw the three internal bisectors, intersecting at the incenter I, the center of the incircle.

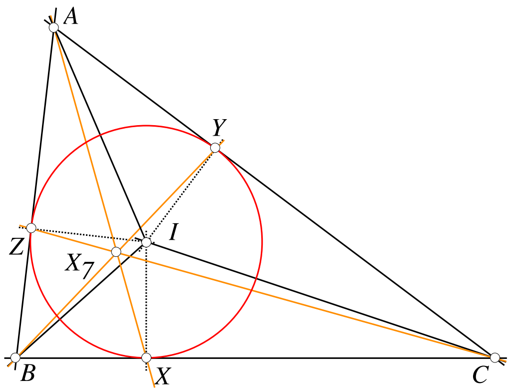

FIGURE 1

If the incircle touches the sides BC, CA, AB at X, Y, Z respectively and we write $YA = AZ = u$, $ZB = BX = v$, $XC = CY = w$ then, if we call s the semiperimeter of ABC, we have

$$\begin{cases} v+w=a \\ w+u=b \\ u+v=c \end{cases} \Rightarrow 2(u+v+w)=2s \Rightarrow u+v+w=s \Rightarrow \begin{cases} u=s-a \\ v=s-b \\ w=s-c \end{cases}.$$

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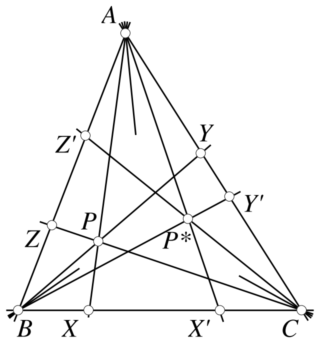
2. Isogonal Conjugate

Let us recall the concept of isogonal conjugate with respect to a triangle.

Problem 1. *Let ABC be a triangle and P a point with cevian triangle XYZ. The reflections of lines AX', BY', CZ' of AX, BX, CX with respect to the corresponding internal bisectors concur at a point P^* called* isogonal conjugate *of P with respect to ABC.*



This can be easily proved by using the trigonometric form of Ceva Theorem, that is, since AX, BY and CY concur we have

$$\frac{\sin\angle BAX}{\sin\angle XAC}\cdot\frac{\sin\angle CBY}{\sin\angle YBA}\cdot\frac{\sin\angle ACZ}{\sin\angle ZCB}=1.$$

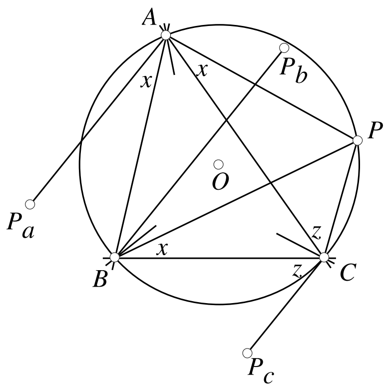
This becomes

$$\frac{\sin\angle X'AC}{\sin\angle BAX'}\cdot\frac{\sin\angle Y'BA}{\sin\angle CBY'}\cdot\frac{\sin\angle Z'CB}{\sin\angle A'CZ}=1,$$

and then AX', BY', CZ' also concur.

3. Isogonal conjugate of an infinite point

Problem 2. *Let ABC be a triangle and P a point on the circumcircle. Call P_a, P_b, P_c the reflection of P with respecto to the internal bisectors of angles A, B, C respectively. Then lines AP_a, BP_b y CP_c are parallel.*



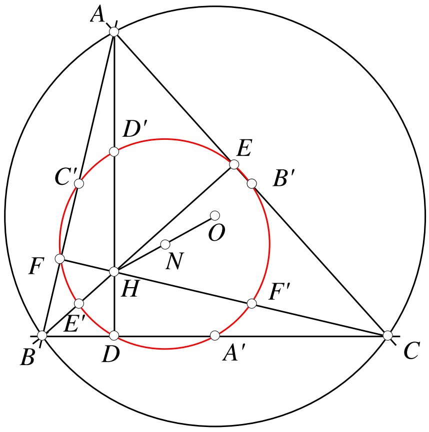
Solution. We use the figure and angle chase. The supplementary angle of $\angle ACP_c$ is $180° - C - z = A + B - z = A + x = \angle CAP_a$, therefore AP_a and AC_a are parallel. We also have $\angle P_aAB = \angle CAP = \angle CBP = \angle P_bBA$, hence AP_a and BP_b are also parallel.

In other words, the isogonal conjugate of a point on the circumcircle is a point at infinity. The reverse is true and can be proved using the same figure. Details are left to the reader.

4. The nine point circle

This result is very well known:

Problem 3. *Let ABC be a triangle with orthocenter H, medial triangle $A'B'C'$ and, orthic triangle DEF. Call D', E', F' the midpoints of HD, HE, HF. The nine points $A', B', C', D, E, F, D', E', F'$ lie on a circle called* the nine point circle *of ABC. Its center N is the midpoint of HO, where O is the circumcenter of ABC.*

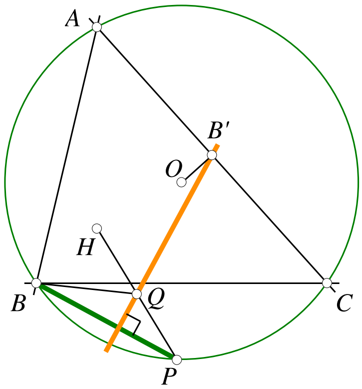


It follows that the nine point circle is the image of the circumcircle by a homothety of center H and ratio $1/2$.

Therefore, if P lies on the circumcircle, the midpoint Q of the segment HP lies on the nine point circle and we can consider the isogonal conjugate of P with respect to ABC and the isogonal conjugate of Q with respect to $A'B'C'$.

What is the relationship between these two infinite points?

Problem 4. *Let ABC be a triangle with orthocenter H, P a point on its circumcircle, and Q the midpoint of HP. Prove that if B' is the midpoint of CA, $B'Q$ is perpendicular to BP.*



Solution. We use complex numbers, in lowercase letters representing the corresponding affixes in uppercase letters. The triangle is inscribed in the unit circle, with the circumcenter at the origin. The centroid G corresponds to $g = \frac{1}{3}(a+b+c)$ and the orthocenter to $h = a+b+c$. Therefore, since

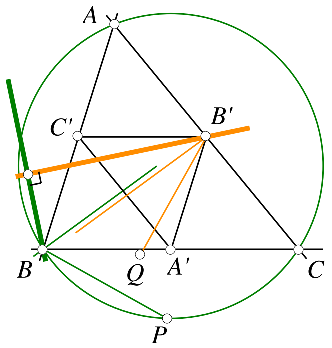
$$q - b' = \frac{p+h}{2} - \frac{c+a}{2} = \frac{p+a+b+c}{2} - \frac{a+c}{2} = \frac{p+b}{2}.$$

The complex numbers $p+b$ and $p-b$ are perpendicular, because

$$\overline{\frac{p+b}{p-b}} = \frac{\overline{p}+\overline{b}}{\overline{p}-\overline{b}} = \frac{\frac{1}{p}+\frac{1}{b}}{\frac{1}{p}-\frac{1}{b}} = \frac{p+b}{b-p} = -\frac{p+b}{p-b}$$

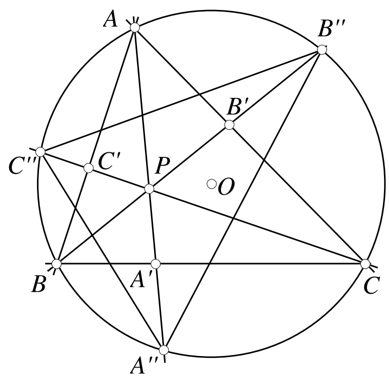
and then the quotient is pure imaginary.

As a result the reflection of BP with respect to the internal bisector of B in ABC and the reflection of $B'Q$ with respect to the internal bisector of B' in $A'B'C'$ are also perpendicular, since these internal bisectors are parallel.

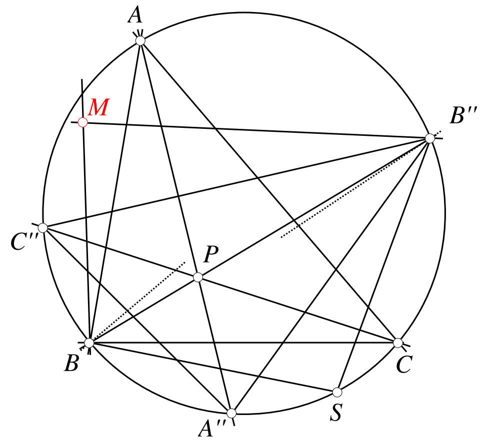


5. Cevian triangle and circumcevian triangle

Given a triangle ABC and a point of P, the intersections A', B', C' of AP, BP, CP with BC, CA and AB are the vertices of the *cevian triangle* of P, whereas the second intersections AP, BP, CP and the circumcircle form the circumcevian triangle de P.



Now, given a triangle ABC and the circumcevian triangle $A''B''C''$ of some point P, for any other point S on the circumcircle we can consider two infinite points: the isogonal conjugate of S with respect to ABC and the isogonal conjugate of S with respect to $A''B''C''$. What is the the relationship between them?



To visualize this we draw the reflection of line BS with respect to the internal bisector of angle B of ABC and the reflection of line $B''S$ with respect to the internal bisector of angle B'' of $A''B''C''$, intersecting at M.

In other words we have defined a function from the line at infinity to itself by which the infinite point of BM goes to the infinite point of $B''M$. By using barycentric coordinates and *Mathematica*, we calculate the angle θ formed by these two lines.

If $P = (x : y : z)$ in barycentric coordinates we find

$$\cos\theta = \frac{\left|\sum_{\text{cyclic}} a^2\left(c^2 S_C y - b^2 S_B z\right) yz\right|}{abc \cdot |x+y+z|^3 \cdot PA \cdot PB \cdot PC},$$

that **does not depend** on S.

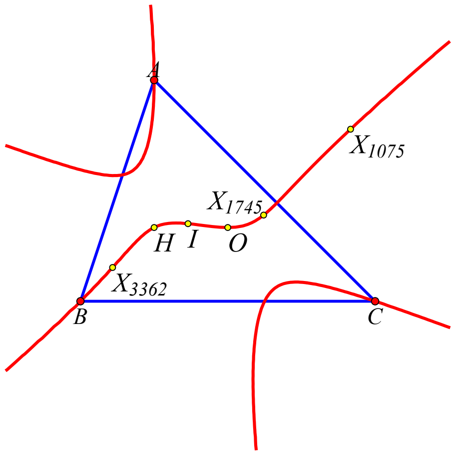
Therefore the locus of M when S traverses the circumcircle is a circle through B and B''.

6. The McCay cubic

The McCay cubic of a triangle ABC has equation

$$\sum_{\text{cyclic}} a^2\left(c^2 S_C y - b^2 S_B z\right) yz = 0,$$

exactly the expression that appears in the numerator of $\cos\theta$, therefore this cubic is the locus of points P for which lines BM and $B''M$ are perpendicular.

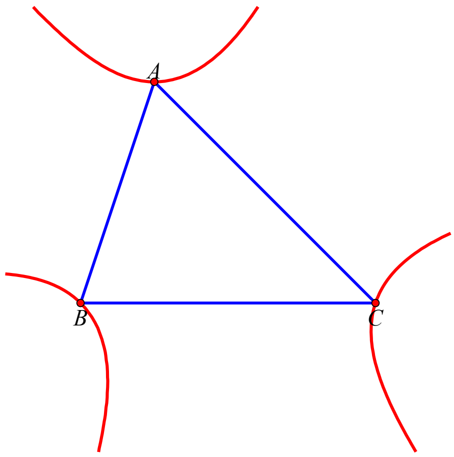


The McCay cubic goes through point $X(n)$ in ETC for $n = 1$, 3, 4, 1075, 1745, 3362, 13855, 46357, 46358. Note that pairs $(1, 1)$, $(3, 4)$, $(1745, 3362)$, $(1075, 13855)$, $(46357, 46358)$ are isogonal conjugates. This suggests an interesting property:

Problem 5. *The value of* $\cos\theta$ *is invariant if we replace* P *by the isogonal conjugate of* P*.*

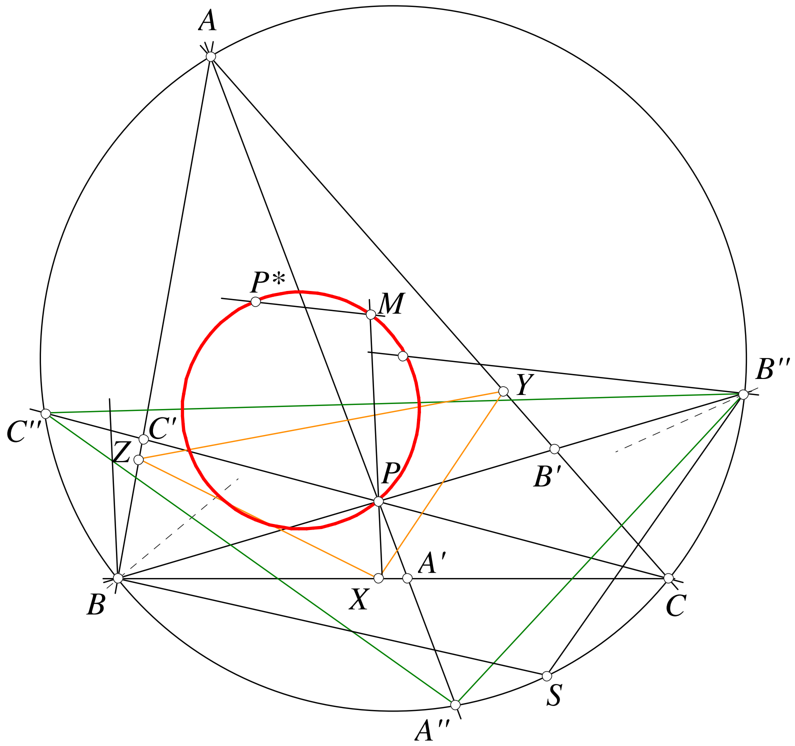
If we calculate the value of $\sin\theta$ instead, we get another cubic, K024, with equation

$$\sum_{cyclic} a^2(c^2y + b^2z)yz = 0.$$



7. Circle through P and P^*

We finish with a circle that does not depend on vertices B and B''. Draw parallels through P to BS and through P^* (the isogonal conjugate of P) to $B''S$. If these lines intersect at M, then M describes a circle trough P and P^*.



The radius ρ of this circle can be obtained with the formula

$$\rho = \left|\frac{PP*}{2k}\right| \cdot \sqrt{\frac{(A''B''C'')}{(XYZ)}}.$$

- PP^* is the distance between P and P^*.
- $k = \frac{A''A}{AA'} + \frac{B''B}{BB'} + \frac{C''C}{CC'} + 2.$
- $(A''B''C'')$ and (XYZ) are the areas of the circumcevian and pedal triangles of P.

8. Conclusion

It is amazing the fact that the angle formed by the two directions that are isogonal conjugates of a variable point on the circumcircle is a constant angle.

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DEPARTAMENT OF MATHEMATICS
I.E.S. ÁLVAREZ CUBERO,
14800 PRIEGO DE CÓRDOBA, SPAIN
E-mail address: garciacapitan@gmail.com