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Geometry of φ-Sasaki metric on tangent bundle

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Abstract. In this paper, we introduce the φ-Sasaki metric on the tangent bundle. Firstly, we characterize the Riemannian curvature, the sectional curvature and the scalar curvature on the tangent bundle with the φ-Sasaki metric. Secondly, we study the geodesics on the tangent bundle with respect to this metric. Finally, we study the geometry of φ-unit tangent bundle equipped with φ-Sasaki metric.

1. Introduction

The tangent bundle equipped with the Sasaki metric has been studied by many authors such as Sasaki [17], Yano and Ishihara [19], Dombrowski [7], Salimov, Gezer and Cengiz [5, 8, 12, 15], Bejan and Crasmareanu [3] etc... The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on tangent bundle. Musso and Tricerri [11] has introduced the notion of Cheeger-Gromoll metric, this metric has been studied also by many authors (see [1, 6, 9, 16, 18]).

In a previous work, [21], we proposed the φ-Sasaki metric on a tangent bundle where we studied the para-Kahler-Norden properties on a tangent bundle with this metric. In this paper, we investigate the geometry of the φ-Sasakian metrics. Firstly we give the Levi-Civita connection of this metric (Theorem 3.1) and we establish the Riemannian curvature (Theorem 4.1 and Proposition 4.1) and we characterize the sectional curvature (Theorem 4.2 and Proposition 4.2) also the scalar curvature (Theorem 4.3, Theorem 4.4 and Proposition 4.3). Secondly, we study the geodesics on the tangent bundle, where we establish the necessary and sufficient conditions under which a curve be geodesic with respect to this metric. (Theorem 5.1, Theorem 5.2, Corollary 5.1 and Corollary 5.2). In the last section, we study the geometry of φ-unit tangent bundle equipped with φ-Sasaki metric, where we presented the formulas of the Levi-Civita connection (Theorem 6.1) and all formulas of the Riemannian curvature tensors (Theorem 6.2).

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2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, u^i)_{i=\overline{1,m}}$ on TM, where (u^i) is the Cartesian coordinates in each tangent space T_xM at $x \in M$ with respect to the natural base $\left(\frac{\partial}{\partial x^i}|_x\right)$, x being an arbitrary point in U whose coordinates are (x^i). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM. \tag{1}$$

of the tangent bundle to TM at any $(x, u) \in TM$ into vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial u^i}|_{(x,u)},\ \xi^i \in \mathbb{R}\}, \tag{2}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}|_{(x,u)},\ \xi^i \in \mathbb{R}\}. \tag{3}$$

.

Note that the map $X \to {}^HX$ is an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. Similarly, the map $X \to {}^VX$ is an isomorphism between the vector spaces T_xM and $V_{(x,u)}TM$. Obviously, each tangent vector $Z \in T_{(x,u)}TM$ can be written in the form $Z = {}^HX + {}^VY$, where $X, Y \in T_xM$ are uniquely determined vectors.

Note that the map $\xi \to {}^H\xi = \xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. Similarly, the map $\xi \to {}^V\xi = \xi^i \frac{\partial}{\partial u^i}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $V_{(x,u)}TM$. Obviously, each tangent vector $Z \in T_{(x,u)}TM$ can be written in the form $Z = {}^HX + {}^VY$, where $X, Y \in T_xM$ are uniquely determined vectors.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$ {}^VX = X^i \frac{\partial}{\partial u^i}, \tag{4}$$

$$ {}^HX = X^i\left(\frac{\partial}{\partial x^i} - u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}\right). \tag{5}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial u^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TTM.

In particular if U be a local vector field constant on each fiber T_xM such that $(U = u)$, the vertical lift ${}^VU$ is called the canonical vertical vector field or Liouville vector field on TM.

The bracket operation of vertical and horizontal vector fields is given by the formulas: [7, 19]

$$\begin{cases} [{}^HX, {}^HY] = {}^H[X, Y] - {}^V(R(X, Y)u), \\ [{}^HX, {}^VY] = {}^V(\nabla_X Y), \\ [{}^VX, {}^VY] = 0, \end{cases} \tag{6}$$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}.$$

Definition 2.1. *Let (M,g) be a Riemannian manifold and $F:TM\to TM$ be a smooth bundle endomorphism of TM. Then the vertical and horizontal vector fields ${}^VF$ and ${}^HF$ are defined on TM by*

$$\begin{aligned}{}^VF:TM &\rightarrow TTM\\ (x,u)&\mapsto {}^V(F(u)),\end{aligned}$$

$$\begin{aligned}{}^HF:TM &\rightarrow TTM\\ (x,u)&\mapsto {}^H(F(u)),\end{aligned}$$

locally we have

$$ {}^VF(\eta)=\eta^{i\,V}(F(\frac{\partial}{\partial x^i}))\quad \textit{and}\quad {}^HF(\eta)=\eta^{i\,H}(F(\frac{\partial}{\partial x^i})). \tag{7}$$

where $\eta=\eta^i\frac{\partial}{\partial x^i}$ is a local representation of vector field η on M.

3. φ-Sasaki metric

An almost product structure φ on a manifold M is a $(1,1)$ tensor field on M such that $\varphi^2=id_M$, $\varphi\neq\pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M,φ) is called an almost product manifold.

An almost para-complex manifold is an almost product manifold (M,φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost par-acomplex manifold is necessarily even [4].

An almost para-complex Norden manifold (M^{2m},φ,g) is a $2m$-dimensional differentiable manifold M with an almost para-complex structure φ and a Riemannian metric g such that:

$$g(\varphi X,Y)=g(X,\varphi Y)\quad\Leftrightarrow\quad g(\varphi X,\varphi Y)=g(X,Y), \tag{8}$$

for any vector fields X and Y on M, in this case g is called a pure metric with respect to φ or Norden metric (B-metric)[14].

Also note that

$$G(X,Y)=g(\varphi X,Y), \tag{9}$$

is a bilinear, symmetric tensor field of type $(0,2)$ on (M,φ) and pure with respect to the paracomplex structure φ, which is called the twin (or dual) metric of g, and it plays a role similar to the Kähler form in Hermitian Geometry. Some properties of twin Norden metric are investigated in [10, 14].

A para-Kähler-Norden manifold is an almost para-complex Norden manifold (M^{2m},φ,g) such that φ is integrable i.e. $\nabla\varphi=0$ (B-manifold), where ∇ is the Levi-Civita connection of g [13, 14].

It is well known that if (M^{2m}, φ, g) is a para-Kähler-Norden manifold, the Riemannian curvature tensor is pure [14], and for any vector fields Y and Z on M.

$$(10) \qquad \begin{cases} R(\varphi Y, Z) = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) = R(Y, Z), \end{cases}$$

Definition 3.1. [21] *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold. On the tangent bundle TM, we define a φ-Sasaki metric noted g^{φ} by*

$$\begin{array}{lrcl} (1) & g^{\varphi}({}^{H}X, {}^{H}Y)_{(x,u)} & = & g_x(X, Y), \\ (2) & g^{\varphi}({}^{H}X, {}^{V}Y)_{(x,u)} & = & 0, \\ (3) & g^{\varphi}({}^{V}X, {}^{V}Y)_{(x,u)} & = & g_x(X, \varphi Y) = G(X, Y), \end{array}$$

for any vector fields X and Y on M and $(x, u) \in TM$, where G is the twin Norden metric of g defined by (9).

Subsequently, $|.|$ denotes the norm with respect to (M^{2m}, φ, g).

Lemma 3.1. [22] *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, then we have the following:*

$$\begin{array}{lrcl} (1) & {}^{H}X(g(u, \varphi u)) & = & 0, \\ (2) & {}^{V}X(g(u, \varphi u)) & = & 2g(X, \varphi u), \\ (3) & {}^{H}X(g(Y, \varphi u)) & = & g(\nabla_X Y, \varphi u), \\ (4) & {}^{V}X(g(Y, \varphi u)) & = & g(X, \varphi Y), \end{array}$$

for any vector fields X and Y on M.

Lemma 3.2. [21] *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, we have the following*

$$\begin{array}{lrcl} (1) & {}^{H}Xg^{\varphi}({}^{H}Y, {}^{H}Z) & = & Xg(Y, Z), \\ (2) & {}^{V}Xg^{\varphi}({}^{H}Y, {}^{H}Z) & = & 0, \\ (3) & {}^{H}Xg^{\varphi}({}^{V}Y, {}^{V}Z) & = & g^{\varphi}({}^{V}(\nabla_X Y), {}^{V}Z) + g^{\varphi}({}^{V}Y, {}^{V}(\nabla_X Z)), \\ (4) & {}^{V}Xg^{\varphi}({}^{H}Y, {}^{H}Z) & = & 0, \end{array}$$

for any vector fields X, Y and Z on M.

The Levi-Civita connection $\widetilde{\nabla}$ of TM with φ-Sasaki metric g^{φ} is given by the following theorem.

Theorem 3.1. [21] *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^{φ}. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M^{2m}, φ, g) (resp (TM, g^{φ})), then we have*

$$\begin{array}{lrcl} (1) & (\widetilde{\nabla}_{{}^{H}X}{}^{H}Y)_{(x,u)} & = & {}^{H}(\nabla_X Y)_{(x,u)} - \dfrac{1}{2}{}^{V}(R_x(X, Y)u), \\ (2) & (\widetilde{\nabla}_{{}^{H}X}{}^{V}Y)_{(x,u)} & = & {}^{V}(\nabla_X Y)_{(x,u)} + \dfrac{1}{2}{}^{H}(R_x(\varphi u, Y)X), \\ (3) & (\widetilde{\nabla}_{{}^{V}X}{}^{H}Y)_{(x,u)} & = & \dfrac{1}{2}{}^{H}(R_x(\varphi u, X)Y), \\ (4) & (\widetilde{\nabla}_{{}^{V}X}{}^{V}Y)_{(x,u)} & = & 0, \end{array}$$

for all vector fields X and Y on M and $(x,u) \in TM$, where R denote the curvature tensor of (M^{2m}, φ, g).

The proof of Theorem 3.1 follows directly from Kozul formula, (6) and Lemma 3.2.

Lemma 3.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ, then we have*

$$\widetilde{\nabla}_{^HX}{}^VU = 0 \quad , \quad \widetilde{\nabla}_{^VU}{}^HX = 0 \quad , \quad \widetilde{\nabla}_{^VX}{}^VU = {}^VX$$
$$\widetilde{\nabla}_{^VU}{}^VX = 0 \quad , \quad \widetilde{\nabla}_{^VU}{}^VU = {}^VU,$$

for all vector field X on M, where ^VU is the canonical vertical vector field on TM.

From Definition 2.1 and Theorem 3.1 we have:

Proposition 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M^{2m}, φ, g) (resp (TM, g^φ)) and F is a tensor field of type $(1,1)$ on M, then:*

$$\begin{array}{lrcl}
(1) & (\widetilde{\nabla}_{^HX}{}^HF)_{(x,u)} & = & {}^H(\nabla_X F)_{(x,u)} - \frac{1}{2}{}^V(R_x(X, F(u))u), \\
(2) & (\widetilde{\nabla}_{^HX}{}^VF)_{(x,u)} & = & {}^V(\nabla_X F)_{(x,u)} + \frac{1}{2}{}^H(R_x(\varphi u, F(u))X), \\
(3) & (\widetilde{\nabla}_{^VX}{}^HF)_{(x,u)} & = & {}^H(F(X))_{(x,u)} + \frac{1}{2}{}^H(R_x(\varphi u, X)F(u)), \\
(4) & (\widetilde{\nabla}_{^VX}{}^VF)_{(x,u)} & = & {}^V(F(X))_{(x,u)},
\end{array}$$

for all vector fields X and Y on M and $(x,u) \in TM$, where R denote the curvature tensor of (M^{2m}, φ, g).

4. Curvatures of φ-Sasaki metric

We shall calculate the Riemannian curvature tensor of TM with the φ-Sasaki metric g^φ. This curvature tensor is characterized by the formula:

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \tag{11}$$

for any vector fields $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ on TM.

Theorem 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. If R (resp $\widetilde{R}$) denote the Riemann curvature tensor of (M^{2m}, φ, g) (resp (TM, g^φ)), then we have the following formulas*

$$\begin{aligned}
\widetilde{R}({}^HX, {}^HY){}^HZ \;=\; & {}^H(R(X,Y)Z) + \frac{1}{2}{}^H(R(\varphi u, R(X,Y)u)Z)) \\
& + \frac{1}{4}{}^H(R(\varphi u, R(X,Z)u)Y) - \frac{1}{4}{}^H(R(\varphi u, R(Y,Z)u)X) \\
& + \frac{1}{2}{}^V((\nabla_Z R)(X,Y)u),
\end{aligned} \tag{12}$$

$$\widetilde{R}(^{H}X, {}^{V}Y)^{V}Z = -\frac{1}{2}{}^{H}(R(\varphi Y, Z)X) - \frac{1}{4}{}^{H}(R(u,Y)R(u,Z)X), \tag{13}$$

$$\begin{aligned}\widetilde{R}(^{V}X, {}^{V}Y)^{H}Z &= {}^{H}(R(\varphi X, Y)Z) + \frac{1}{4}{}^{H}(R(u,X)R(u,Y)Z) \\ &\quad -\frac{1}{4}{}^{H}(R(u,Y)R(u,X)Z)),\end{aligned} \tag{14}$$

$$\begin{aligned}\widetilde{R}(^{H}X, {}^{V}Y)^{H}Z &= \frac{1}{2}{}^{H}((\nabla_X R)(\varphi u, Y)Z) + \frac{1}{2}{}^{V}(R(X,Z)Y) \\ &\quad +\frac{1}{4}{}^{V}(R(R(\varphi u, Y)Z, X)u),\end{aligned} \tag{15}$$

$$\begin{aligned}\widetilde{R}(^{H}X, {}^{H}Y)^{V}Z &= \frac{1}{2}{}^{H}((\nabla_X R)(\varphi u, Z)Y) - \frac{1}{2}{}^{H}((\nabla_Y R)(\varphi u, Z)X)) \\ &\quad +\frac{1}{4}{}^{V}(R(R(\varphi u, Z)Y, X)u) - \frac{1}{4}{}^{V}(R(R(\varphi u, Z)X, Y)u) \\ &\quad +{}^{V}(R(X,Y)Z),\end{aligned} \tag{16}$$

$$\widetilde{R}(^{V}X, {}^{V}Y)^{V}Z = 0, \tag{17}$$

for any vector fields X, Y and Z on M.

Proof. In the proof, we will use the Definition 3.1, Lemma 3.2, Theorem 3.1 and Proposition 3.1 we have:
1) Let $F : TM \to TM$ be the bundle endomorphism given by
$F(u) = R(Y,Z)u$ and ${}^{V}F(u) = {}^{V}(R(Y,Z)u)$, from direct calculation we get:

$$\begin{aligned}\widetilde{\nabla}_{^{H}X}\widetilde{\nabla}_{^{H}Y}{}^{H}Z &= \widetilde{\nabla}_{^{H}X}{}^{H}(\nabla_Y Z) - \frac{1}{2}\widetilde{\nabla}_{^{H}X}{}^{V}F \\ &= {}^{H}(\nabla_X\nabla_Y Z) - \frac{1}{2}{}^{V}(R(X, \nabla_Y Z)u) \\ &\quad -\frac{1}{2}{}^{V}(\nabla_X(R(Y,Z)u)) + \frac{1}{2}{}^{V}(R(Y,Z)(\nabla_X U)) \\ &\quad -\frac{1}{4}{}^{H}(R(\varphi u, R(Y,Z)u)X).\end{aligned} \tag{18}$$

From which, with permutation of X by Y in the formula (18) we get

$$\begin{aligned}\widetilde{\nabla}_{^{H}Y}\widetilde{\nabla}_{^{H}X}{}^{H}Z &= {}^{H}(\nabla_Y\nabla_X Z) - \frac{1}{2}{}^{V}(R(Y, \nabla_X Z)u) \\ &\quad -\frac{1}{2}{}^{V}(\nabla_Y(R(X,Z)u)) + \frac{1}{2}{}^{V}(R(X,Z)(\nabla_Y U)) \\ &\quad -\frac{1}{4}{}^{H}(R(\varphi u, R(X,Z)u)Y).\end{aligned} \tag{19}$$

Also, we find

$$\begin{aligned}\widetilde{\nabla}_{[^{H}X, {}^{H}Y]}{}^{H}Z &= \widetilde{\nabla}_{^{H}[X,Y]}{}^{H}Z - \widetilde{\nabla}_{^{V}(R(X,Y)u)}{}^{H}Z \\ &= {}^{H}(\nabla_{[X,Y]}Z) - \frac{1}{2}{}^{V}(R([X,Y],Z)u) \\ &\quad -\frac{1}{2}{}^{H}(R(\varphi u, R(X,Y)u)Z).\end{aligned} \tag{20}$$

From(18), (19), (20) and the second Bianchi identity

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

We obtain the formula (12).

(2) From $\widetilde{\nabla}_{^VY}{}^VZ = 0$, we have directly $\widetilde{\nabla}_{^HX}\widetilde{\nabla}_{^VY}{}^VZ = 0$.
Let $F : TM \to TM$ be the bundle endomorphism given by
$F(u) = R(\varphi u, Z)X$ and ${}^HF(u) = {}^H(R(\varphi u, Z)X)$, from direct calculation we get,

$$\begin{aligned}\widetilde{\nabla}_{^VY}\widetilde{\nabla}_{^HX}{}^VZ &= \widetilde{\nabla}_{^VY}\big({}^V(\nabla_X Z) + \frac{1}{2}{}^HF\big)\\ &= \frac{1}{2}{}^H(R(\varphi Y, Z)X) + \frac{1}{4}{}^H(R(\varphi u, Y)R(\varphi u, Z)X),\end{aligned}$$

using and from (10), we get

$$\widetilde{\nabla}_{^VY}\widetilde{\nabla}_{^HX}{}^VZ = \frac{1}{2}{}^H(R(\varphi Y, Z)X) + \frac{1}{4}{}^H(R(u, Y)R(u, Z)X),$$

we have directly:

$$\widetilde{\nabla}_{[^HX,{}^VY]}{}^VZ = \widetilde{\nabla}_{^V(\nabla_X Y)^V}{}^VZ = 0,$$

we obtain the formula (13).

(3) Applying formula (13) and 1^{st} Bianchi identity.

$$\widetilde{R}({}^VX, {}^VY){}^HZ = \widetilde{R}({}^HZ, {}^VY){}^VX - \widetilde{R}({}^HZ, {}^VX){}^VY,$$

we get

$$\widetilde{R}({}^HZ, {}^VY){}^VX = -\frac{1}{2}{}^H(R(\varphi Y, X)Z) - \frac{1}{4}{}^H(R(u, Y)R(u, X)Z)$$

and

$$\widetilde{R}({}^HZ, {}^VX){}^VY = -\frac{1}{2}{}^H(R(\varphi X, Y)Z) - \frac{1}{4}{}^H(R(u, X)R(u, Y)Z).$$

Which give the formula (14).

The other formulas are obtained by a similar calculation.

Proposition 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. Then TM is flat if and only if M is flat.*

Proof. (i) It is a direct consequence of Theorem 4.1 that $R = 0$ implies $\widetilde{R} = 0$.
(ii) If we assume that $\widetilde{R} = 0$, using the formula 12 and calculate $\widetilde{R}({}^HX, {}^HY){}^HZ$ at $(x, 0) \in TM$ we get $(R(X,Y)Z)_x = (\widetilde{R}({}^HX, {}^HY){}^HZ)_{(x,0)} = 0$.

For any vector fields V and W on TM and $(x,u) \in TM$ such that $V_{(x,u)}$ and $W_{(x,u)}$ are linearly independent, the sectional curvature of the plane spanned by $V_{(x,u)}$ and $W_{(x,u)}$ is given by

$$\widetilde{K}(V,W) = \frac{g^\varphi(\widetilde{R}(V,W)W,V)}{g^\varphi(V,V)g^\varphi(W,W) - g^\varphi(V,W)^2} \tag{21}$$

Theorem 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^{φ}. If K (resp., $\widetilde{K}$) denote the sectional curvature of (M^{2m}, φ, g) (resp., (TM, g^{φ})), then for any vector fields X and Y on M, we have*

$$
\begin{aligned}
(1)\ \ \widetilde{K}({}^{H}X, {}^{H}Y) &= K(X,Y) - \frac{3g(R(X,Y)u, \varphi R(X,Y)u)}{4(|X|^2|Y|^2 - g(X,Y)^2)},\\
(2)\ \ \widetilde{K}({}^{H}X, {}^{V}Y) &= \frac{|R(u,Y)X|^2}{4|X|^2 g(Y,\varphi Y)}\\
(3)\ \ \widetilde{K}({}^{V}X, {}^{V}Y) &= 0.
\end{aligned}
$$

Proof. (1) From the formula (12), we have

$$
\begin{aligned}
\widetilde{K}({}^{H}X, {}^{H}Y) &= \frac{g^{\varphi}(\widetilde{R}({}^{H}X, {}^{H}Y){}^{H}Y, {}^{H}X)}{g^{\varphi}({}^{H}X, {}^{H}X)g^{\varphi}({}^{H}Y, {}^{H}Y) - g^{\varphi}({}^{H}X, {}^{H}Y)^2}\\
&= \frac{1}{|X|^2|Y|^2 - g(X,Y)^2}\big(g(R(X,Y)Y,X)\\
&\quad + \frac{1}{2}g(R(\varphi u, R(X,Y)u)Y,X) + \frac{1}{4}g(R(\varphi u, R(X,Y)u)Y,X)\big)\\
&= \frac{1}{|X|^2|Y|^2 - g(X,Y)^2}\big(g(R(X,Y)Y,X)\\
&\quad + \frac{3}{4}g(R(Y,X)\varphi u, R(X,Y)u)\big)\\
&= K(X,Y) - \frac{3g(R(X,Y)u, \varphi R(X,Y)u)}{4(|X|^2|Y|^2 - g(X,Y)^2)}.
\end{aligned}
$$

(2) From the formula (13), we have

$$
\begin{aligned}
\widetilde{K}({}^{H}X, {}^{V}Y) &= \frac{g^{\varphi}(\widetilde{R}({}^{H}X, {}^{V}Y){}^{V}Y, {}^{H}X)}{g^{\varphi}({}^{H}X, {}^{H}X)g^{\varphi}({}^{V}Y, {}^{V}Y) - g^{\varphi}({}^{H}X, {}^{V}Y)^2}\\
&= \frac{1}{4|X|^2 g(Y,\varphi Y)}\big[-\frac{1}{4}g(R(u,Y)R(u,Y)X,X)\big]\\
&= \frac{|R(u,Y)X|^2}{4|X|^2 g(Y,\varphi Y)},
\end{aligned}
$$

(3) The result follows immediately from the formula (17)

$$
\widetilde{K}({}^{V}X, {}^{V}Y) = \frac{g^{\varphi}(\widetilde{R}({}^{V}X, {}^{V}Y){}^{V}Y, {}^{V}X)}{g^{\varphi}({}^{V}X, {}^{V}X)g^{\varphi}({}^{V}Y, {}^{V}Y) - g^{\varphi}({}^{V}X, {}^{V}Y)^2} = 0.
$$

Proposition 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold of constant sectional curvature λ and TM its tangent bundle equipped with the φ-Sasaki metric g^{φ}. If $\widetilde{K}$ denote the sectional curvature of (TM, g^{φ}), then for any orthonormal vector fields X and Y on M, we have*

$$
\begin{aligned}
(1)\ \ \widetilde{K}({}^{H}X, {}^{H}Y) &= \lambda - \frac{3\lambda^2}{4}\big(g(X,u)g(X,\varphi u) + g(Y,u)g(Y,\varphi u)\big),\\
(2)\ \ \widetilde{K}({}^{H}X, {}^{V}Y) &= \frac{\lambda^2 g(X,u)^2}{4g(Y,\varphi Y)},\\
(3)\ \ \widetilde{K}({}^{V}X, {}^{V}Y) &= 0.
\end{aligned}
$$

Proof. M has constant curvature λ, for any vector fields U, V and W on M, we have

$$R(U,V)W = \lambda\big(g(V,W)U - g(U,W)V\big),$$

then a direct calculations, we get

$$\begin{aligned} g(R(X,Y)u, \varphi R(X,Y)u) &= \lambda^2\big(g(X,u)g(X,\varphi u) + g(Y,u)g(Y,\varphi u)\big), \\ |R(u,Y)X|^2 &= \lambda^2 g(X,u)^2, \end{aligned}$$

thus, the proof is complete.

Theorem 4.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. If σ (resp., $\widetilde{\sigma}$) denote the scalar curvature of (M^{2m}, φ, g) (resp., (TM, g^φ)), then we have*

$$\widetilde{\sigma} = \sigma - \frac{1}{4}\sum_{i,j=1}^{2m} g(R(E_i,E_j)u, \varphi R(E_i,E_j)u). \tag{22}$$

where, $(E_i)_{i=\overline{1,2m}}$ be local orthonormal frame on (M^{2m}, φ, g).

Proof. Let $(E_i)_{i=\overline{1,2m}}$ (resp.$(e_i)_{i=\overline{1,2m}}$) be local orthonormal frame on (M^{2m}, φ, g) (resp. (M^{2m}, φ, G)) and G is the twin Norden metric of g defined by (9). then $(E_i^H, e_j^V)_{i,j=\overline{1,2m}}$ is a local orthonormal frame on TM. Using Theorem 4.2 and direct calculations give.

$$\begin{aligned} \widetilde{\sigma} &= \sum_{\substack{i,j=1 \\ i\neq j}} \widetilde{K}(E_i^H, E_j^H) + 2\sum_{i,j=1}^{2m} \widetilde{K}(E_i^H, e_j^V) + \sum_{\substack{i,j=1 \\ i\neq j}} \widetilde{K}(e_i^V, e_j^V) \\ &= \sum_{\substack{i,j=1 \\ i\neq j}} \Big(K(E_i,E_j) - \frac{3}{4} g(R(E_i,E_j)u, \varphi R(E_i,E_j)u) + \frac{1}{2}|R(u,e_j)E_i|^2\Big) \\ &= \sigma - \frac{3}{4}\sum_{i,j=1}^{2m} g(R(E_i,E_j)u, \varphi R(E_i,E_j)u) + \frac{1}{2}\sum_{i,j=1}^{2m} |R(u,e_j)E_i|^2. \end{aligned}$$

In order to simplify this last expression, we have

$$\begin{aligned} \sum_{i,j=1}^{2m} |R(u,e_j)E_i|^2 &= \sum_{i,j=1}^{2m} g(R(u,e_j)E_i, R(u,e_j)E_i)) \\ &= \sum_{i,j,s=1}^{2m} g(R(u,e_j)E_i, E_s) g(R(u,e_j)E_i, E_s) \\ &= \sum_{i,j,s=1}^{2m} g(R(E_i,E_s)u, e_j) g(R(E_i,E_s)u, e_j) \end{aligned}$$

Using (8) and (9), we have

$$\begin{aligned}\sum_{i,j=1}^{2m}|R(u,e_j)E_i|^2 &= \sum_{i,j,s=1}^{2m} G(\varphi R(E_i,E_s)u,e_j)G(\varphi R(E_i,E_s)u,e_j)\\ &= \sum_{i,j,s=1}^{2m} G(R(E_i,E_s)u,e_j)G(R(E_i,E_s)u,e_j)\\ &= \sum_{i,s=1}^{2m} G(R(E_i,E_s)u,R(E_i,E_s)u)\\ &= \sum_{i,j=1}^{2m} g(R(E_i,E_j)u,\varphi R(E_i,E_j)u).\end{aligned}$$

Which implies that

$$\widetilde{\sigma} = \sigma - \frac{1}{4}\sum_{i,j=1}^{2m} g(R(E_i,E_j)u,\varphi R(E_i,E_j)u).$$

Theorem 4.4. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. Then (TM,g^φ) has constant scalar curvature if and only if (M^{2m},φ,g) is flat.*

Proof. The result is an immediate consequence of Theorem 4.3.

Corollary 4.1. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. Then (TM,g^φ) Then has constant scalar curvature if and only if the scalar curvature is zero.*

Proposition 4.3. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold of constant sectional curvature λ and (TM,g^φ) its tangent bundle equipped with the φ-Sasaki metri. Si $\widetilde{\sigma}$ denote the scalar curvature of (TM,g^φ), then, we have*

$$\widetilde{\sigma} = (2m-1)\lambda\big(2m - \frac{1}{2}\lambda g(u,\varphi u)\big). \tag{23}$$

Proof. Using the formulas of curvature and scalar curvature of a Riemannian manifold with constant sectional curvature λ we have,

$$R(X,Y)Z = \lambda\big(g(Y,Z)X - g(X,Z)Y\big),$$

for any vector fields X,Y and Z on M, and

$$\sigma = 2m(2m-1)\lambda,$$

hence

$$\sum_{i,j=1}^{2m} g(R(E_i,E_j)u,\varphi R(E_i,E_j)u) = 2(2m-1)\lambda^2 g(u,\varphi u).$$

From Theorem 4.3, we deduce the result.

5. Geodesics of φ-Sasaki metric.

Let (M, g) be a Riemannian manifold and $x : I \to M$ be a curve on M. We define a curve

$$\begin{array}{rcl} C : I & \to & TM \\ t & \mapsto & C(t) = (x(t), y(t)) \end{array}$$

where $y(t) \in T_{x(t)}M$ (i.e. $y(t)$ is a vector field along $x(t)$).

Definition 5.1. [19] *Let (M, g) be a Riemannian manifold, if $x(t)$ is a curve on M. The curve $C(t) = (x(t), \dot{x}(t))$ is called the natural lift of curve $x(t)$.*

Definition 5.2. [19] *Let (M, g) be a Riemannian manifold and ∇ denote the Levi-Civita connection of (M, g). A curve $C(t) = (x(t), y(t))$ is said to be a horizontal lift of the cure $x(t)$ if and only if $\nabla_{\dot{x}} y = 0$.*

Lemma 5.1. [20] *Let (M, g) be a Riemannian manifold and $x : I \to M$ be a curve on M. If $C : t \in I \to C(t) = (x(t), y(t)) \in TM$ is a curve on TM such that $y(t) \in T_{x(t)}M$, then*

$$\dot{C} = {}^H\dot{x} + {}^V(\nabla_{\dot{x}} y) \tag{24}$$

where ∇ denote the Levi-Civita connection of M.

Theorem 5.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, TM its tangent bundle equipped with the φ-Sasaki metric g^φ and $C(t) = (x(t), y(t))$ be a the cure on (TM, g^φ), then*

$$\widetilde{\nabla}_{\dot{C}}\dot{C} \;=\; {}^H\big(\nabla_{\dot{x}}\dot{x} + R(\varphi y, \nabla_{\dot{x}} y)\dot{x}\big) + {}^V(\nabla_{\dot{x}}\nabla_{\dot{x}} y). \tag{25}$$

Proof. Using Theorem 3.1 and Lemma 5.1 we obtain,

$$\begin{aligned}
\widetilde{\nabla}_{\dot{C}}\dot{C} &= \widetilde{\nabla}_{[{}^H\dot{x} + {}^V(\nabla_{\dot{x}} y)]}[{}^H\dot{x} + {}^V(\nabla_{\dot{x}} y)] \\
&= \widetilde{\nabla}_{{}^H\dot{x}}{}^H\dot{x} + \widetilde{\nabla}_{{}^H\dot{x}}{}^V(\nabla_{\dot{x}} y) + \widetilde{\nabla}_{{}^V(\nabla_{\dot{x}} y)}{}^H\dot{x} + \widetilde{\nabla}_{{}^V(\nabla_{\dot{x}} y)}{}^V(\nabla_{\dot{x}} y) \\
&= {}^H(\nabla_{\dot{x}}\dot{x}) - \frac{1}{2}{}^V(R(\dot{x}, \dot{x})y) + {}^V(\nabla_{\dot{x}}\nabla_{\dot{x}} y) + \frac{1}{2}{}^H(R(\varphi y, \nabla_{\dot{x}} y)\dot{x}) \\
&\quad + \frac{1}{2}{}^H(R(\varphi y, \nabla_{\dot{x}} y)\dot{x}) \\
&= {}^H\big(\nabla_{\dot{x}}\dot{x} + R(\varphi y, \nabla_{\dot{x}} y)\dot{x}\big) + {}^V(\nabla_{\dot{x}}\nabla_{\dot{x}} y).
\end{aligned}$$

From the Theorem 5.1 we obtain

Theorem 5.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, TM its tangent bundle equipped with the φ-Sasaki metric g^φ and $C(t) = (x(t), y(t))$ be a curve on TM. Then $C(t)$ is a geodesic on TM if and only if*

$$\left\{\begin{array}{l} \nabla_{\dot{x}}\dot{x} = R(\nabla_{\dot{x}} y, \varphi y)\dot{x} \\ \\ \nabla_{\dot{x}}\nabla_{\dot{x}} y = 0 \end{array}\right. \tag{26}$$

Using Theorem 5.2 we deduce

Corollary 5.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. The natural lift $C(t) = (x(t), \dot{x}(t))$ of any geodesic $x(t)$ on (M^{2m}, φ, g) is a geodesic on (TM, g^φ).*

Corollary 5.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, TM its tangent bundle equipped with the φ-Sasaki metric g^{φ} and $C(t) = (x(t), y(t))$ be the horizontal lift of the curve $x(t)$. Then $C(t)$ is a geodesic on (TM, g^{φ}) if and only if $x(t)$ is a geodesic on (M^{2m}, φ, g).*

Remark 5.1. *If $x(t)$ is a geodesic on (M^{2m}, φ, g) locally we have:*

$$\nabla_{\dot{x}}\dot{x} = 0 \quad \Leftrightarrow \quad \frac{d^2x^h}{dt^2} + \sum_{i,j=1}^{2m} \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^h = 0, \quad h = \overline{1, 2m}.$$

If $C(t) = (x(t), y(t))$ is a horizontal lift of the curve $x(t)$, locally we have:

$$\nabla_{\dot{x}}y = 0 \quad \Leftrightarrow \quad \frac{dy^h}{dt} + \sum_{i,j=1}^{2m} \frac{dx^j}{dt} y^i \Gamma_{ij}^h = 0, \quad h = \overline{1, 2m}.$$

Remark 5.2. *Using the Remark 5.1 we can costruct an infinity of examples of geodesics on (TM, g^{φ}).*

Example 5.1. *Let $(\mathbb{R}^2, \varphi, g)$ be a para-Kähler-Norden manifold such that*

$$g = e^{2x^1}(dx^1)^2 + e^{2x^2}(dx^2)^2$$

and

$$\varphi\frac{\partial}{\partial x^1} = \frac{e^{x^1}}{e^{x^2}}\frac{\partial}{\partial x^2} \quad , \quad \varphi\frac{\partial}{\partial x^2} = \frac{e^{x^2}}{e^{x^1}}\frac{\partial}{\partial x^1}$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = 1.$$

1) *The geodesics $x(t) = (x^1(t), x^2(t))$ such that $x(0) = (a, b) \in \mathbb{R}^2$ and $\dot{x}(0) = (v, w) \in \mathbb{R}^2$ satisfy the system of equations,*

$$\frac{d^2x^h}{dt^2} + \sum_{i,j=1}^{2} \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^h = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{d^2x^1}{dt^2} + (\dfrac{dx^1}{dt})^2 = 0 \\[2ex] \dfrac{d^2x^2}{dt^2} + (\dfrac{dx^2}{dt})^2 = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x^1(t) = a + \ln(1 + vt) \\[1ex] x^2(t) = b + \ln(1 + wt) \end{cases}$$

Hence
$\dot{x}(t) = \dfrac{v}{1+vt}\dfrac{\partial}{\partial x^1} + \dfrac{w}{1+wt}\dfrac{\partial}{\partial x^2}$ *and* $x(t) = (a + \ln(1 + vt), b + \ln(1 + wt))$.
Then from Corollary 5.1, the curve $C(t) = (x(t), \dot{x}(t))$ is a geodesic on $T\mathbb{R}^2$.
2) *If $C(t) = (x(t), y(t))$ is horizontal lift of the curve $x(t)$ i.e. $\nabla_{\dot{x}}y = 0$ then,*

$$\frac{dy^h}{dt} + \sum_{i,j=1}^{2m} \frac{dx^j}{dt} y^i \Gamma_{ij}^h = 0 \Leftrightarrow \begin{cases} \dfrac{dy^1}{dt} + \dfrac{dx^1}{dt}y^1 = 0 \\[2ex] \dfrac{dy^2}{dt} + \dfrac{dx^2}{dt}y^2 = 0 \end{cases} \Leftrightarrow \begin{cases} y^1(t) = \dfrac{k_1}{1+vt} \\[2ex] y^2(t) = \dfrac{k_2}{1+wt} \end{cases}$$

Hence $y(t) = \dfrac{k_1}{1+vt}\dfrac{\partial}{\partial x^1} + \dfrac{k_2}{1+wt}\dfrac{\partial}{\partial x^2}$, *where* $k_1, k_2 \in \mathbb{R}$.
From Corollary 5.2, the curve $C(t) = (x(t), y(t))$ is a geodesic on $T\mathbb{R}^2$.

6. φ-unit tangent bundle $T_1^\varphi M$ with φ-Sasaki metric

The φ-tangent sphere bundle of radius $r > 0$ over a para-Kähler-Norden manifold (M^{2m}, φ, g), is the hypersurface

$$T_r^\varphi M = \{(x, u) \in TM,\ g(u, \varphi u) = r^2\}.$$

When $r = 1$, $T_1^\varphi M$ is called the φ-unit tangent (sphere) bundle.

$$T_1^\varphi M = \{(x, u) \in TM,\ g(u, \varphi u) = 1\}. \tag{27}$$

If we set

$$\begin{array}{rcl} F : TM & \to & \mathbb{R} \\ (x, u) & \mapsto & F(x, u) = g(u, \varphi u) - 1, \end{array}$$

then the hypersurface $T_1^\varphi M$ is given by

$$T_1^\varphi M = \{(x, u) \in TM,\ F(x, u) = 0\},$$

and $grad_{g^\varphi} F$ (the gradient of F with respect to g^φ) is a normal vector field to $T_1^\varphi M$. From the Lemma 3.1, we have

$$g^\varphi({}^H X, grad_{g^\varphi} F) = {}^H X(F) = {}^H X(g(u, \varphi u) - 1) = 0,$$
$$g^\varphi({}^V X, grad_{g^\varphi} F) = {}^V X(F) = {}^V X(g(u, \varphi u) - 1) = 2g(X, \varphi u) = 2g^\varphi({}^V X, {}^V U),$$

So

$$grad_{g^\varphi} F = 2\,{}^V U.$$

Then the unit normal vector field to $T_1^\varphi M$ is given by

$$\mathcal{N} = \frac{grad_{g^\varphi} F}{\sqrt{g^\varphi(grad_{g^\varphi} F, grad_{g^\varphi} F)}} = \frac{{}^V U}{\sqrt{g^\varphi({}^V U, {}^V U)}} = {}^V U.$$

The tangential lift ${}^T X$ with respect to g^φ of a vector $X \in T_x M$ to $(x, u) \in T_1^\varphi M$ as the tangential projection of the vertical lift of X to (x, u) with respect to $\mathcal{N}$, that is

$${}^T X = {}^V X - g^\varphi_{(x,u)}({}^V X, \mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} = {}^V X - g_x(X, \varphi u){}^V U.$$

For the sake of notational clarity, we will put $\bar{X} = X - g(X, \varphi u)U$, then ${}^T X = {}^V \bar{X}$.

From the above, we get the direct sum decomposition

$$\begin{array}{rcl} T_{(x,u)}TM & = & T_{(x,u)}T_1^\varphi M \oplus span\{\mathcal{N}_{(x,u)}\} \\ & = & T_{(x,u)}T_1^\varphi M \oplus span\{{}^V U_{(x,u)}\}. \end{array} \tag{28}$$

where $(x, u) \in T_1^\varphi M$.

Indeed, if $W \in T_{(x,u)}TM$, then they exist $X, Y \in T_x M$, such that

$$\begin{array}{rcl} W & = & {}^H X + {}^V Y \\ & = & {}^H X + {}^T Y + g^\varphi_{(x,u)}({}^V Y, \mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} \\ & = & {}^H X + {}^T Y + g_x(Y, u){}^V U_{(x,u)}. \end{array}$$

Then, we can say that the tangent space $T_{(x,u)}T_1^\varphi M$ of $T_1^\varphi M$ at (x, u) is given by

$$T_{(x,u)}T_1^\varphi M = \{{}^H X + {}^T Y \,/\, X \in T_x M, Y \in u^\perp \subset T_x M\}.$$

where $u^{\perp} = \{Y \in T_xM,\, g(Y,\varphi u) = 0\}$. Hence $T_{(x,u)}T_1^{\varphi}M$ is spanned by vectors of the form ${}^{H}X$ and ${}^{T}Y$.

Given a vector field X on M, the tangential lift ${}^{T}X$ of X is given by

$$(29)\qquad {}^{T}X_{(x,u)} = \left({}^{V}X - g^{\varphi}({}^{V}X,\mathcal{N})\mathcal{N}\right)_{(x,u)} = {}^{V}X_{(x,u)} - g_x(X_x,\varphi u)^{V}U_{(x,u)}.$$

For any vector field X on M, we have the followings

(1) $g^{\varphi}({}^{H}X,\mathcal{N}) = 0$,
(2) $g^{\varphi}({}^{T}X,\mathcal{N}) = 0$,
(3) ${}^{T}X = {}^{V}X$ if and only if $g(X,\varphi u) = 0$,
(4) ${}^{T}U = 0$.

Definition 6.1. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and (TM,g^{φ}) its tangent bundle equipped with the φ-Sasaki metric. The Riemannian metric $\hat{g}^{\varphi}$ on $T_1^{\varphi}M$, induced by g^{φ}, is completely determined by the identities*

$$\begin{aligned}
\hat{g}^{\varphi}({}^{H}X,{}^{H}Y) &= g(X,Y),\\
\hat{g}^{\varphi}({}^{T}X,{}^{H}Y) &= \hat{g}^{\varphi}({}^{H}X,{}^{T}Y) = 0,\\
\hat{g}^{\varphi}({}^{T}X,{}^{T}Y) &= g(\bar{X},\varphi\bar{Y}),
\end{aligned}$$

For any vector fields X and Y on M and $\bar{X} = X - g(X,\varphi u)U$.

We shall calculate the Levi-Civita connection $\widehat{\nabla}$ of $T_1^{\varphi}M$ with φ-Sasaki metric $\hat{g}^{\varphi}$. This connection is characterized by the formula:

$$(30)\qquad \widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widetilde{\nabla}_{\widehat{X}}\widehat{Y} - g^{\varphi}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y},\mathcal{N})\mathcal{N}.$$

for all vector fields $\widehat{X}$ and $\widehat{Y}$ on $T_1^{\varphi}M$.

Theorem 6.1. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric $\hat{g}^{\varphi}$, then we have the following formulas.*

$$\begin{aligned}
1.\ \widehat{\nabla}_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_XY) - \frac{1}{2}{}^{T}(R(X,Y)u),\\
2.\ \widehat{\nabla}_{{}^{H}X}{}^{T}Y &= {}^{T}(\nabla_XY) + \frac{1}{2}{}^{H}(R(\varphi u,Y)X),\\
3.\ \widehat{\nabla}_{{}^{T}X}{}^{H}Y &= \frac{1}{2}{}^{H}(R(\varphi u,X)Y),\\
4.\ \widehat{\nabla}_{{}^{T}X}{}^{T}Y &= -g(Y,\varphi u)^{T}X,
\end{aligned}$$

for all vector fields X and Y on M, where ∇ is the Levi-Civita connection and R is its curvature tensor of (M^{2m},φ,g).

Proof. In the proof, we will use the Theorem 3.1, Lemma 3.3 and the formula (30).

1. By direct calculation, we have

$$\begin{aligned}
\widehat{\nabla}_{{}^{H}X}{}^{H}Y &= \widetilde{\nabla}_{{}^{H}X}{}^{H}Y - g^{\varphi}(\widetilde{\nabla}_{{}^{H}X}{}^{H}Y,\mathcal{N})\mathcal{N}\\
&= {}^{H}(\nabla_XY) - \frac{1}{2}{}^{V}(R(X,Y)u) + \frac{1}{2}g^{\varphi}({}^{V}(R(X,Y)u),\mathcal{N})\mathcal{N}\\
&= {}^{H}(\nabla_XY) - \frac{1}{2}{}^{T}(R(X,Y)u).
\end{aligned}$$

2. We have $\widehat{\nabla}_{^HX}{}^TY = \widetilde{\nabla}_{^HX}{}^TY - g^{\varphi}(\widetilde{\nabla}_{^HX}{}^TY, \mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\widetilde{\nabla}_{^HX}{}^TY = \frac{1}{2}{}^H(R(\varphi u, Y)X) + {}^T(\nabla_X Y) \ and \ \ g^{\varphi}(\widetilde{\nabla}_{^HX}{}^TY, \mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{^HX}{}^TY \ = \ \frac{1}{2}{}^H(R(\varphi u, Y)X) + {}^T(\nabla_X Y).$$

3. Also, we have $\widehat{\nabla}_{^TX}{}^HY = \widetilde{\nabla}_{^TX}{}^HY - g^{\varphi}(\widetilde{\nabla}_{^TX}{}^HY, \mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\widetilde{\nabla}_{^TX}{}^HY = \frac{1}{2}{}^H(R(\varphi u, Y)X) \ \ and \ g^{\varphi}(\widetilde{\nabla}_{^HX}{}^TY, \mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{^HX}{}^TY \ = \ \frac{1}{2}{}^H(R(\varphi u, Y)X)\,.$$

4. In the same way above, we have $\widehat{\nabla}_{^TX}{}^TY = \widetilde{\nabla}_{^TX}{}^TY - g^{\varphi}(\widetilde{\nabla}_{^TX}{}^TY, \mathcal{N})\mathcal{N}$,

$$\widetilde{\nabla}_{^TX}{}^TY \ = \ -g(Y, \varphi u)^TX - g(\bar{X}, \varphi\bar{Y})^VU.$$

and

$$g^{\varphi}(\widetilde{\nabla}_{^TX}{}^TY, \mathcal{N})\mathcal{N} \ = \ -g(\bar{X}, \varphi\bar{Y})^VU.$$

Hence

$$\widehat{\nabla}_{^TX}{}^TY \ = \ -g(Y, \varphi u)^TX.$$

We shall calculate the Riemannian curvature tensor of $T_1^{\varphi}M$ with the φ-Sasaki metric $\hat{g}^{\varphi}$.

Denoting by $\widehat{R}$ the Riemannian curvature tensors of $(T_1^{\varphi}M, \hat{g}^{\varphi})$, from the Gauss equation for hypersurfaces we deduce that $\widehat{R}(\widehat{X}, \widehat{Y})\widehat{Z}$ satisfies

(31) $\widehat{R}(\widehat{X}, \widehat{Y})\widehat{Z} = {}^t(\widetilde{R}(\widehat{X}, \widehat{Y})\widehat{Z}) - B(\widehat{X}, \widehat{Z}).A_{\mathcal{N}}\widehat{Y} + B(\widehat{Y}, \widehat{Z}).A_{\mathcal{N}}\widehat{X},$

for all $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ vector fields on $T_1^{\varphi}M$. where ${}^t(\widetilde{R}(\widehat{X}, \widehat{Y})\widehat{Z})$ is the tangential component of $\widetilde{R}(\widehat{X}, \widehat{Y})\widehat{Z}$ with respect to the direct sum decomposition (28), $A_{\mathcal{N}}$ is the shape operator of $T_1^{\varphi}M$ in (TM, g^{φ}) derived from $\mathcal{N}$, and B is the second fundamental form of $T_1^{\varphi}M$ (as a hypersurface immersed in TM), associated to $\mathcal{N}$ on $T_1^{\varphi}M$.

$A_{\mathcal{N}}\widehat{X}$ is the tangential component of $(-\widetilde{\nabla}_{\widehat{X}}\mathcal{N})$ i.e.

$$A_{\mathcal{N}}\widehat{X} = -{}^t(\widetilde{\nabla}_{\widehat{X}}\mathcal{N}),$$

$B(\widehat{X}, \widehat{Y})$ is given by Gauss's formula, $\widetilde{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla}_{\widehat{X}}\widehat{Y} + B(\widehat{X}, \widehat{Y}).\mathcal{N}$, so

$$B(\widehat{X}, \widehat{Y}) = g^{\varphi}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y}, \mathcal{N}).$$

Theorem 6.2. *Let* (M^{2m},φ,g) *be a para-Kähler-Norden manifold and* $T_1^{\varphi}M$ *its* φ*-unit tangent bundle equipped with the* φ*-Sasaki metric* $\hat{g}^{\varphi}$*, then we have the following formulas.*

$$
\begin{aligned}
(1)\ \ \widehat{R}({}^HX,{}^HY){}^HZ &= {}^H(R(X,Y)Z)+\frac{1}{2}{}^H(R(\varphi u,R(X,Y)u)Z)\\
&\quad +\frac{1}{4}{}^H(R(\varphi u,R(X,Z)u)Y)-\frac{1}{4}{}^H(R(\varphi u,R(Y,Z)u)X)\\
&\quad +\frac{1}{2}{}^T((\nabla_Z R)(X,Y)u),\\
(2)\ \ \widehat{R}({}^HX,{}^TY){}^TZ &= -\frac{1}{2}{}^H(R(\varphi\bar{Y},\bar{Z})X)-\frac{1}{4}{}^H(R(u,Y)R(u,Z)X),\\
(3)\ \ \widehat{R}({}^TX,{}^TY){}^HZ &= {}^H(R(\varphi\bar{X},\bar{Y})Z)+\frac{1}{4}{}^H(R(u,X)R(u,Y)Z)\\
&\quad -\frac{1}{4}{}^H(R(u,Y)R(u,X)Z),\\
(4)\ \ \widehat{R}({}^HX,{}^TY){}^HZ &= \frac{1}{2}{}^H((\nabla_X R)(\varphi u,Y)Z)+\frac{1}{2}{}^T(R(X,Z)\bar{Y})\\
&\quad +\frac{1}{4}{}^T(R(R(\varphi u,Y)Z,X)u),\\
(5)\ \ \widehat{R}({}^HX,{}^HY){}^TZ &= \frac{1}{2}{}^H((\nabla_X R)(\varphi u,Z)Y)-\frac{1}{2}{}^H(\nabla_Y R)(\varphi u,Z)X)\\
&\quad +\frac{1}{4}{}^T(R(R(\varphi u,Z)Y,X)u)-\frac{1}{4}{}^T(R(R(\varphi u,Z)X,Y)u)\\
&\quad +{}^T(R(X,Y)\bar{Z}),\\
(6)\ \ \widehat{R}({}^TX,{}^TY){}^TZ &= g(\bar{Y},\varphi\bar{Z}){}^TX-g(\bar{X},\varphi\bar{Z}){}^TY,
\end{aligned}
$$

for all vector fields X,Y *and* Z *on* M*, where* $\bar{X}=X-g(X,\varphi u)u$*.*

Proof. Using Theorem 3.1 and Lemma 5.1, we obtain

$$A_{\mathcal{N}}{}^HX=0\ ,\quad A_{\mathcal{N}}{}^TX=-{}^TX, \tag{32}$$

$$B({}^HX,{}^HY)=B({}^HX,{}^TY)=B({}^TX,{}^HY)=0, \tag{33}$$

$$B({}^TX,{}^TY)\ \ =\ \ -g(\bar{X},\varphi\bar{Y}). \tag{34}$$

It now suffices to using Theorem 4.1 and (31)-(34), we obtain the required formulae for the curvature tensor (see [2]).

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