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ON MÖBIUS TRANSFORMATIONS AND POLYGONS CIRCUMSCRIBING CONICS

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Abstract. In this article we study the relations of some particular Möbius transformations related to vertices of polygons circumscribing a conic. We explore their geometric properties and apply them to prove the theorems of Siebeck-Marden, Siebeck-Linfield. We show also some necessary and sufficient conditions for three Möbius transformations which allow to define any conic inscribed in a triangle.

1. Introduction

The aim of this article is to study by elementary geometric means a particular kind of Möbius transformations related to a general polygon circumscribing a conic. There are two core facts lying on the foundations of this study. The first is the following general property of conics formulated as a theorem ([9, p.4]) (see Figure 1). Chasles in his magnificent treatise, cited above, noticed its power and derived from it all possible projective and metrical properties of conics.

Theorem 1.1. *The intersection points $\{X, Y\}$ of two fixed tangents $\{\varepsilon, \varepsilon'\}$ of a conic κ with a variable third tangent of the same are related by a homography between the two lines. This is equivalent with the property of four fixed tangents to intersect a variable fifth in four points whose cross ratio is constant.*

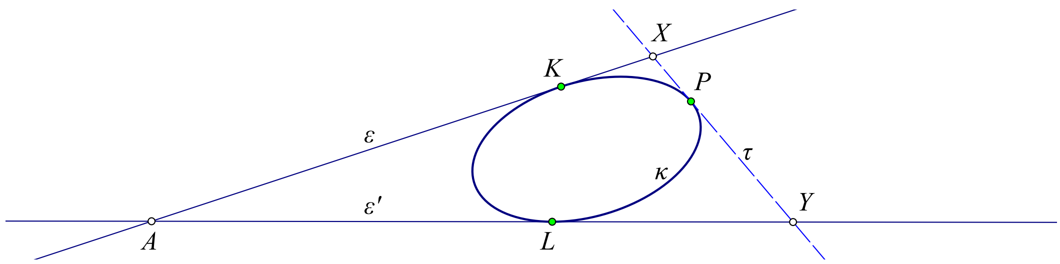

Figure 1. Homography $h : X \mapsto Y$ between the tangent lines $\{\varepsilon, \varepsilon'\}$

Representing the points $\{X, Y\}$ through *"projective coordinates"* $\{x, y\}$ ([3, p.45], [23]) of the corresponding lines $\{\varepsilon, \varepsilon'\}$, a general homography between these lines is expressed ([22, I, p.122], [23]) through a relation of the form

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$$(1) \qquad y = \frac{ax+b}{cx+d} \quad \text{with constants} \quad ad-bc \neq 0 ,$$

and is completely determined by prescribing the values $\{y_1, y_2, y_3 \in \varepsilon'\}$ at three arbitrary points $\{x_1, x_2, x_3 \in \varepsilon\}$ ([29, I, p.59]). In our case, in order to define the above homography, it suffices to consider the intersections $\{X_i(x_i), Y(y_i), i = 1, 2, 3\}$ of $\{\varepsilon, \varepsilon'\}$ with three particular tangents of the conic.

Remark 1.1. *When the contact point P of the variable tangent comes in coincidence with the contact point K of the fixed tangent ε, then also X becomes coincident with K and Y with A. Analogously, when P comes in coincidence with L, then X becomes coincident with A and Y coincident with point L.*

Thus, the above homography maps $\{K \xmapsto{h} A \xmapsto{h} L\}$ and, in order to completely describe h, we need one only additional pair of points $X \xmapsto{h} Y$ defined by some particular tangent τ of the conic (see Figure 1). By the very definition of h, it follows that different choices of this particular additional tangent will lead to the same transformation.

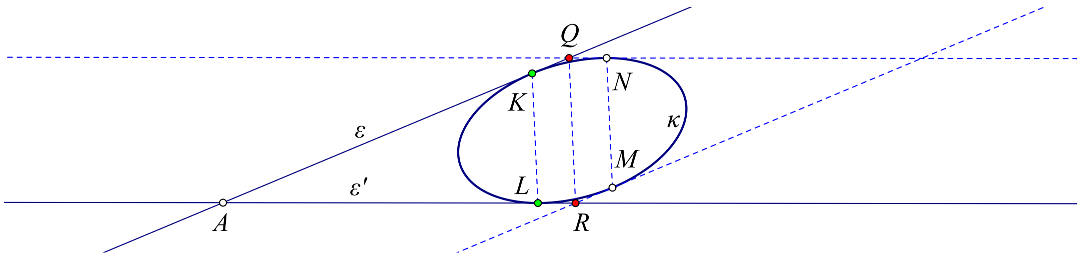

FIGURE 2. The limit points, Q sent to infinity, R image of infinity

Figure 2 suggests a fundamental property of the transformation h. Point Q is the intersection of line ε with the tangent parallel to ε', consequently Q maps via h to the point at infinity of the line ε'. Analogously the intersection point R of ε' with the tangent parallel to ε is the image point R of the point at infinity of line ε. The points $\{Q, R\}$ are traditionally called *"limit points"* of the homography and in its representation by equation (1) their coordinates $\{q, r\}$ are given by

$$(2) \qquad q = -\frac{d}{c} \quad \text{and} \quad r = \frac{a}{c} .$$

It is also readily seen that the lines $\{QR, KL, MN\}$ are parallel, the last two equidistant from the first. The existence of parallel tangents presupposes that the conic is a *central* one. In fact, most of the time we'll deal with this kind of conics. The somewhat simpler case of parabolas is handled in separate sections (8, 12) below.

The second fact lying on the foundations of this discussion is a theorem by Siebeck ([28], [12, p.154], theorem 5.1) dealing with properties of *"Möbius transformations"* which we analyze in the next section.

In this article, identifying the Euclidean plane with the set of complex numbers, we extend the (real) homography h to a (complex) Möbius transformation ([12, p.126], [27, p.41]) and study the relations of this extension to the conic κ. We deal mainly with the question of the fixed points of this transformation, which Siebeck's theorem identifies with the focal points of κ. Möbius transformations are described by the same formula (1) with the only difference that all the entities appearing there are now complex numbers. Their *fixed points* result by solving a (complex) quadratic equation

$$(3) \qquad z = \frac{az+b}{cz+d} \quad \Leftrightarrow \quad cz^2 + (d-a)z - b = 0 ,$$

which, according to Siebeck's theorem, offers a convenient way to locate the focal points of a conic inscribed in a polygon.

In the first part of the article we review some elementary properties of conics needed for a geometric synthetic proof of this theorem, which in the original article [28] by Siebeck is given using the representation of conics with complex numbers and in [12, p.126] is given using the definition of focal points as intersections of tangents to the conic from the cyclic points at infinity ([10, p.161], [8, p.311]).

In section 2 we discuss in short the way the homography h extends to a complex Möbius transformation and how, conversely, each Möbius transformation restricted on certain lines defines correspondingly (real) line homographies.

In section 3 we review some well known properties of conics needed subsequently. In section 4 we analyze two examples of a special kind of Möbius transformation, the *"Möbius involution"*, which in our context represent the building blocks of our extended transformation h. In section 5 we show that h is indeed a composition of Möbius involutions of the kinds studied in the preceding section and give a proof of Siebeck's theorem. In section 6 we examine the composition of all the analogous transformations $\{h_A, h_B, \ldots\}$ for the successive vertices of a polygon $p = ABC \ldots Z$ circumscribing the conic κ. In section 7 we discuss the properties of the *characteristic constant* or *"invariant"* of h_A and a related way to distinguish the kind of a central conic κ inscribed in a polygon p. In section 8 we discuss the case of polygons circumscribing a parabola, for which the Möbius transformations h_A turn out to be a *similarities.* In section 9 we discuss the case of conics inscribed in triangles and show, how the theorems of Siebeck-Marden and Siebeck-Linfield are intimately related to transformations of the type h_A and result from the properties of the latter. In section 10 we discuss the case of trapezia to which reduce also some particular kinds of polygons. Finally, in sections 11 and 12 we discuss the converse procedure of definition of a triangle and a particular inscribed in it conic from certain triples of Möbius transformations.

2. From line homographies to Möbius transformations

First, we notice the coincidence of the cross ratio $(AB;CD) = \frac{a-c}{b-c} : \frac{a-d}{b-d}$ of four points *on a line* ε using for its definition two different parameterizations of its points. In the first $\{a, b, c, d\}$ denote line coordinates on ε and in the second definition the same symbols denote complex numbers. That the two definitions of the cross ratio lead to the same result is immediately seen in the case ε coincides with the real axis of the complex plane. For a general line ε of the plane, we can use a similarity and map ε to the real axis. Since similarities preserve the cross ratio in both cases, its independence from the kind of parameterization is proved.

Also the extension of a line homography h to a Möbius transformation is easily understood if we consider the preservation of cross ratios of four collinear points by both, h and its extension h' ([27, p.47]). Three points $\{X_i\}$ on line ε and their images $\{Y_i\}$ on line ε' determine a unique homography $h : \varepsilon \to \varepsilon'$ and also a unique Möbius transformation $h' : \mathcal{C} \to \mathcal{C}$, $W = h'(Z)$, through the equality of cross ratios

$$(Y_1Y_2; Y_3W) = (X_1X_2; X_3Z) . \tag{4}$$

Here $\mathcal{C}$ denotes the extension ([27, p.15]) of the complex plane $\mathbb{C}$ to which we have added the *"point at infinity"* ∞. The same formal equation defines the homography h using projective line coordinates $\{X_i(x_i), Y_i(y_i), Z(z), W(w)\}$ in the case the points lie on the two lines $\{\varepsilon, \varepsilon'\}$ and the Möbius transformation h', the symbols now denoting complex numbers. In both cases the transformation is defined by solving equation (4) w.r.t. $w = h(z)$ respectively $W = h'(Z)$, latter defined for all points of $\mathcal{C}$ and reducing to h for points on ε. In the sequel we'll use the same symbol for both the homography between the lines $\{\varepsilon, \varepsilon'\}$ and its extension in $\mathcal{C}$. The precise meaning in each case will be evident from the context.

It is easily seen also a converse property ([27, p.55]), namely that every Möbius transformation defines, by its restriction on certain lines, line homographies. Figure 3 shows how this is done. The generic Möbius transformation h is defined by corresponding to the vertices of a triangle UVW the vertices of another triangle $U'V'W'$, the triangles being genuine or degenerate with distinct vertices. There is then defined a *"characteristic parallelogram"* ([27, p.69]) having pairs of opposite vertices $\{(S,S'),(Q,R)\}$ respectively the fixed points and the limit points. Every line ε passing through the point Q sent to infinity by h, maps to a line ε' through the other limit point R. The restriction of h on ε defines then a line homography $h : \varepsilon \ni X \to Y \in \varepsilon'$ between the lines $\{\varepsilon, \varepsilon'\}$. There are again two points $\{K \in \varepsilon, L \in \varepsilon'\}$, the homography mapping $\{K \xmapsto{h} A \xmapsto{h} L\}$. Again the line homography and its extension, coinciding with the initial Möbius transformation, are completely determined by these three points and two additional corresponding points $\{X, Y = h(X).\}$ Also fixing $\{\varepsilon, \varepsilon'\}$, the lines $\tau = XY$ joining corresponding points enve-

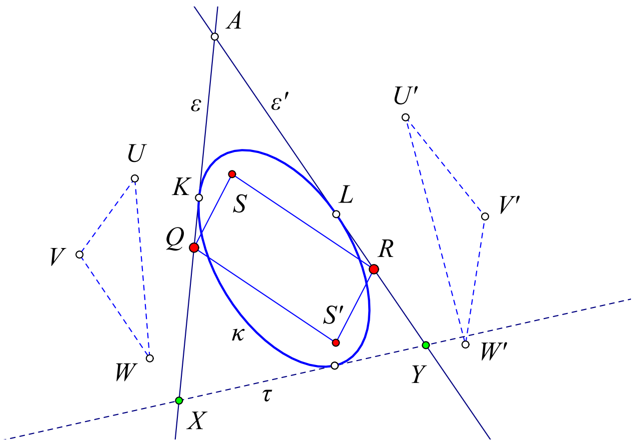

FIGURE 3. Producing conversely, line homographies from Möbius transformations

lope, according to the Chasles-Steiner theorem ([9, p.6]), a conic κ creating a configuration like the one standing in the focus of our discussion. Siebeck's theorem guarantees that the fixed points of h coincide with the focal points of κ. Next three sections proceed to a review of some elementary properties of conics and an elementary proof of this theorem.

3. On Newton's theorem and isogonal properties of conics

One of Newton's theorems for conics deals with products of segments intercepted on conics by lines through a point. For the convenience of reference I formulate the well known property as a theorem ([8, p.168], [19, I,p.371]) (see Figure 4).

Theorem 3.1. *Two lines $\{\alpha, \beta\}$ through the point P intersect the central conic κ respectively at the points $\{(A,B),(C,D)\}$. Then, it is*

$$\frac{PA \cdot PB}{PC \cdot PD} = \frac{(A'B')^2}{(C'D')^2},$$

where $\{A'B', C'D'\}$ are the diameters of κ parallel respectively to $\{\alpha, \beta\}$.

The theorem has several interesting consequences, since the above ratio remains constant if we vary the location of P but maintain the directions of the lines $\{\alpha, \beta\}$. In particular, the ratio of the tangents from a point (see Figure 4), a case in which $A = B$

and $C = D$, is equal to the ratio of the corresponding parallel diameters

$$\frac{PA}{PC} = \frac{A'B'}{C'D'} \ . \tag{5}$$

The ratio of the theorem on the left remains constant also in the case of parabolas, but

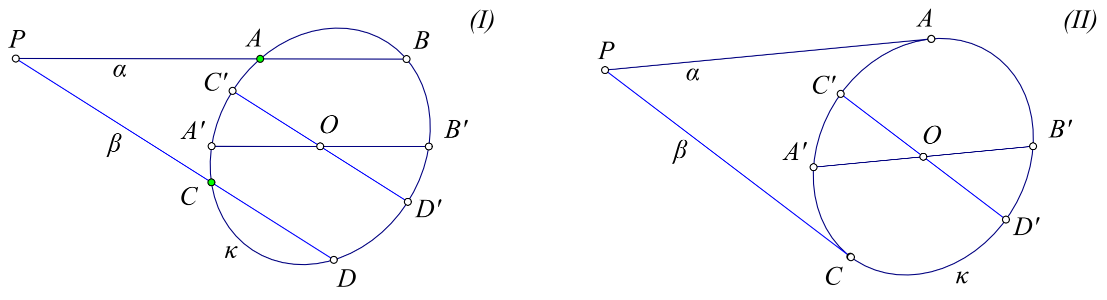

FIGURE 4. Newton's theorem The case of tangents

in this case we have not an expression like the right side. Another consequence, of use below, often formulated as an exercise ([19, I, p.264]), to which I supply a short proof, is the following (see Figure 5):

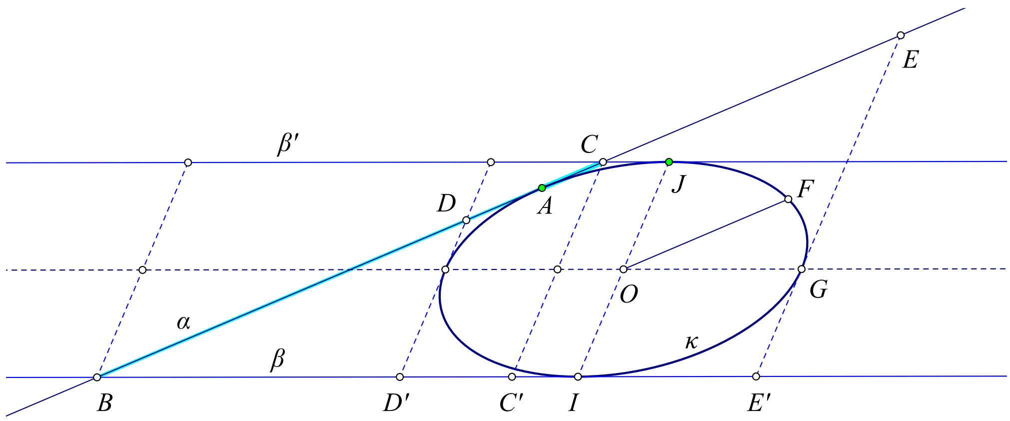

FIGURE 5. Segments on a tangent between two parallel tangents

Theorem 3.2. *$\{\beta, \beta'\}$ are two parallel tangents of the central conic κ at the points $\{I, J \in \kappa\}$. The variable tangent α at the point A meets $\{\beta, \beta'\}$ correspondingly at the points $\{B, C\}$. Then it is*

$$BA \cdot AC \;=\; OF^2 \quad \text{and} \quad BI \cdot CJ \;=\; OG^2 \ ,$$

where OF is the semi-diameter of κ parallel to line α and OG is the semi-diameter parallel to the tangents $\{\beta, \beta'\}$.

Proof. From equation (5) we have the relations

$$\frac{BA}{BI} = \frac{OF}{OG} = \frac{AC}{CJ} \ \Rightarrow\ \frac{BA}{BI} \cdot \frac{AC}{CJ} = \frac{(OF)^2}{(OG)^2} \ \Rightarrow\ BA \cdot AC = \frac{(OF)^2}{(OG)^2} \cdot BI \cdot CJ \ .$$

Thus, it suffices to show that $BI \cdot CJ = OG^2$. If we project the points of α parallel to IJ onto line β, the relation becomes $BI \cdot C'I = IE'^2$. This is characteristic of a harmonic quadruple $(BC'; D'E') = -1$, which is true and proves the claim. The reason for latter relation is a consequence of theorem 1.1, guaranteeing that the cross ratio $(BC; DE)$

intercepted on the variable tangent α by the four tangents $\{\beta, \beta'\}$ and their conjugate tangents $\{DD', EE'\}$, is a constant independent of the location of the tangent α. In our case letting α take a special position, e.g. letting it coincide with β', we see that this constant cross ratio is -1. Since the parallel projection along IJ to points of line β preserves the cross ratio, we have the desired proof.

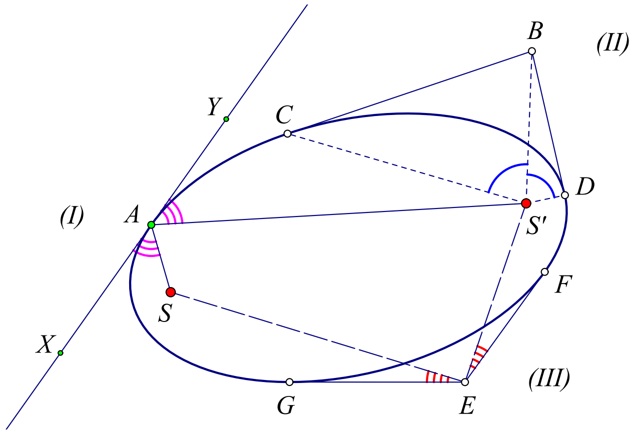

FIGURE 6. Three isogonal properties of central conics

Figure 6, in its three parts $(I), (II), (III)$, shows correspondingly three *"isogonal properties of central conics"* ([1, pp.10,12], [15, pp. 42,43]), which will be of use below:

(I) *The tangent at a point* A *is equal inclined to the focal radii* $\{AS, AS'\}$: $\widehat{XAS} = \widehat{S'AY}$.

(II) *The tangents from a point* B *are seen from a focus under equal angles* $\widehat{CS'B} = \widehat{BS'D}$.

(III) *The tangents from a point* E *are equal inclined to the focal radii* $\widehat{GES} = \widehat{S'EF}$.

Notice that (II), in the case of hyperbolas, must be considered under the aspect of *"directed angles"* ([16, p.11]), by which two angles are considered equal also if they differ by $180°$. The properties remain true also in the case of parabolas with somewhat modified wording due to the fact that one of the focal points, S' say, is at infinity. Then, focal rays, such as AS' must be replaced with lines through A parallel to the axis of the parabola.

Lemma 3.1. *Assume that the tangent at point* A *of a central conic* κ *with focal points* $\{S, S'\}$ *intersects two other parallel tangents* $\{\beta, \beta'\}$ *at the points* $\{B, C\}$. *Then triangles* $\{ABS, AS'C\}$ *are similar (see Figure 7).*

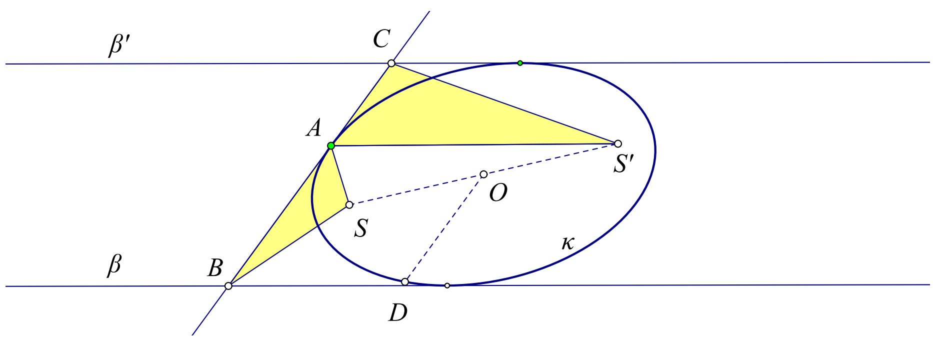

FIGURE 7. Two similar triangles defined by two parallel tangents

Proof. The isogonal property (I) implies that the angles at A are equal $\widehat{BAS} = \widehat{S'AC}$. It is also well known that the product of the focal distances $|AS||AS'| = OD^2$, latter being the half diameter parallel to the tangent ([19, I,p.257]). From theorem 3.2 we have

$BA \cdot AC = OD^2$ too, which together with the preceding relation produces the equality of ratios $\frac{BA}{AS} = \frac{S'A}{AC}$. Hence the triangles are similar as claimed.

The preceding discussion supports next lemma ([25], [14, p.77]), which is essential in our proof of Siebeck's theorem. For the convenience of the reader I reproduce here the short proof by Rouse.

Lemma 3.2. *The central conic κ with focal points $\{S, S'\}$ touches the sides of the triangle ABC at the points $\{A' \in BC, \quad B' \in CA, \quad C' \in AB\}$. And line $\{EF, \quad E \in AB, \ F \in AC\}$ is tangent and parallel to BC. Then the triangles $\{ACS, AS'E\}$ are similar, the triangles $\{ABS', ASF\}$ are also similar, and the products are equal*

$$AC \cdot AE = AB \cdot AF = AS \cdot AS' . \tag{6}$$

As a consequence, if we fix the tangents $\{AC, AB\}$ and vary the direction of the parallel tangents $\{BC, EF\}$ these products remain constant (see Figure 8).

Proof. The isogonal property (III) implies that the angles of the triangles at A are equal. From lemma 3.1 $\widehat{S'EC'} = \widehat{C'SB}$. From the isogonal property (II) we know also that

$$\widehat{C'SB} = \frac{1}{2}\widehat{C'SA'} = \frac{1}{2}\widehat{B'SC'} - \frac{1}{2}\widehat{B'SA'} = \widehat{B'SA} - \widehat{B'SC} = \widehat{CSA},$$

and the triangles, having two angles respectively equal are similar. Analogously is proved

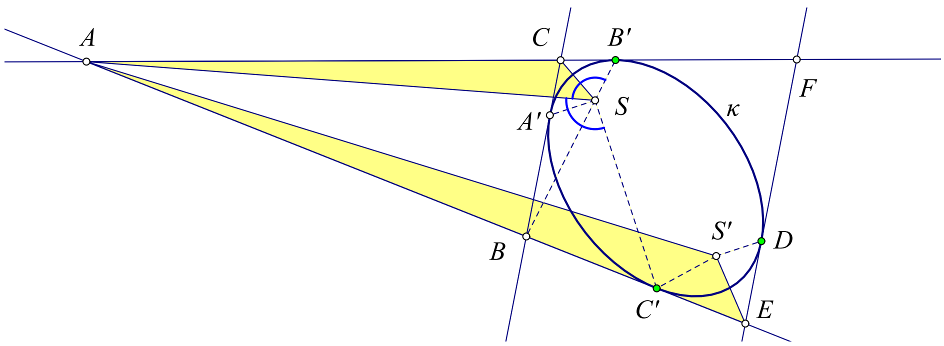

FIGURE 8. The product $AC \cdot AE$ is independent of the direction of the parallels

the similarity of the triangles $\{ABS', ASF\}$. The relation of the products results directly from these similarities.

4. Two Möbius involutions

Möbius involutions ([12, p.158], [27, p.49]) are characterized by their property $h \circ h = e \Leftrightarrow h^{-1} = h$, equivalent to $a + d = 0$ in the analytic description through equation (1). To completely define a Möbius involution it suffices to give two related pairs $(A, A' = h(A))$ and $(B, B' = h(B))$, therefore in this case we see the often used notation $h = (AA', BB')$. One or both of these points may be fixed points of the involution, the symbol being then correspondingly (AA, BB') or (AA, BB).

In any case, given two pairs of related points $\{(A, A'), (B, B')\}$, fixed or not, it is easily seen, that, identifying the symbols with corresponding complex numbers, the involution is described by formula ([27, p.52])

$$w = \frac{(AA' - BB')z + BB'(A + A') - AA'(B + B')}{(A + A' - B - B')z + BB' - AA'} . \tag{7}$$

In case we are given the two fixed points $\{A, B\}$, the corresponding Möbius involution $h = (AA, BB)$ is defined geometrically using the fourth vertex $Y = h(X)$ of the corresponding *harmonic* quadrangle ([8, p.206], [16, p.100,p.306]) $XAYB$, which is completely determined by the three points $\{A, B, X\}$ (see Figure 9). Having these three points, the fourth vertex Y of the corresponding harmonic quadrangle is determined as intersection of the circumcircle of triangle AXB with the *symmedian* from X.

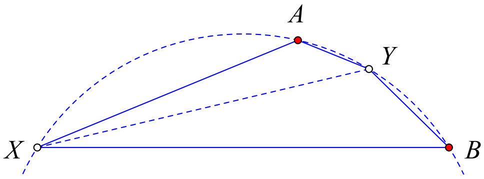

FIGURE 9. Determination of the involution $X \overset{h}{\longmapsto} Y$ from its fixed points $\{A, B\}$

We'll see that our transformation h, defined in section 1, is a product (i.e. composition) of two special Möbius involutions. Their properties are formulated in [12, Ex.125, p.202] without a proof. Here I state them in the form of the subsequent two lemmata, supply their proofs and draw some consequences.

Lemma 4.1. *Lines $\{\varepsilon = AD, \varepsilon' = AE\}$ are two fixed tangents at points $\{D, E\}$ of the central conic κ with focal points $\{S, S'\}$ (see Figure 10). Then, the Möbius involution $h = (AA, DE)$ fixing A and interchanging $\{D, E\}$ interchanges also the focal points $\{S, S'\}$.*

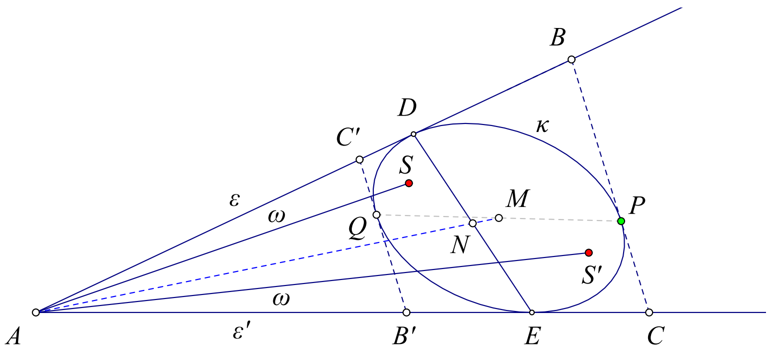

FIGURE 10. Studying the involution fixing A and interchanging $\{D, E\}$

Proof. We represent the points of the plane by complexes $\{A(a), B(b), C(c), \ldots\}$ assuming also that point A is the origin. The Möbius involution $w = h(z) = (pz + q)/(rz - p)$ fixing $A(0)$ has $q = 0$ and the constants $\{p, r\}$ result trivially from the requirement

$$h(D) = E \quad \Rightarrow \quad w = \frac{(de)z}{(d+e)z - (de)} .$$

Should h fix A and interchange $\{S, S'\}$ too, then we could represent it also formally in the same way

$$w = \frac{(ss')z}{(s+s')z - ss'} ,$$

and the two representations should coincide for every z. Equating the two expressions and simplifying we obtain the equation

$$de(s + s') = ss'(d + e) \quad \Leftrightarrow \quad de \cdot m = ss' \cdot n ,$$

where $\{M(m), N(n)\}$ are respectively the middles of the segments $\{SS', DE\}$. Last equation reduces to an equation of the distances of the corresponding points. To see this, write the complexes in polar form assuming AE coincident with the real axis

$$d = |AD|e^{i\theta} \quad , \quad s' = |AS'|e^{i\omega} \quad , \quad s = |AS|e^{i(\theta-\omega)} \quad , \quad m = |AM|e^{i\phi} \quad , \quad n = |AN|e^{i\phi} .$$

Here θ is the angle $\widehat{EAD}$, the equal angles ω result from the isogonal property (III) of section 3, and ϕ is the angle $\widehat{EAM}$, the points $\{M, N\}$ being collinear on AM, which is the conjugate diameter of κ to the direction of DE. Introducing these into the last equation we obtain the necessary and sufficient condition for our property to hold

$$(8) \qquad |AD||AE||AM| \;=\; |AS||AS'||AN| \;\stackrel{(6)}{=\!=}\; |AB||AB'||AN|\ .$$

According to lemma 3.2 the equalities remain valid if we change the direction of the parallel tangents $\{BC, B'C'\}$ (see Figure 10). So we choose the direction parallel to the chord DE getting at figure 11. Projecting all the points parallel to the direction of DE, onto points

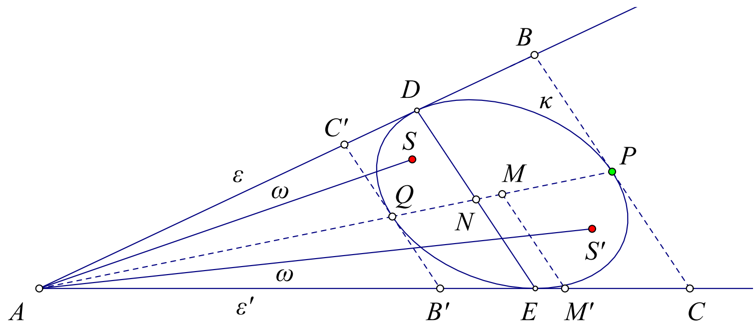

FIGURE 11. Projecting all points parallel to DE to points of line ε'

of the line ε', last equation becomes equivalent to

$$1 \;=\; \frac{|AD|}{|AB|}\cdot\frac{|AE|}{|AB'|}\cdot\frac{|AM|}{|AN|} \;=\; \frac{|AE|}{|AC|}\cdot\frac{|AE|}{|AB'|}\cdot\frac{|AM'|}{|AE|} \;=\; \frac{|AE|}{|AC|}\cdot\frac{|AM'|}{|AB'|}\ ,$$

seen easily to be equivalent to the fact that (A, E) are harmonic conjugate to (B', C). Latter though is true, since DE is the polar of A and parallel projection preserves the harmonic relation between points on a line. This completes the proof of the lemma.

Remark 4.1. *Taking into account the remarks at the beginning of the section about fixed points and harmonic quadrangles, the first equality of equations* (8) *is equivalent with the fact, that the circumcircles* $\{\lambda, \mu\}$ *of the triangles* $\{SAS', DAE\}$ *intersect at the second fixed point* A^* *of the Möbius involution* h*. For both triangles line* AA^* *is the symmedian from* A *and the quadrangles* $\{ASA^*S', ADA^*E\}$ *are harmonic.*

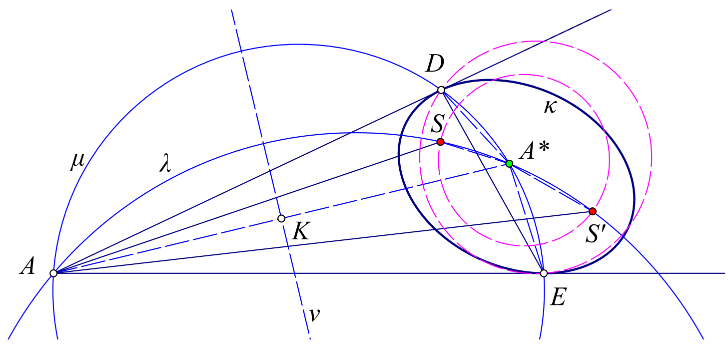

FIGURE 12. The other than A fixed point A^* of the involution h

Further, the transformation h *leaves the circles through* $\{A, A^*\}$ *invariant, mapping each member of the pencil of all these circles to itself, and also leaves invariant every member circle of the pencil which is orthogonal to the previous one. Since through each point* X *of the plane passes a unique member of each pencil, the image point* $Y = h(X)$

can be found also by considering the other than X intersection point of the member-circles through X of these two orthogonal pencils (see Figure 12).

In particular the radical axes $\{AA^*, \nu\}$ of the two pencils intersecting at the middle K of AA^* remains invariant under h, consequently K maps to infinity and is a limit point of the transformation. Since involutions are inverse to itself the other limit point of h is the point at infinity.

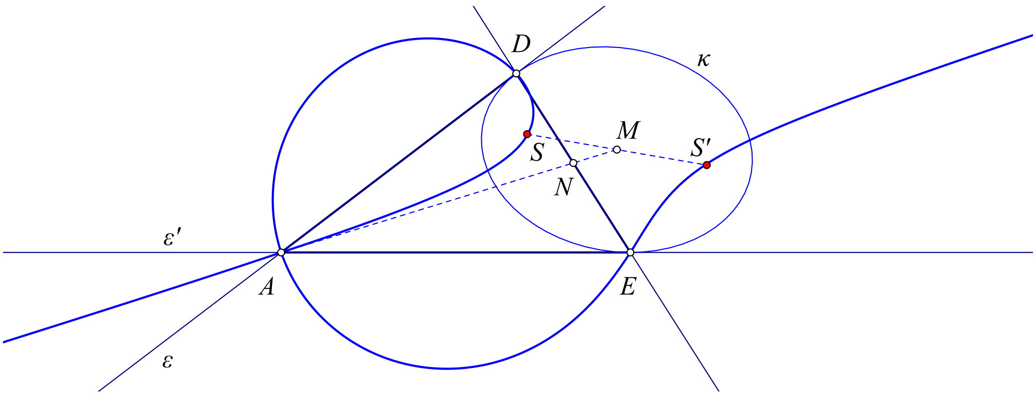

FIGURE 13. Locus of focal points of conics κ tangent to $\{\varepsilon, \varepsilon'\}$ at $\{D, E\}$

Remark 4.2. *The first of equations* (8) *can be written in the form*

$$\frac{|AS||AS'|}{|AM|} = \frac{|AD||AE|}{|AN|},$$

which, since the right member is constant, expresses a property of the focal points of all members of the "bitangent pencil" consisting of conics $\{\kappa\}$ tangent to lines $\{AD, AE\}$ respectively at $\{D, E\}$. Figure 13 shows the geometric locus of the focal points of the members of this pencil. It is a singular at A cyclic cubic, passing through the vertices of the triangle ADE. These curves are called "isoptic" ([24]*), because their points view the segments $\{AD, AE\}$ under equal or supplementary angles, a property satisfied by the focal points because of the isogonal property (II) of section 3.*

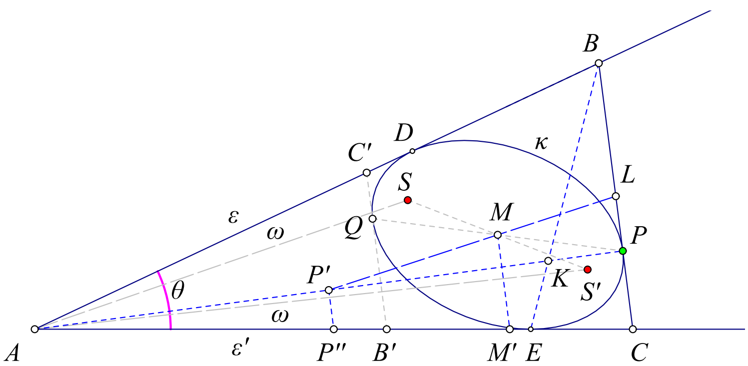

FIGURE 14. Studying the involution interchanging the couples $\{(A, P), (B, C)\}$

Lemma 4.2. *In the triangle ABC is inscribed the central conic κ touching the sides correspondingly at the points $\{P \in BC, E \in CA, D \in AB\}$. Then, the Möbius involution $h = (AP, BC)$ interchanging $A \leftrightarrow P$ and $B \leftrightarrow C$, interchanges also the focal points $S \leftrightarrow S'$ (see Figure 14).*

Proof. Besides the involution in question, we consider also the involution (AP, SS') interchanging $A \leftrightarrow P$ and $S \leftrightarrow S'$. Assume the vertex A is at the origin and AC is the real axis. Applying formula (7) with our hypothesis and representing with the symbols complex numbers, we get the description of the involutions

$$(AP, BC): \quad w \;=\; \frac{-BCz + BCP}{(P - B - C)z + BC} \quad , \quad (AP, SS'): \quad w \;=\; \frac{-SS'z + SS'P}{(P - S - S')z + SS'} \; .$$

It suffices to show that the two involutions coincide. Since they coincide at $A = 0$ it suffices to show that they coincide also at another point, equivalently, that the first involution coincides with the second at S, i.e.

$$S' \;=\; \frac{-BCS + BCP}{(P - B - C)S + BC} \qquad \Leftrightarrow \qquad (P - B - C)SS' + BCS' = -BCS + BCP \; .$$

Writing the complex numbers in polar form we have (see Figure 14)

$$A = 0 \;\; , \quad B = |B|e^{i\theta} \;\; , \quad C = |C| \;\; , \quad S = |S|e^{i(\theta-\omega)} \;\; , \quad |S'| = |S|e^{i\omega} \; .$$

Introducing these into last equation we obtain

$$\begin{aligned}
(P - B - C)|S||S'| + |B||C|S' = -|B||C|S + |B||C|P \qquad \Leftrightarrow \\
P(|S||S'| - |B||C|) \;=\; (B + C)|S||S'| - (S' + S)|B||C| \; , \quad \text{but by (6)} \\
|S||S'| = |B||B'| \qquad \Rightarrow \qquad P(|B||B'| - |B||C|) \;=\; (B + C)|B||B'| - (S' + S)|B||C| \\
\Leftrightarrow \quad P(|B'| - |C|) \;=\; (B + C)|B'| - (S' + S)|C| \qquad \Leftrightarrow \\
P(|B'| - |C|) \;=\; 2L|B'| - 2M|C| \qquad \Leftrightarrow \quad P'(|B'| - |C|) \;=\; L|B'| - M|C| \; ,
\end{aligned}$$

where $\{P' = \frac{1}{2}P \;,\; L = \frac{1}{2}(B + C) \;,\; M = \frac{1}{2}S + S'\}$ are the middles of the corresponding segments $\{AP, BC, SS'\}$. An inspection of figure 14 suggests that the points $\{P', L, M\}$ are collinear. Assuming for the moment that this is true, last equation expresses P' as linear combination of the points $\{B', C\}$. Since linear combinations are preserved by parallel projections, we project everything parallel to BC onto line $\varepsilon' = AC$ obtaining

$$\begin{aligned}
|P''|(|B'| - |C|) \;=\; |C||B'| - |M'||C| \qquad \Rightarrow \\
\frac{1}{2}|C|(|B'| - |C|) \;=\; |C||B'| - \frac{1}{2}(|B'| + |C|)|C| \; ,
\end{aligned}$$

which is an identity. Thus, to finish the proof we need still to prove the collinearity of $\{P', M, L\}$. For this we use *"barycentric coordinates"* w.r.t. the triangle ABC ([31, p.25], [21]). In the framework of these coordinates the joins of the vertices with the contacts of the conic κ with the opposite sides are seen to pass through a common point $K(p : q : r)$ (see Figure 14), called *perspector* of the conic ([31, p.119]). The conic is expressed through equation

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} - 2\frac{yz}{qr} - 2\frac{zx}{rp} - 2\frac{xy}{pq} \;=\; 0.$$

The points of interest have respectively the following barycentric coordinates

$$L(0 : 1 : 1) \; , \; P(0 : q : r) \; , \; P'(q + r : q : r) \; , \; M(p(q + r) : q(r + p) : r(p + q)) \; .$$

Last point M is the center of the conic and also the pol of the line at infinity whose equation is $x + y + z = 0$. Its coordinates are determined, up to multiplicative constant, from the equation expressing the relation pol-polar in terms of the matrix of the conic κ

$$\begin{pmatrix} \frac{1}{p^2} & \frac{1}{pq} & \frac{1}{pr} \\ \frac{1}{pq} & \frac{1}{q^2} & \frac{1}{qr} \\ \frac{1}{pr} & \frac{1}{qr} & \frac{1}{r^2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = k \begin{pmatrix} p(q + r) \\ q(r + p) \\ r(p + q) \end{pmatrix} \; .$$

The collinearity of the points $\{P', M, L\}$ results by proving the vanishing of the determinant of the corresponding coordinate-vectors, seen trivially to be true:

$$\begin{vmatrix} q+r & q & r \\ p(q+r) & q(r+p) & r(p+q) \\ 0 & 1 & 1 \end{vmatrix} = 0 \ .$$

5. A composition of two involutions

Assuming the polygon $p = ABC\dots Z$ circumscribes a central conic κ, we isolate a vertex A and its adjacent side-lines $\{\varepsilon, \varepsilon'\}$ together with some other side-line $\tau = XY$ of the polygon, creating a triangle AXY circumscribing the central conic too (see Figure 15). Next theorem is a slightly different but equivalent formulation of Siebeck's theorem.

Theorem 5.1. *The fixed points $\{S, S'\}$ of the Möbius transformation h_A mapping the triple $\{X, K, A \in \varepsilon\}$ correspondingly to $\{Y, A, L \in \varepsilon'\}$, where $\{K, L\}$ are the contacts of $\{\varepsilon, \varepsilon'\}$ with the conic κ, coincide with the focal points $\{S, S'\}$ of the central conic κ.*

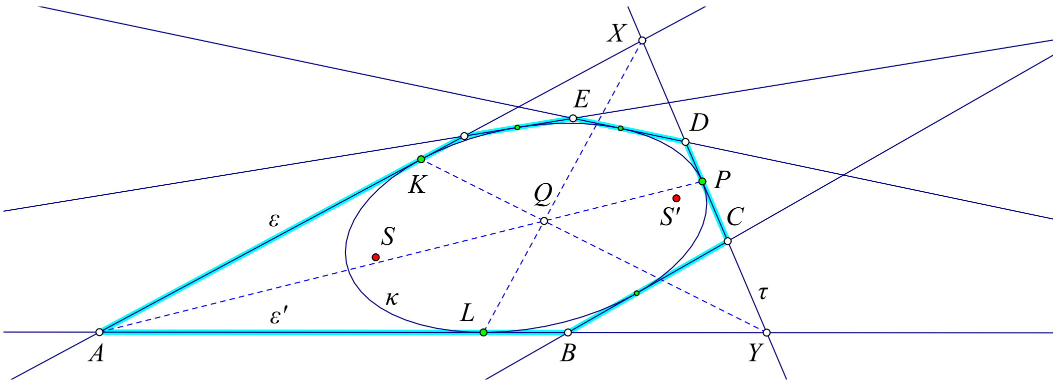

Figure 15. A triangle with a vertex at A defined by the circumscribed polygon

Proof. We show that the Möbius transformation h_A is the product of two Möbius involutions $h_A = g \circ f$, where $f = (AP, XY)$ and $g = (YY, LP)$. This implies the proof, since f is of the kind handled in lemma 4.2 and g is of the kind handled in lemma 4.1 and both interchange the points $\{S, S'\}$, hence their composition leaves both fixed.

By the general properties of Möbius transformations, to prove the coincidence of the transformations $h_A = g \circ f$, it suffices to show their coincidence at three points. This is easy for the two points $\{A, X\}$, since by the definition of these transformations

$$A \overset{f}{\longmapsto} P \overset{g}{\longmapsto} L \quad \text{and} \quad X \overset{f}{\longmapsto} Y \overset{g}{\longmapsto} Y \ .$$

We show that the two transformations coincide also at K : $h_A(K) = A = (g \circ f)(K)$. Identifying the symbols with the complex numbers representing the points and taking A at the origin and AY coincident with the real axis we examine the images of the points lying on line ε

$$tX \in \varepsilon \ , \ (g \circ f)(tX) \quad \text{for} \quad t \in \mathbb{R} \ .$$

Applying formula (7) we have the corresponding representations of $\{f, g\}$

$$w = f(z) = \frac{-XYz + XYP}{(P - X - Y)z + XY} \quad , \quad w' = g(w) = \frac{(Y^2 - LP)w + 2LPY - Y^2(L + P)}{(2Y - L - P)w + LP - Y^2} \ .$$

Setting $z = tX$ and $L = kY$ we obtain after simplification

$$
\begin{aligned}
w = f(tX) &= \frac{-XYt + YP}{(P - X - Y)t + Y} ,\\
w' = g(f(tX)) &= \frac{(k(Y + X - 2P) + P - X)t + k(P - Y)}{(k(X - P) + Y - X)t + (P - Y)} Y\\
&= \frac{(k((X - P) - (P - Y)) - (X - P))t + k(P - Y)}{(k(X - P) - (X - Y))t + (P - Y)} Y\\
&= \frac{(k(\lambda(X - Y) - \mu(X - Y)) - \lambda(X - Y))t + k\mu(X - Y)}{(k\lambda(X - Y) - (X - Y))t + \mu(X - Y)} Y\\
&= \frac{(k(\lambda - \mu) - \lambda)t + k\mu}{(k\lambda - 1)t + \mu} Y \;=\; t'Y, \quad \text{with } t' \in \mathbb{R} ,
\end{aligned}
$$

where we have set $\{X - P = \lambda(X - Y) \ , \ P - Y = \mu(X - Y) \ , \ \lambda \ , \ \mu \in \mathbb{R} \ , \ \lambda + \mu = 1\}$. Last equation shows that $g \circ f$ maps line ε onto ε'. Point $A = 0$ is obtained for

$$
(k(\lambda - \mu) - \lambda)t + k\mu = 0 \quad \Rightarrow \quad t = \frac{k(\lambda - 1)}{k(2\lambda - 1) - \lambda} \quad \Rightarrow \quad \frac{zX}{zA} = \frac{\lambda(1 - k)}{k(\lambda - 1)} ,
$$

which together with

$$
(*) \qquad \frac{LA}{LY} = \frac{k}{k - 1} \quad , \quad \frac{PY}{PX} = \frac{\lambda - 1}{\lambda} \quad \Rightarrow \quad \frac{LA}{LY} \cdot \frac{PY}{PX} \cdot \frac{zX}{zA} = -1 .
$$

Since the segments $\{AP, YK, XL\}$ pass through the perspector Q of the conic w.r.t. the triangle AXY, by Ceva's theorem we have also

$$
\frac{LA}{LY} \cdot \frac{PY}{PX} \cdot \frac{KX}{KA} = -1 ,
$$

which together with $(*)$ implies $z = K$ and completes the proof of the theorem showing that $(g \circ f)(K) = A$.

6. About the compositions

According to the discussion in the preceding sections, a polygon $p = A_1A_2 \dots A_n$ circumscribing a central conic defines a series of Möbius transformations $\{h_1, h_2, \dots, h_n\}$, where we shortened the notation to $h_i = h_{A_i}$. According to theorem 5.1, these transformations share the same fixed points $\{S, S'\}$ and, as we notice below (section 7, *nr-9*), they are non-involutive. This implies ([12, p. 169]) that they commute pairwise. Next theorem determines their composition (see Figure 16).

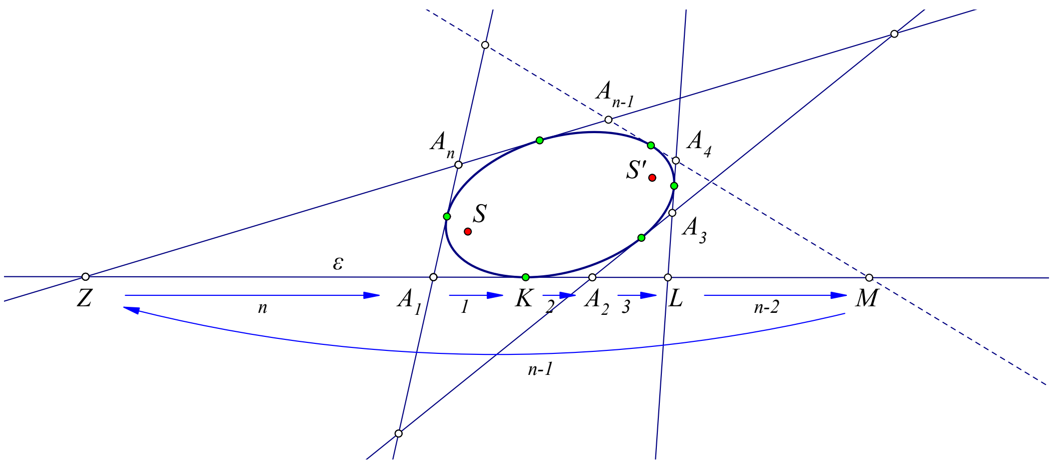

Figure 16. $K = h_1(A_1)$, $A_2 = h_2(K)$, $L = h_3(A_2)$, ...

Theorem 6.1. *With the notation and conventions adopted so far the composition h of the transformations is the identity transformation e.*

$$h = h_n \circ h_{n-1} \circ \cdots \circ h_1 = e , \tag{9}$$

Proof. Since Möbius transformations build a group under composition, the above composition h is also a Möbius transformation. Since a Möbius transformation fixing three points is the identity, it suffices to show the existence of three fixed points of h. Since the transformations h_i share the same fixed points $\{S, S'\}$ their composition h does the same. Hence, in order to show $h = e$ it suffices to find one more fixed point of h.

It is though readily seen that $h(A_1) = A_1$. In fact, consider the contact point K of the conic with the side $\varepsilon = A_1A_2$ and the intersections $\{L, M, \ldots\}$ of ε with the successive other sides $\{A_3A_4, A_4A_5, \ldots\}$ of the polygon. From remark 1.1 follows that $h_1(A_1) = K$ and $h_2(K) = A_2$ (see Figure 16). From the way the transformations h_i are defined, follows also that they map $h_3(A_2) = L$ and more general, for $i > 2$, each intersection L of ε with a side A_iA_{i+1} to the intersection M of ε with next side $A_{i+1}A_{i+2}$. It follows that the last transformation h_n maps the intersection $Z = \varepsilon \cap A_{n-1}A_n$ to A_1 realizing the identification of points $h(A_1) = A_1$, as claimed.

Remark 6.1. *Partial compositions out of the set of transformations $\{h_A, h_B, \cdots, h_Z\}$, for successive vertices, like for example $\{h_{ABC\ldots M} = h_M \circ \cdots \circ h_C \circ h_B \circ h_A\}$, have the same properties and can be defined in the same way as the h_X's. They correspond to a polygon p' resulting from the original p from which the successive sides $\{AB, BC, \ldots, LM\}$ have been deleted. Figure 17 illustrates the case of $h = h_{AB} = h_B \circ h_A$. It coincides with $h_{A'}$ where $A' = ZA \cap BC$ corresponding to polygon $p' = A'C \ldots Z$ resulting from the original*

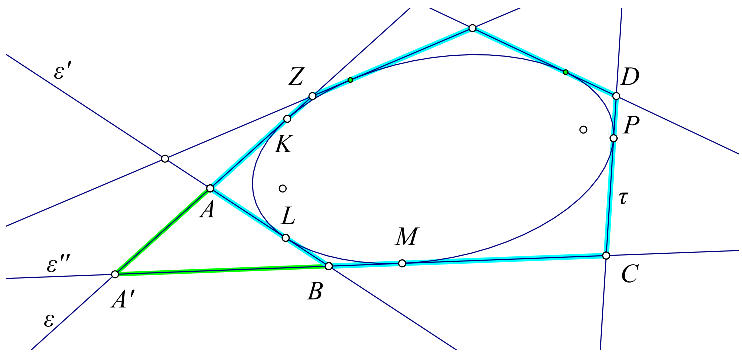

Figure 17. The partial composition $h_{AB} = h_B \circ h_A$ mapping ε to ε''

p from which side AB has been deleted and the vertices $\{A, B\}$ have been replaced with A'. Also $h_B \circ h_A$ maps A to B and composing analogous successive such transformations we obtain the identity transformation through a product, in which each factor maps a vertex of the polygon, considered in a fixed orientation, to an adjacent one

$$(h_A \circ h_Z) \circ \cdots \circ (h_D \circ h_C) \circ (h_C \circ h_B) \circ (h_B \circ h_A)$$
$$\overset{(9)}{=\!=} (h_Z)^2 \circ \cdots \circ (h_B)^2 \circ (h_A)^2 = e ,$$

Thus, a polygon circumscribing a conic defines a group of commuting Möbius transformations with generators $\{h_A, h_B, \ldots\}$ satisfying the relation (9). This rises the question for the converse, namely, whether such a group defines some polygons circumscribing appropriate conics. This will be handled in sections 11 and 12 for the case of triangles.

7. The characteristic constant or invariant

The *"characteristic constant"* ([27, p.64]) or *"invariant"* ([12, p.150]) k of a Möbius transformation $w = h(z)$ with two fixed points $\{S, S'\}$ is defined by its representation

using the fixed points
(10)
$$w = h(z) \quad \Leftrightarrow \quad \frac{S-z}{S'-z} = k\frac{S-w}{S'-w} \quad \Leftrightarrow \quad (SS';zw) = k \quad \Leftrightarrow \quad (S'S;wz) = \frac{1}{k}\ .$$
In particular $k = 1$ and $k = -1$ characterizes correspondingly the identity and the involutive Möbius transformations. Using this constant we can find a criterion for the kind of the conic inscribed in the polygon. We discuss here the case of a central conic and the Möbius transformation h_A fixing a vertex A of the circumscribed polygon, which together with the adjacent to A side-lines and some other side-line of the polygon forms a circumscribed to the conic triangle ABC. In order to simplify the calculations we assume that the coordinate axes coincide with the axes of the conic (see Figure 18).

We work for the case of an inscribed ellipse, the case of hyperbola being analogous with small differences on which we comment later. We assume also that the construction of such a genuine triangle is possible, leaving the case of a parallelogram, for which the construction of a genuine triangle with these properties is impossible, to be discussed in section 10. We assume that $\{D \in AC, E \in AB\}$ are the contact points of the sides with the conic. From the preceding discussion we know that the Möbius transformation $h = h_A$ is completely determined by the correspondences
$$D \xmapsto{h} A \xmapsto{h} E \quad \text{and} \quad C \xmapsto{h} B\ .$$
Having in mind the definition of the invariant k, we define the dependent on E Möbius transformation
$$w = k_E(z) = (SS';zE) = \frac{S-z}{S'-z} : \frac{S-E}{S'-E}\ . \tag{11}$$
We use this to produce the circle η shown in figure 18. In fact, we see immediately that $z = S'$ is the limit point send to infinity by k_E and line $\varepsilon = AB$ does not pass through it. Hence, by the general properties of Möbius transformations, the corresponding image of this line, $\eta = k_E(\varepsilon)$, is a circle. The circle, by its definition, contains the invariants $\{k_A, A \in \varepsilon\}$ of the various Möbius transformations $\{h_A, A \in \varepsilon\}$ resulting by fixing the tangent to the conic ε and varying the point D of the conic and its tangent. In the

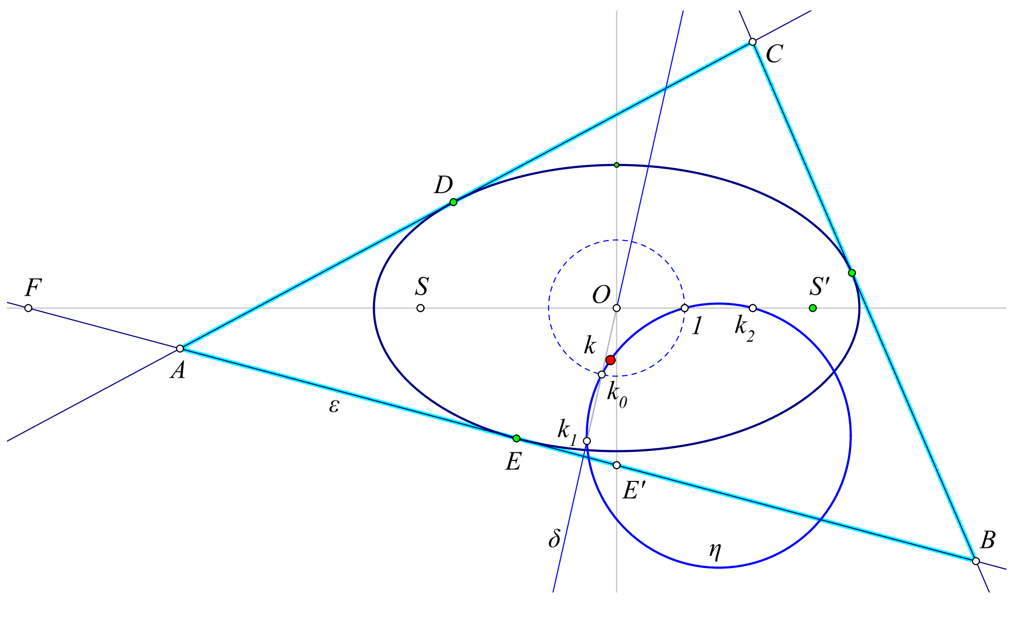

FIGURE 18. Circle η carrying the invariants k of $\{h_A, A \in \varepsilon\}$

framework of the circumscribed polygon, whose figure 18 is a part, this means a variation of its side-line AC maintaining its tangency to the inscribed conic. This change affects the transformation h_A and its invariant by changing the location of A along the line ε, in contrast to the tangent BC, whose variation has no effect on the definition of f_A. This circle has some properties on which we comment.

(1) The circle η contains the invariant k of $h = h_A$, since this is obtained for $z = A$ $k_E(A) = (SS'; AE) = k$.
(2) It passes through $z = 1$ since $k_E(E) = 1$.
(3) Its second intersection point with the unit circle $\{S^1 : |z| = 1\}$ is point $k_0 = k_E(F)$, where F is the intersection of ε with the real axis. In fact, to see this it suffices to show that $|k_0| = |k_E(F)| = \frac{|SF|}{|S'F|} : \frac{|SE|}{|S'E|} = 1 \Leftrightarrow \frac{|S'E|}{|SE|} = \frac{|SF|}{|S'F|}$. But this is a well known property of points on the axes of the conic. The property follows by reflecting S' in FE to a point S'' collinear with $\{S, E\}$ because of the isogonal property (I) of section 3. Then, FE is the bisector at F of triangle SFS'' and this implies the last equality.
(4) Obviously $k_E(S) = 0$, $k_E(S') = \infty$ and $k_1 = k_E(\infty) = \frac{S'-E}{S-E} \in \eta$.
(5) The intersection point E' of the imaginary axis with line ε maps to $k_2 = k_E(E')$ which is the other than $z = 1$ intersection point of the circle η with the real axis. This is proved by showing that k_2 is real. For this reflect S' in ε to the point S'' collinear to $\{S, E\}$. It is formed an isosceles triangle $SE'S''$ showing that the angles $\widehat{ESE'} = \widehat{ES'E'}$ are equal. From this follows easily the reality of k_2.
(6) The real axis maps via k_E to a line δ through the origin and points $\{k_0, k_1\}$ are on this line. This follows from the fact that the real axis passes through the limit point S' sent to infinity by k_E. Hence its image is a line δ. Since $\{F, S, \infty\}$ are on the real axis, their images $\{k_0, 0, k_1\}$ are on δ.
(7) If the conic is an ellipse, its center O is outside the circle η. For this consider the ratio

$$k_0/k_1 = k_E(F)/k_E(\infty) = \left(\frac{S-F}{S'-F} : \frac{S-E}{S'-E}\right) : \left(\frac{S'-E}{S-E}\right) = \left(\frac{S-F}{S'-F}\right) > 0 \ ,$$

which is a condition for the collinear segments $\{\overline{Ok_0}, \overline{Ok_1}\}$ to be equally oriented, thereby proving that O is outside the circle.
(8) The chords $\{[1k_2], [k_0k_1]\}$ of the circle η have the same length. This follows directly by measuring the power of O w.r.t. the circle η correspondingly along the real axis and the line δ.
(9) The circle η does not pass through $z = -1$, which is the invariant characterizing the involutive Möbius transformations. This because in the contrary case, the circle η would degenerate to the real axis and line ε would pass through S', which is impossible. Thus, all the Moebius transfrormations h_A are non-involutive.

Figure 19 shows the analogous configuration for the case of the hyperbola. Properties *nr-1*, *nr-2* are equally obvious. For *nr-3* the argument is the same, differing only in the bisector FE, which now is the external of the angle $\widehat{SFS''}$. *Nr-4* is equally obvious and *nr-5* differs in that the angles $\{\widehat{ESE'}, \widehat{ES'E'}\}$ are now supplementary. *Nr-6* holds verbatim true and *nr-7*, using the same arguments, proves now that $k_0/k_1 < 0$ and the origin is inside the circle η. *Nr-8* and *nr-9* follow using the same arguments.

Next theorem results from the preceding remarks and formulates a criterion allowing the determination of the kind of the inscribed in the polygon conic in terms of the Möbius transformation h_A and its related structures.

Theorem 7.1. *With the notation and conventions adopted so far, the inscribed central conic κ of the circumscriptible to it polygon p is a hyperbola/ellipse, if and only if, its center O is an inner/outer point of the circle $\eta = k_E(\varepsilon)$.*

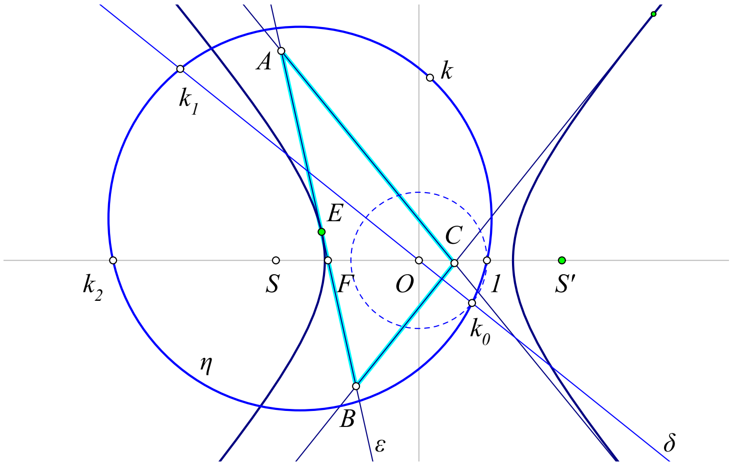

FIGURE 19. Circle η carrying the invariants k of $\{h_A, A \in \varepsilon\}$

Remark 7.1. *One could object that the preceding result is somewhat superficial, since one has the much simpler criterion, for the kind of the central conic inscribed in a polygon, according to which every side-line of the polygon leaves always the focal points on the same side in the case of ellipse, and on different sides in the case of hyperbola.*

The point is that one may not know the location of the focal points $\{S, S'\}$, whereas using the transformation h_A for a single vertex of the polygon one can quickly find the center of the conic as the middle O of the segment SS' expressed directly through the coefficients of the Möbius transformation $O = \frac{a-d}{c}$ and also the circle $\eta = h_A(\varepsilon)$ and carry out the test suggested by the preceding theorem. This, even in the case where the location of the focal points using the quadratic equation (3) *may present difficulties if the coefficients $\{a, b, c, d\}$ are some involved expressions of other variables.*

Remark 7.2. *From our discussion follows, that the invariants $\{k_A, k_B, \dots\}$ of the Möbius transformations $\{h_A, h_B, \dots\}$ corresponding to the vertices of the circumscribed to the conic polygon $p = ABC\dots$ are, excepting the cases $\{k = 1, k = k_2\}$, genuine complex numbers. Excepting also the case $k = k_0$, they have measure $|k| \neq 1$. This implies, that in all cases, excepting the preceding three, the Möbius transformations $\{h_X\}$ are "loxodromic"* ([27, p.65]*).*

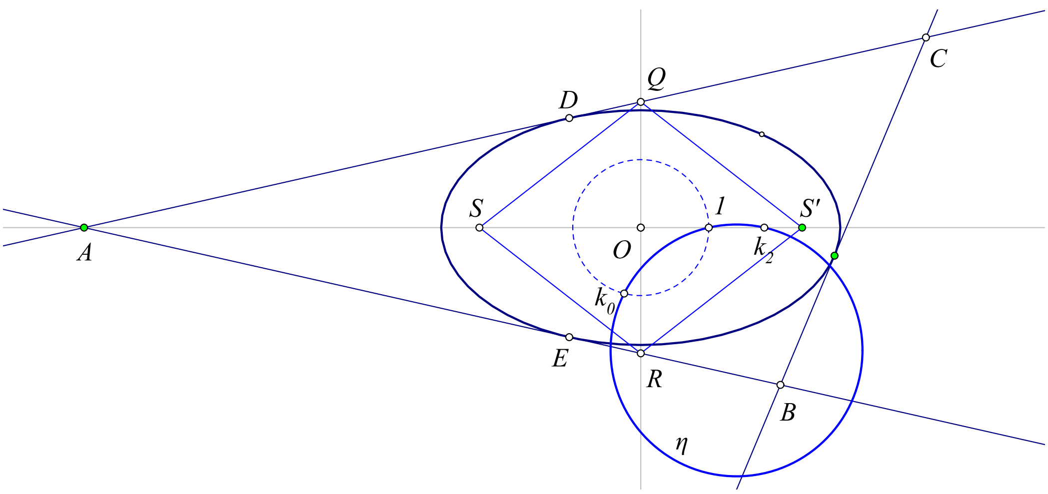

FIGURE 20. Case of vertex lying on the principal axis of the inscribed conic

From the exceptional cases, the one with $k = 1$ cannot occur, since it would imply that the polygon at the vertex A has a "flat ($180°$)" or "zero ($0°$)" angle, which we exclude. The other real case $k = k_2$ is obtained when the vertex A of the polygon falls into the conjugate axis of the conic. From our discussion follows that $k_2 > 0$ in the case of the ellipse and $k_2 < 0$ in the case of hyperbola, showing that the transformation h_A is correspondingly "proper / improper hyperbolic" ([27, p.65]*).*

Figure 20 shows the case with non-real $k = k_0$ and $|k_0| = 1$. The vertex A of the polygon is in this case on the principal axis of the conic and the Möbius transformation is "elliptic". It can be proved that it is conjugate, in the group of Möbius transformations, to a rotation determined by the angle ϕ of the polar form of $k = e^{i\phi}$. The figure shows the location of the invariant $k = k_0$ and also the corresponding characteristic parallelogram which is a rhombus, as is the case for all elliptic Möbius transformations.

8. The case of the parabola

The identification of the fixed point of the Möbius transformation h_A with the focus in the case of a parabola κ is quite simple. Figure 21 shows the corresponding configuration and suggests next theorem.

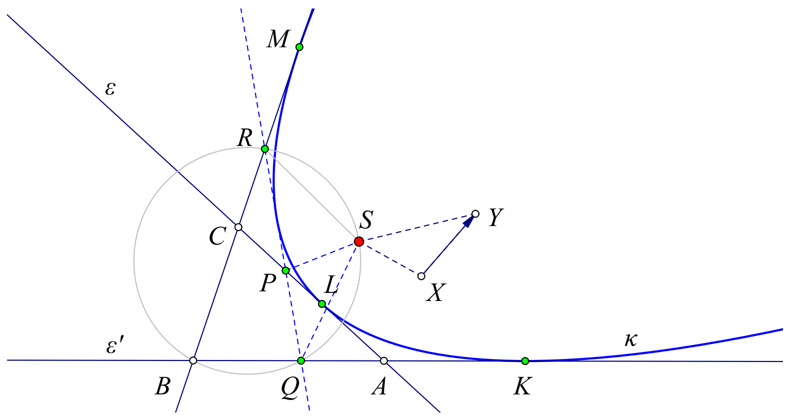

FIGURE 21. $h_A : P \mapsto Q$ is a similarity with center S the focus of the parabola

Theorem 8.1. *In the case of a parabola, the Möbius transformation h_A is a similarity represented with complex numbers in the form $w = h_A(z) = az + b$, with constant complex numbers $\{a, b\}$. The center S of the similarity coincides with the focus of the parabola.*

Proof. In fact, assume the Möbius transformation h_A in the most general form

$$w = \frac{az + b}{cz + d}.$$

From its general definition h_A maps points $\{C, L, A\}$ on the tangent ε at L correspondingly to points $\{B, A, K\}$ of the tangent ε' at K. Identifying the symbols with the corresponding complex numbers representing the points, and taking into account, that *"an arbitrary tangent of the parabola intersects three fixed tangents at points $\{R, P, Q\}$ such that the ratio RP/PQ is constant"* ([26, p.299]). In figure 21 all the ratios are equal: $\{RP/PQ = MC/CB = BA/AK = CL/LA\}$ and we have.

$$A = \frac{aL + b}{cL + d}\ ,\ K = \frac{aA + b}{cA + d}\ ,\ B = \frac{aC + b}{cC + d} \qquad \Rightarrow$$
$$\frac{LC}{LA} = \frac{AB}{AK} \quad \Leftrightarrow \quad \frac{c(L - C)(dC + cCL - b - aL)}{(d + cC)(dL + cL^2 - b - aL)} = 0\ .$$

It is though always $C \neq L$ and

$$dC + cCL - b - aL = 0 \quad \Leftrightarrow \quad L = \frac{dC - b}{-cC + a} = h_A^{-1}(C) \ ,$$

which is not possible, since $h_A(L) = A$. Thus, the only possibility to have this equation is $c = 0$, implying that the Möbius transformation is a similarity. We rewrite it in the form

$$(12) \qquad w = az + b \quad \Rightarrow \quad a = \frac{K - A}{A - L} \ , \ b = \frac{KL - A^2}{L - A} \quad \text{and} \quad S = \frac{KL - A^2}{L + K - 2A} \ ,$$

latter being the fixed point of h_A. By the general properties of similarities ([6, ch.IV], [30, II, p.36]), for every point X of the plane and $Y = h_A(X)$ the triangle SXY remains similar to itself and the angle $\widehat{XSY} = \widehat{PSQ} = \widehat{CAB}$. The identification of S with the focus of the parabola κ results now from the fact that all triangles $\{SPQ\}$ for $\{P \in \varepsilon\}$ are similar to each other and consequently $\{Q = f_A(P) \in \varepsilon'\}$. This, according to the well known property ([13, p.51]) *If the triangle SPQ with fixed vertex S and P varying on a line ε remains similar to itself, then the line PQ envelopes a parabola κ' with focus at S* . Thus the lines $t_P = PQ$ are simultaneously tangent to the parabola of reference κ and to the parabola-envelope κ' with focus S. Hence the two parabolas coincide and S is the focus of κ.

For a polygon $p = ABC \dots Z$ circumscribing a parabola κ, the composition of similarities $h = h_Z \circ \dots h_B \circ h_A$ is proven to be the identity transformation in a similar way, as this was done for central conics. In fact, since all these similarities have the same center S, this is also the center of the similarity h. In order to show that h is the identity transformation it suffices to find one more point fixed by h.

In fact, it is $h(A) = A$. This is seen by watching the sequence $\{A, A', K, L \dots\}$, where A' is the contact point of the first side $\varepsilon = AB$, K is the intersection of the first side $\varepsilon = AB$ with side CD and $\{L, \dots\}$ are the intersections of ε with the subsequent sides $\{DE, EF, \dots\}$ of the polygon. It is readily seen that the various similarities map each term of this sequence to the next, recurring at last to A (see Figure 22).

$$A \xrightarrow{h_A} A' \xrightarrow{h_B} B \xrightarrow{h_C} K \xrightarrow{h_D} L \dots \xrightarrow{h_Z} A \ .$$

This proves next theorem.

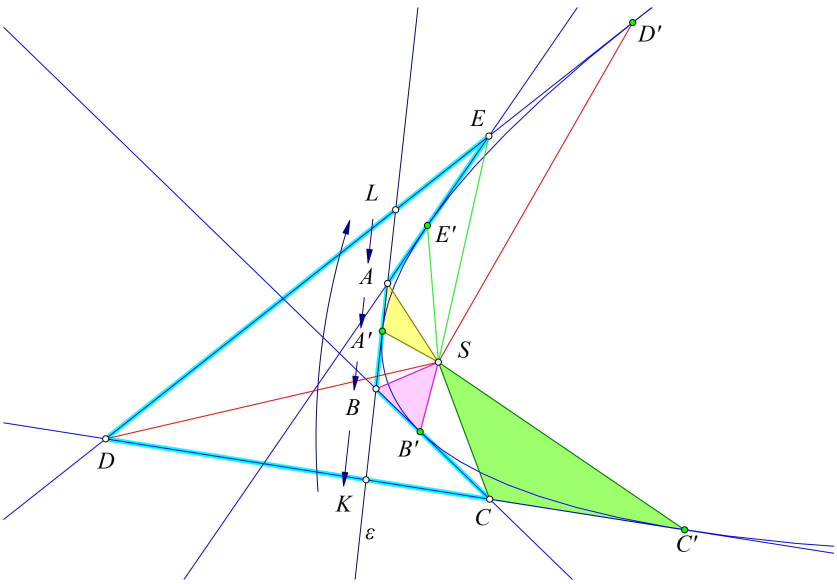

Figure 22. $\quad A \xrightarrow{h_A} A' \xrightarrow{h_B} B \xrightarrow{h_C} K \xrightarrow{h_D} L \xrightarrow{h_L} A$

Theorem 8.2. *For a polygon $p = ABC\dots Z$ circumscribing a parabola the composition of the similarities corresponding to the vertices $\{h = h_Z \circ \dots \circ h_B \circ h_A\}$ is the identity transformation.*

Remark 8.1. *Assuming that the polygon circumscribes a genuine parabola, the various similarities $\{h_A, h_B, \dots\}$ corresponding to its vertices are also a genuine, i.e. different from the identity or a translation. They have two fixed points, their similarity center coinciding with the focus of the parabola and the point at infinity ∞. Obviously the fixed point of the the similarity $w = az + b$ is $z_0 = \frac{b}{1-a}$, and it is readily seen that transformation can be expressed in the form $w = a(z - z_0) + z_0$. From our definition of the invariant follows then that $k = (z_0\infty; zw) = \frac{1}{a}$.*

9. Conics inscribed in a triangle

The case of conics inscribed in triangles is, under the present viewpoint, in some sense *universal* and can be used for all other kinds of polygons circumscribing a conic. This, because selecting some vertex A of the polygon together with the adjacent to it side-lines $\{\varepsilon, \varepsilon'\}$ and any one of the remaining side-lines of the polygon, non-parallel to $\{\varepsilon, \varepsilon'\}$, we create a triangle circumscribing the conic. In this section we discuss this case, starting with some examples and proceeding to the general conic inscribed in a circle, defined through its perspector. Figure 23 shows the basic configuration, with the triangle of reference

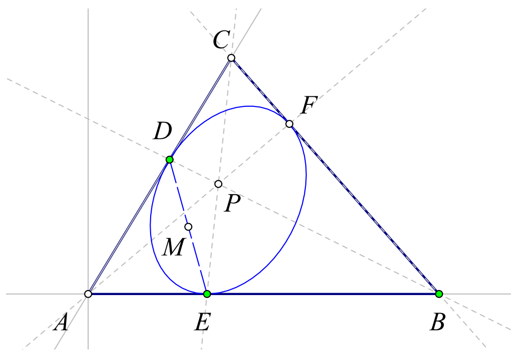

Figure 23. Inscribed conic and its perspector

ABC having its vertex A at the origin and side-line AB coinciding with the $x-$axis. Point P is assumed to be the *perspector* of the inscribed conic, which contacts the sides at the *traces* $\{D, E, F\}$ of P. The Möbius transformation $h = h_A$ corresponds $\{D, A, C\}$ to $\{A, E, B\}$ and, identifying the labels with complex numbers, we see easily that the Möbius transformation $w = h(z) = (az + b)/(cz + d)$ is given by

$$w \;=\; h(z) \;=\; -EBC\frac{z - D}{(D(E - B) - CE)z + BCD}\;.$$

Writing the vertices in the form $\{C = \mu \cdot D, B = \nu \cdot E\}$ we come to the expression

$$w \;=\; -\mu\nu E\frac{z - D}{(1 - \mu - \nu)z + \nu\mu D} \;=\; -\frac{E(z - D)}{\lambda z + D} \quad \text{with} \quad \lambda = \frac{1 - \mu - \nu}{\mu\nu}\;. \tag{13}$$

The fixed points of the transformation are the roots of the quadratic equation

$$\lambda z^2 + (D + E)z - DE \;=\; \lambda z^2 + 2Mz - DE \;=\; 0 \quad \text{with} \quad M = \frac{1}{2}(D + E) \tag{14}$$

the middle of the segment DE.

In the case of the *"incircle"* the fixed points or focal points of the inscribed conic coincide and this leads to a well known triangle formula. In fact, the vanishing of the discriminant

$$M^2 + \lambda DE = 0 \quad \Leftrightarrow \quad \frac{M^2}{DE} = -\lambda = \frac{\mu + \nu - 1}{\mu\nu} \tag{*}$$

in this case implies, with $\{a = |BC|\ ,\ b = |CA|\ ,\ c = |AB|\ ,\ \tau = (a+b+c)/2\}$

$$\mu = \frac{AC}{AD} = \frac{b}{\tau - a} \quad , \quad \nu = \frac{AB}{AE} = \frac{c}{\tau - a} \quad \text{and} \quad \frac{M^2}{AE} = \cos^2\left(\frac{\widehat{A}}{2}\right) \ ,$$

since in this case the triangle ADE is isosceles. Combining these with $(*)$ we obtain

$$\cos^2\left(\frac{\widehat{A}}{2}\right) \quad = \quad \frac{\tau(\tau - a)}{bc} \ .$$

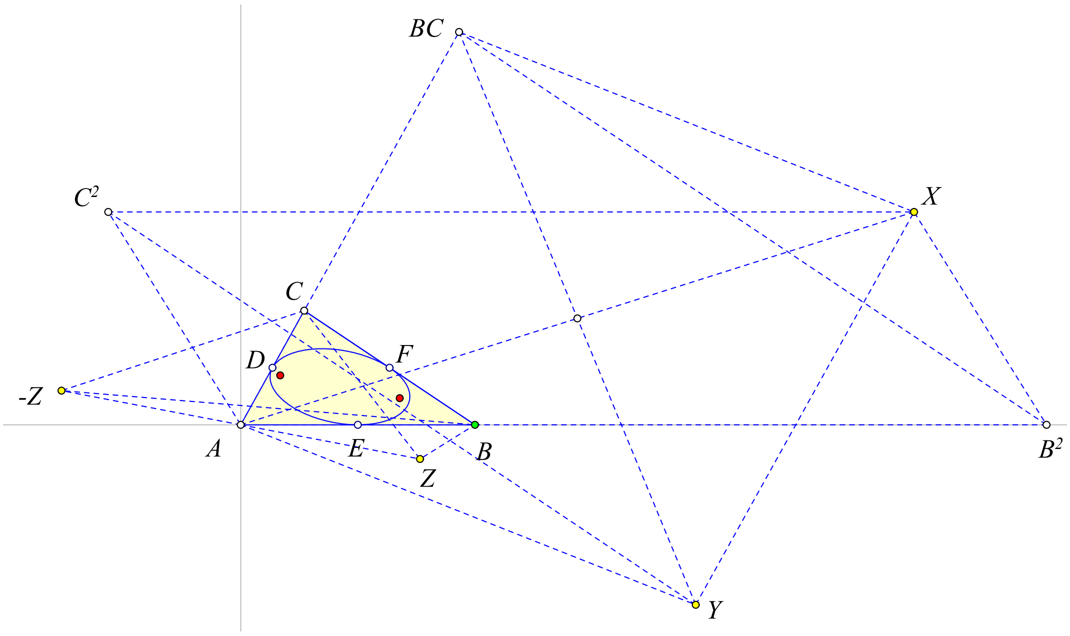

FIGURE 24. The focal points as centroids of triangles $\{ZBC, (-Z)BC\}$

Next example is the *"Steiner in-ellipse"* of the triangle seen in figure 24. In this case $\mu =\ \nu = 2, \lambda = -3/4$ and it is easily seen that the equation (14) for the fixed points of the Möbius transformation takes the form

$$3z^2 - 2(B+C) + BC = 0 \ , \tag{15}$$

implying for the focal points the expressions

$$S, S' = \frac{1}{3}(B + C \pm Z) \quad \text{with} \quad Z = \sqrt{B^2 + C^2 - BC} \ .$$

The operations with the complexes are suggested by the figure, and can be carried out easily geometrically requiring constructions using only straightedge and compasses. Points $\{X = B^2 + C^2 \ \ ,\ \ Y = B^2 + C^2 - BC \ \ ,\ \ Z = \sqrt{B^2 + C^2 - BC}\ \}$, lead to the triangles $\{ZBC, (-Z)BC\}$ and the focal points are the centroids of these two triangles

$$S = \frac{1}{3}(B + C + Z) \ \ ,\ \ S' = \frac{1}{3}(B + C - Z) \ .$$

This construction of the focal points of Steiner's in-ellipse can be used to give a proof of the ([18]) *"most marvelous theorem in Mathematics"* attributed to Siebeck-Marden ([7], [17], [2], [6]) which states:

Theorem 9.1. *For the non collinear complex numbers $\{A, B, C\}$, the roots of the derivative of the complex polynomial $P(z) = (z - A)(z - B)(z - C)$ coincide with the focal points of the Steiner in-ellipse of the triangle ABC.*

Proof. In fact, taking A at the origin and line AB coincident with the $x-$axis we get for the derivative $P'(z)$ the same equation (15).

The general conic inscribed in a triangle, seen in figure 23, is determined by its *perspector* $P(u:v:w)$, whose barycentric coordinates are related to the ratios on the sides of the traces $\{D,E,F\}$ of P

$$\frac{FB}{FC}=-\frac{w}{v} \quad , \quad \frac{DC}{DA}=-\frac{u}{w} \quad , \quad \frac{EA}{EB}=-\frac{v}{u} \quad \Rightarrow$$
$$\mu=\frac{u+w}{w} \quad , \quad \nu=\frac{u+v}{v} \quad , \quad \lambda=-\frac{vw+wu+uv}{(v+u)(w+u)} \; .$$

Equation (14) becomes then

$$\begin{aligned} &-(vw+wu+uv)z^2+(Bv(w+u)+Cw(u+v))z-vwBC=0 \quad \stackrel{A=0}{\Longleftrightarrow} \\ &-(vw+wu+uv)z^2 \\ &\qquad +(Au(v+w)+Bv(w+u)+Cw(u+v))z \\ &\qquad\qquad -(vwBC+wuCA+uvAB)=0 \; . \end{aligned}$$

This, setting $\{p=1/u, q=1/v, r=1/w\}$ is seen equivalent to

$$(p+q+r)z^2-(A(q+r)+B(r+p)+C(p+q))z+(BCp+CAq+ABr)=0 \;\Leftrightarrow$$
$$\frac{p}{z-A} + \frac{q}{z-B} + \frac{r}{z-C} = 0 \; , \tag{16}$$

proving the Siebeck-Linfield generalization of the preceding theorem ([20]):

Theorem 9.2. *For non collinear complex numbers $\{A,B,C\}$ and non-zero real numbers $\{p,q,r: pqr\neq 0\}$, the roots of equation (16) are the focal points of the conic inscribed in the triangle ABC and touching the sides $\{AB,BC,CA\}$ correspondingly at points $\{C',A',B'\}$ such that*

$$\frac{AC'}{C'B}=\frac{p}{q} \quad , \quad \frac{BA'}{A'C}=\frac{q}{r} \quad , \quad \frac{CB'}{B'A}=\frac{r}{p} \; .$$

10. Conics inscribed in a trapezium

We start with a parallelogram. Genuine conics κ inscribed in a parallelogram $p=ABCD$ must be central. The diagonals of the parallelogram are conjugate diameters of κ. To simplify calculations we set the vertex A at the origin and the side AB on the $x-$axis. The inscribed conics are created by drawing a line $\alpha=KL$ parallel to the diagonal BD and considering the conic tangent to $\{\beta=AB, \gamma=AD\}$ respectively at L, K and passing through the symmetric L' of L w.r.t. the center of the parallelogram (see Figure 25). The Möbius transformation h_A in this case takes a simple form. Taking into account that

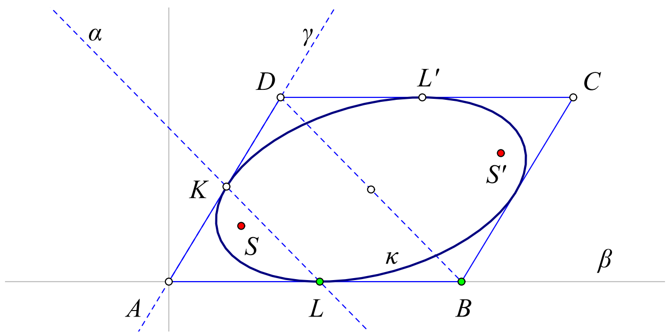

Figure 25. Conic κ inscribed in the parallelogram $ABCD$

the points $\{D, K, A\}$ map via h_A to $\{\infty, A, L\}$, using the special position of $ABCD$, and identifying points with complex numbers we see easily that

$$(17) \qquad h_A(z) = B \cdot \frac{z-K}{z-D}\ , \quad \text{with fixed points satisying} \quad z^2-(B+D)z+BK=0\ .$$

Since $B+D=C$, this reduces to $z^2-Cz+BK=0$ and an easy location of the focal points.

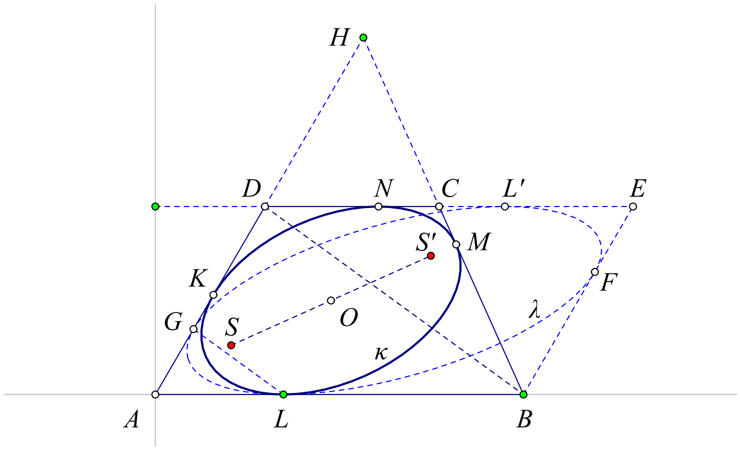

FIGURE 26. Conic κ inscribed in a trapezium

The case of trapezium is also in some sense universal, since it occurs also in the case of polygons $p = ABC\dots$ with three consecutive sides $\{CD, DA, AB, \dots\}$ from which $\{CD, AB\}$ happen to be parallel. Any other side of the polygon complements the three sides to a trapezium. Using the same coordinate system as before, the homography h_A maps again the triple $\{D, K, A\}$ to $\{\infty, A, L\}$ and leads to an analogous equation

$$(18) \qquad w = \frac{DL}{K} \cdot \frac{z-K}{z-D} \quad \Leftrightarrow \quad w = sB \cdot \frac{z-K}{z-D} \quad \text{where} \quad \frac{D}{K} = s\frac{B}{L}\ .$$

This leads to equation $z^2-(D+sB)z+sBK=0$ for the focal points, which reduces to (17) in the case of parallelograms, for which the real number $s=1$.

Figure 26 suggests the way to obtain all conics inscribed in a trapezium by reducing the problem to that of parallelograms. In fact, $q = ABED$ is the parallelogram created by extending the side DC of the trapezium. Having the inscribed in the trapezium conic κ, the *"affinity"*([11, p.199]) g of the plane fixing the points $\{D, A, B\}$ and mapping C to E maps κ to a conic $\lambda = g(\kappa)$ inscribed in the parallelogram and tangent to AB at the same point L with κ. We obtain all conics κ as $g^{-1}(\lambda)$ from the conics λ inscribed in the parallelogram q and from equation (18) results a convenient way to determine their focal points.

11. Triangles from three Möbius transformations

From our discussion so far follows that each polygon $p = ABC\dots$ circumscribed to a conic defines some series of commuting Möbius transformations $\{h_A, h_B, \dots\}$, sharing the same fixed points and satisfying the condition (9) of section 5. Also an easy calculation using the common fixed points shows that the invariant k_{BA} of a composition $h_B \circ h_A$ is equal to the product of the respective invariants $k_{BA} = k_B \cdot k_A$.

All these facts rise the question of a converse possibility, to construct a polygon from a finite series $\{h_1, \dots h_n\}$ of Möbius transformations satisfying these compatibility conditions. We postpone the examination of the general case to a future occasion and handle in this and the next section the case of triangles which answers the question in the affirmative. Here we consider Möbius transformations with distinct fixed points *"lying in the* *finite plane"*, i.e. different from ∞, which lead to central conics. In the next section we consider transformations still with two distinct fixed points, one of which however is ∞, leading to parabolas.

Theorem 11.1. *Three non-involutive Möbius transformations $\{h_1, h_2, h_3\}$ with distinct common fixed points $\{S, S'\}$ lying in the finite plane and invariants $\{k_1, k_2, k_3\}$ satisfying $k_1k_2k_3 = 1$, define a triangle ABC and an inscribed in it central conic with focal points $\{S, S'\}$. Conversely, every central conic inscribed in a triangle results from a triple of Möbius transformations satisfying the above conditions.*

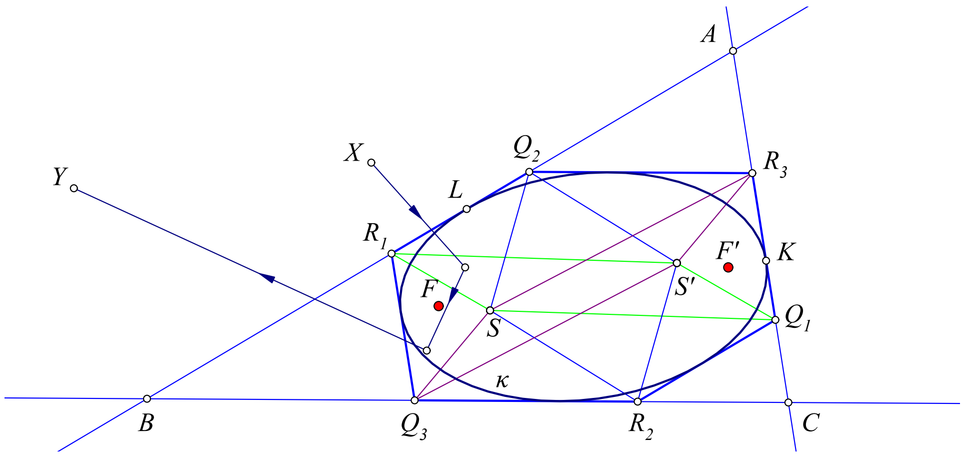

Figure 27. Configuration of three pairwise commuting Möbius transformations

Proof. We show the first part of the theorem, since the converse part follows from the preceding discussion.

Figure 27 shows a configuration resulting from three commuting Möbius transformations $\{h_1, h_2, h_3\}$, whose invariants do not satisfy the condition $k_1k_2k_3 = 1$. This implies that $h = h_3 \circ h_2 \circ h_1 \neq e$ is not the identity. For an X different from the assumed common fixed points $\{S, S'\}$ of these transformations we have then $Y = h(X) \neq X$, as this is suggested by the figure.

The limit points $\{(Q_i, R_i)$ of $h_i, i = 1, 2, 3\}$ define three *characteristic parallelograms* and these through their vertices define a symmetric hexagon $R_1Q_3R_2Q_1R_3Q_2$. By the inverse of Brianchon's theorem ([1, p.66]) there is a conic κ inscribed in this hexagon. The hexagon defines through its non-consecutive side-lines two congruent, symmetric w.r.t. to the center of the conic, triangles and we stick to one of them ABC as shown in the figure.

The conic κ, in general, has focal points $\{F, F'\}$ different from the common points $\{S, S'\}$ of our initial transformations $\{h_1, h_2, h_3\}$. We define the Möbius transformations $\{h_A, h_B, h_C\}$, as we have done in the preceding sections. For example h_A will map, through the tangents of κ, points $\{K, A, C\}$ of line AC correspondingly to points $\{A, L, B\}$ of line AB, with fixed points $\{F, F'\}$ and limit points $\{Q_1, R_1\}$, the same with those of h_1. We show, that if $k_1k_2k_3 = 1$, then the fixed points $\{F, F'\}$ coincide with $\{S, S'\}$ and consequently the maps coincide too, i.e. $\{h_1 = h_A, h_2 = h_B, h_3 = h_C\}$. For example we'll then have $h_1 = h_A$, because the two maps will share the same fixed points and will coincide at Q_1 and R_1.

Assuming that $\{m_1, m_2, m_3\}$ are respectively the invariants of $\{h_A, h_B, h_C\}$, we have the expressions through the fixed points

$$k_i = \frac{S - Q_i}{S' - Q_i} \quad \text{and} \quad m_i = \frac{F - Q_i}{F' - Q_i} \quad \text{for } i = 1, 2, 3\ .$$

Because of the symmetry w.r.t. the center of the conic, which we may assume at the origin, we have $S' = -S$ and

$$\text{there is an} \quad x \in \mathbb{C} \quad \text{with} \quad F = S + x \quad \text{and} \quad F' = -S - x \ .$$

The valid condition $m_1 m_2 m_3 = 1$ implies

$$(F' - Q_1)(F' - Q_2)(F' - Q_3) = (F - Q_1)(F - Q_2)(F - Q_3) \quad \Rightarrow$$
$$-(S + x + Q_1)(S + x + Q_2)(S + x + Q_3) = (S + x - Q_1)(S + x - Q_2)(S + x - Q_3) \quad \Rightarrow$$
$$2(x + S)(x^2 + 2Sx + S^2 + Q_1Q_2 + Q_2Q_3 + Q_3Q_1) = 0 \ . \tag{19}$$

The assumed condition $k_1 k_2 k_3 = 1$ implies

$$(S' - Q_1)(S' - Q_2)(S' - Q_3) = (S - Q_1)(S - Q_2)(S - Q_3) \quad \Rightarrow$$
$$-(S + Q_1)(S + Q_2)(S + Q_3) = (S - Q_1)(S - Q_2)(S - Q_3) \quad \Rightarrow$$
$$2S(S^2 + Q_2Q_3 + Q_1Q_3 + Q_1Q_2) = 0 \ . \tag{20}$$

The two equations imply, taking into account our assumption $S \neq 0$:

$$(x + S)(x^2 + 2Sx) = 0 \quad \Rightarrow \quad x = -S \quad \text{or} \quad x = 0 \quad \text{or} \quad x = -2S \ .$$

The first solution $x = -S$ is rejected since it produces $F = F' = 0$ i.e. the case κ being a circle, which we have excluded. The second $x = 0$ implies the coincidence of $\{S, S'\}$ with $\{F, F'\}$ and the third $x = -2S$ is not acceptable producing $\{F = -S, F' = -3S\}$ non-symmetric w.r.t. to the origin. Thus, the only acceptable solution $x = 0$ leads to identification of $\{S, S'\}$ with $\{F, F'\}$ and completes the proof of the theorem for the case of central conics.

12. Parabolas from three similarities

There are three basic facts differentiating the case of parabolas from the central conics with regard to circumscribed polygons and in particular triangles. The first is the existence of one only focal point lying in the finite plane, the fact that the Möbius transformations $\{h_A, h_B, \ldots\}$ are similarities, and the fact ([1, p.22]) that *the circumcircle κ of the circumscribed to a parabola triangle ABC passes through the focus S of the parabola* (see Figure 28).

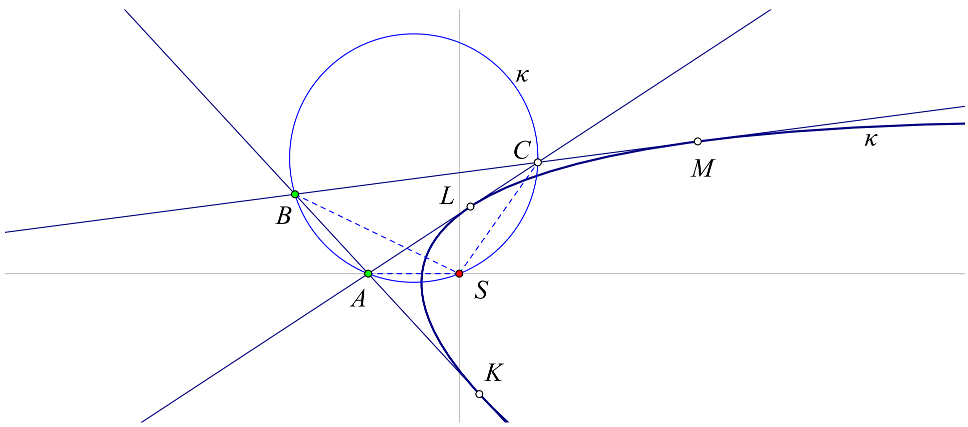

Figure 28. The circumcircle of $\triangle ABC$ passes through the focus S

Our discussion so far could suggest the conjecture that three similarities $\{h_1, h_2, h_3\}$ possessing a common fixed point S define a triangle and a parabola inscribed in it. But this is not so. There is one additional condition for the $\{h_i\}$ that must be satisfied in order to be able to define through them a triangle and an inscribed in it parabola. The additional condition is due to the third basic fact, stated above, and leads to the following theorem.

Theorem 12.1. *Three similarities* $\{h_i(z) = a_i(z - z_0) + z_0, i = 1, 2, 3\}$*, expressed with complex numbers, define a triangle* ABC *and an inscribed in it parabola with focus* z_0*, if and only if* $a_1a_2a_3 = 1$ *and* $(1 - a_1a_3)/(a_3 - a_1a_3) \in \mathbb{R}$*, i.e.* $\{1, a_3, a_1a_3\}$ *are collinear.*

Proof. In section 8 we saw the necessity of the condition $a_1a_2a_3 = 1$. Assuming $z_0 = 0$, the necessity of the other relation results from the fact that the four points $S(0)$, C and $B = a_1C$, $A = a_1a_3C$ are on a circle κ, hence their cross ratio is real

$$\mathbb{R} \ni (SC; BA) = \frac{S - B}{C - B} : \frac{S - A}{C - A} = \frac{0 - a_1C}{C - a_1C} : \frac{0 - a_1a_3C}{C - a_1a_3C} = \frac{1 - a_1a_3}{a_3 - a_1a_3} .$$

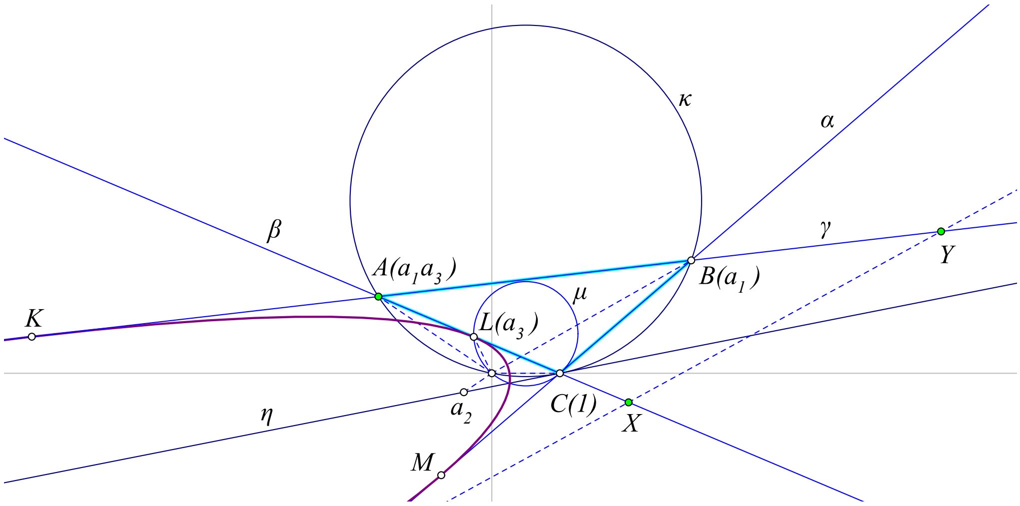

Figure 29. Generating an inscribed parabola from three similarities

To proceed to the converse we use the configuration of figure 29 which shows the location of the complex numbers $\{a_1, a_2, a_3\}$ satisfying the given conditions and a possible definition of the triangle ABC. Because of the general properties of similarities, cited in section 8, applying successively the transformations to any point z of the plane we obtain triangles which are pairwise similar. Thus, we can define the triangle ABC by starting from $z = 1$

$$C = 1 \ , \ B = h_1(C) = a_1 \ , \ A = h_3(B) = a_1a_3 \ .$$

By the general properties of similarities the points $X \in AC$ map to points $Y \in AB$ and the line XY envelopes a parabola with focus at the origin and for appropriate positions of X the tangent XY coincides with the side-lines of the triangle. Analogous arguments for h_2, h_3 lead to the same parabola inscribed in ABC and prove the theorem.

We notice some properties in figure 29, whose simple proofs are left as exercises:

(1) If we fix the location of a_1, hence the location of the circle $\kappa = (01a_1)$, then the possible a_2 satisfying the second relation vary on the line $\eta : a_2 = (1 - t) + t\frac{1}{a_1}$.
(2) This line is tangent to κ at $z = 1$.
(3) Point a_3 varies on a circle μ through $\{0, 1\}$ tangent to the line BC at $C = 1$.
(4) The similarity h_1 maps the circle μ onto κ.
(5) The collinearity of one of the triples $\{(1, a_1, a_1a_2), (1, a_2, a_2a_3), (1, a_3, a_1a_3)\}$ implies the collinearity of the other two.
(6) Point a_2a_3 is the other than 1 intersection of line η and circle μ and $M = a_1a_2$.

References

[1] Akopyan, A. and Zaslavsky, A., *Geometry of Conics*, American. Math. Soc., New York, 2012.
[2] Badertscher, E., *A Simple Direct Proof of Marden's Theorem*, Amer, Math. Monthly, **121(2014)** 547-548.
[3] Baker, H., *An Introduction to plane geometry*, Chelsea publishing company, New York, 1971.
[4] Berger, M., *Geometry I, II*, Springer Verlag, Heidelberg, 1987.
[5] Barker, W. and Howe, R., *Continuous Symmetry, From Euclid to Klein*, American Math. Soc., New York, 2007.
[6] Bogosel, b., *A Geometric Proof of the Siebeck-Marden Theorem*, Amer. Math. Monthly, **124(2017)** 459-463.
[7] Carlson, F., *http://www.su.se/polopoly$_f$s/*1.229312.1426783194!*/menu/standard/file/marden.pdf*, 1943.
[8] Casey, J., *A treatise on the analytical geometry of the point, line, circle and conic sections*, Hodges Figgis and Co., Dublin, 1893.
[9] Chasles, M., *Traite de Sections Coniques*, Gauthier-Villars, Gauthier-Villars, Paris, 1865.
[10] Clebsch, A., *Vorlesungen über Geometrie*, Teubner, Leipzig, 1876.
[11] Coxeter, H., *Introduction to Geometry*, John Wiley and Sons Inc., New York, 1961.
[12] Deaux, R., *Introduction to the Geometry of Complex Numbers*, Dover, New York, 1956.
[13] Filon, L., *An introduction to Projective Geometry*, Edward Arnold, London, 1909.
[14] Gallatly, W., *The modern geometry of the triangle*, Francis Hodgsonn, London, 2013.
[15] Glaeser, G. and Stachel, H. and Odenhal, B., *The Universe of Conics*, Springer, New York, 2016.
[16] Johnson, R. *Advanced Euclidean Geometry*, Dover Publications, New York, 1960.
[17] Kalman, D., *An Elementary Proof of Marden's Theorem*, Amer. Math. Monthly, **115(2008)** 330-338.
[18] Kalman, D., *The Most Marvelous Theorem in Mathematics*, Math Horizons, **15(2008)** 16-17.
[19] Loney, S., *The elements of coordinate geometry I, II*, AITBS publishers, Delhi, 1991.
[20] Marden, M., *A note on the zeroes of the sections of a partial fraction*, Bulletin of the Amer. Math. Soc., **51(1945)** 935-940.
[21] Pamfilos, P., *Barycentric coordinates* , http://users.math.uoc.gr/∼pamfilos/ eGallery/problems/Quadratics.pdf, 2021.
[22] Pamfilos, P., *Homographic relation* , http://users.math.uoc.gr/∼pamfilos/ eGallery/problems/HomographicRelation.pdf, 2022.
[23] Pamfilos, P., *Projective Line* , http://users.math.uoc.gr/∼pamfilos/ eGallery/problems/Projective Line.pdf, 2022.
[24] Pamfilos, P. and Thoma, A., *Apollonian Cubics: An Application of Group Theory to a Problem in Euclidean Geometry*, Math. Magazine,
[25] Rouse, E., *Director Circle of a Conic Inscribed in a Triangle*, The Mathematical Gazette, **2(1894)** 12-13.
[26] Salmon, G., *A treatise on Conic Sections*, Longmans, Green and Co., London, 1917.
[27] Schwerdtfeger, H., *Geometry of complex numbers*, Dover, New York, 1979.
[28] Siebeck, H., *Die Brennpunkte eines Kegelschnitts als solche Punkte der Ebene aufgefasst, in welchen je zwei entsprechende Punkte zweier kreisverwandter Systeme vereinigt sind*, Archiv der Mathematik und Physik, **33(1859)** 462-474.
[29] Oswald Veblen, O. and Young, J., *Projective Geometry vol. I, II*, Ginn and Company, New York, 1910.
[30] Yaglom, I., *Geometric Transformations I, II, III, IV*, Math. Assoc. of America, Washington, 1962.
[31] Yiu, P., *Introduction to the Geometry of the Triangle*, http://math.fau. edu/Yiu/Geometry.html, 2013.

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