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# ISOMETRY GROUPS OF THE SPACES OF TRUNCATED PENTAKIS DODECAHEDRON AND TRUNCATED TRIAKIS ICOSAHEDRON 

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#### Abstract

Many authors have been studied on the relations between Minkowski geometry and convex solids especially polyhedra. In this paper we introduce two new Minkowski geometries derived by $d_{T P D}-$ metric and $d_{T T I}$-metric which unit spheres are truncated pentakis dodecahedron and truncated triakis icosahedron, respectively. These solids are in the class Truncated Catalan solids. We also give some properties of $d_{T P D}-$ metric and $d_{T T I}$-metric and we show that the group of isometries of the 3 -dimensional space covered by $d_{T P D}$-metric or $d_{T T I}$-metric is the semi-direct product of icosahedral group $I_{h}$ the (Euclidean) symmetry group of the icosahedron and $T(3)$ the group of all translations of the 3 -dimensional space.


## 1. Introduction

Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions with the same linear structure of the Euclidean one but distance is not uniform in all directions. What is meant by the similarity of the linear structure is the points, lines and planes are the same and angles are measured in the same way. But changing the distance function changes the concepts related to the distance, for instance instead of the usual sphere in Euclidean space, unit ball is a general symmetric convex set. (See [1] and [2]). Although the convex set theory is ancient it is one of the most interesting and classical field of modern matematics due to rich applications. Convex set theory developed geometrically by introducing some notions, but primarily polyhedra. A simple definition of a polyhedron would be given as the finite, connected set of plane polygons and it is possible to obtain polyhedra by many different ways. For more detail see [3] and [4].

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Polyhedra, like polygons, may be convex or non-convex. A polyhedron is an extremely important special solid in $\mathbb{R}^{n}$ especially when it is convex. Convex polyhedra are mainly classified as Platonic, Archimedean and Catalan solids.Regular convex polyhedra consist of just one type of regular polygon and they are named as Platonic solids, semi-regular convex polyhedra which are called Archimedean solids consist of two or more different types of regular polygons and Catalan solids are dual polyhedra of Archimedean solids and their faces are not regular polygons (see $[3],[15]$ ). By the studies on metric geometry in 3 -dimensional space it has seen that metrics and convex polyhedra are closely related. 3-dimensional analytical space covered with maximum and taxicab metrics are Minkowski geometries whose unit spheres are cube and octahedron, respectively, which are two of Platonic Solids. The taxicab (Manhattan) and the maximum (Chebyshev) norms are defined as $\|X\|_{1}=|x|+|y|+|z|$ and $\|X\|_{\infty}=\max \{|x|,|y|,|z|\}$, respectively and they are special cases of $l_{p}$-norm; $\|X\|_{p}=\left(|x|^{p}+|y|^{p}+|z|^{p}\right)^{1 / p}$, where $X=(x, y, z) \in \mathbb{R}^{3}$. Among $l_{p}$-metrics only crystalline metrics, i.e., metrics having polygonal unit balls are $l_{1}-$ and $l_{\infty}-$ metrics [5]. Another metric is $C C$-metric which is defined as

$$
d_{C C}\left(P_{1}, P_{2}\right)=d_{L}\left(P_{1}, P_{2}\right)+(\sqrt{2}-1) d_{S}\left(P_{1}, P_{2}\right)
$$

where

$$
d_{L}\left(P_{1}, P_{2}\right)=\max \left\{\left|x_{1}-x_{2}\right|,\left|y_{1}-y_{2}\right|,\left|z_{1}-z_{2}\right|\right\}
$$

$d_{S}\left(P_{1}, P_{2}\right)=\min \left\{\left|x_{1}-x_{2}\right|+\left|y_{1}-y_{2}\right|,\left|x_{1}-x_{2}\right|+\left|z_{1}-z_{2}\right|,\left|y_{1}-y_{2}\right|+\left|z_{1}-z_{2}\right|\right\}$, $P_{1}=\left(x_{1}, y_{1}, z_{1}\right)$ and $P_{2}=\left(x_{2}, y_{2}, z_{2}\right)$. Thus Minkowski geometry obtained by covering 3 -dimensional analytical space with $C C$-metric has unit sphere as a deltoidal icositetrahedron, a Catalan Solid. There are many studies in the literature on relations between convex polyhedra and metrics. (See [7], [8], [9], [11], [12], [13], [14], [17], [19], [20], [21], [22]). In the light of the mentioned studies, in this study we first introduce two new metrics, and showed that the spheres of the 3 -dimensional analytical space covered by these metrics are truncated pentakis dodecahedron and truncated triakis icosahedron then we give some properties of these metrics. Furthermore we show that the group of isometries of the 3 -dimensional space covered by $T P D$-metric or $T T I$-metric is the semi-direct product of $I_{h}$ and $T(3)$, where icosahedral group $I_{h}$ is the (Euclidean) symmetry group of icosahedron and $T(3)$ is the group of all translations of the 3 -dimensional space.

## 2. TRUNCATED PENTAKIS DODECAHEDRON METRIC AND SOME PROPERTIES

There are many different operations to obtain a polyhedra. For example eleven of Arhimedean polyhedra can be obtained from Platonic solids by truncation. To truncate means to cut off the vertices [6]. Truncated pentakis dodecahedron is a convex solid obtained by truncating pentakis dodecahedron. Pentakis dodecahedron is a Catalan solid whose faces consist of 60 isosceles triangles and truncated pentakis dodecahedron is a convex solid
with 60 mirror-symmetric pentagonal and 20 regular hexagonal faces, 132 vertices and 210 edges.


Figure 1: Pentakis Dodecahedron and Truncated Pentakis Dodecahedron
First we give some notions that will be used in the descriptions of distance functions we define. For $P_{1}=\left(x_{1}, y_{1}, z_{1}\right), P_{2}=\left(x_{2}, y_{2}, z_{2}\right) \in \mathbb{R}^{3}, M$ denotes $\left\|P_{1}-P_{2}\right\|_{\infty}$ and $S$ denotes $\left\|P_{1}-P_{2}\right\|_{1}$. Moreover $X-Y-Z-X$ and $Z-Y-X-Z$ orientations are called positive $(+)$ direction and negative (-) direction, respectively. $M^{+}$and $M^{-}$expresses the next term in the respective direction according to $M$. For example, if $M=\left|x_{1}-x_{2}\right|$, then $M^{+}=\left|y_{1}-y_{2}\right|$ and $M^{-}=\left|z_{1}-z_{2}\right|$. The metric for which the unit sphere is the truncated tetrakis hexahedron is defined as following:

Definition 2.1. The distance function $d_{T P D}: \mathbb{R}^{3} \times \mathbb{R}^{3} \longrightarrow[0, \infty)$ which is defined by

$$
d_{T P D}\left(P_{1}, P_{2}\right)=\max \left\{\begin{array}{c}
\frac{3+\sqrt{5}}{6} M+\frac{\sqrt{5}-1}{3} M^{+}+\frac{1}{3} M^{-} \\
\frac{\sqrt{5}}{3} M+\frac{2}{3} M^{-}+\frac{\sqrt{5}-1}{6} M^{+}, \\
M+\frac{\sqrt{5}-1}{6} M^{+}, a M+b M^{-}, c S
\end{array}\right\}
$$

where $a=\sqrt{3}\left(\frac{5 \sqrt{5}-13}{33}\right)+\frac{3+9 \sqrt{5}}{22}, b=\sqrt{3}\left(\frac{14 \sqrt{5}-32}{33}\right)+\frac{6 \sqrt{5}-9}{11}$ and $c=$ $\sqrt{6}\left(\frac{53-25 \sqrt{5}}{13}\right)+\frac{155 \sqrt{3}-177 \sqrt{2}-243}{26}+\sqrt{5}\left(\frac{123+83 \sqrt{2}-73 \sqrt{3}}{26}\right)$ is called the truncated pentakis dodecahedron distance between $P_{1}$ and $P_{2}$, where $P_{1}=\left(x_{1}, y_{1}, z_{1}\right)$, $P_{2}=\left(x_{2}, y_{2}, z_{2}\right) \in \mathbb{R}^{3}$.

There are five different paths from $P_{1}$ to $P_{2}$ with the same length with respect to the truncated pentakis dodecaheron distance. These paths are
i) union of three line segments each of them is parallel to a coordinate axis,
ii) union of two line segments one of which is parallel to a coordinate axis and other line segment makes $\arctan \left(\frac{\sqrt{5}}{2}\right)$ angle with another coordinate axis.
iii) union of two line segments one of which is parallel to a coordinate axis and other line segment makes $\arctan \left(\frac{5+4 \sqrt{5}}{6}\right)$ angle with another coordinate axis.
$i v)$ union of three line segments one of which is parallel to a coordinate axis, one of which makes $\arctan \left(\frac{\sqrt{5}}{20}\right)$ angle with another coordinate axis and the last line segment makes $\arctan \left(\frac{10+3 \sqrt{5}}{10}\right)$ angle with the last coordinate axis.
$v)$ union of three line segments one of which is parallel to a coordinate axis, one of which makes $\arctan \left(\frac{10-3 \sqrt{5}}{4}\right)$ angle with another coordinate axis and the last line segment makes arctan $\left(\frac{\sqrt{5}}{2}\right)$ angle with the last coordinate axis.

Thus the truncated tetrakis hexahedron distance between $P_{1}$ and $P_{2}$ is for $(i) c$ times of the sum of Euclidean lengths of the three line segments, for (ii) a times of the sum of Euclidean lengths of the two line segments, for (iii) the sum of Euclidean lengths of the two line segments, for (iv) $\frac{\sqrt{5}}{2}$ times of the sum of Euclidean lengths of the three line segments and for (v) $\frac{3+\sqrt{5}}{6}$ times of the sum of Euclidean lengths of the three line segments, where $a=\sqrt{3}\left(\frac{5 \sqrt{5}-13}{33}\right)+\frac{3+9 \sqrt{5}}{22}$ and $c=\sqrt{6}\left(\frac{53-25 \sqrt{5}}{13}\right)+\frac{155 \sqrt{3}-177 \sqrt{2}-243}{26}+$ $\sqrt{5}\left(\frac{123+83 \sqrt{2}-73 \sqrt{3}}{26}\right)$. Figure 2 illustrates the truncated tetrakis hexahedron path from $P_{1}$ to $P_{2}$ if maximum value is $c\left(\left|x_{1}-x_{2}\right|+\left|y_{1}-y_{2}\right|+\left|z_{1}-z_{2}\right|\right)$, $a\left|y_{1}-y_{2}\right|+b\left|x_{1}-x_{2}\right|,\left|y_{1}-y_{2}\right|+\frac{\sqrt{5}-1}{6}\left|z_{1}-z_{2}\right|, \frac{\sqrt{5}}{3}\left|y_{1}-y_{2}\right|+\frac{2}{3}\left|x_{1}-x_{2}\right|+$ $\frac{\sqrt{5}-1}{6}\left|z_{1}-z_{2}\right|$ or $\frac{3+\sqrt{5}}{6}\left|y_{1}-y_{2}\right|+\frac{\sqrt{5}-1}{3}\left|z_{1}-z_{2}\right|+\frac{1}{3}\left|x_{1}-x_{2}\right|$.


Figure 2: Some TPD way from $P_{1}$ to $P_{2}$
Lemma 2.1. Let $P_{1}=\left(x_{1}, y_{1}, z_{1}\right), P_{2}=\left(x_{2}, y_{2}, z_{2}\right) \in \mathbb{R}^{3}$ be distinct two points, $M=\left\|P_{1}-P_{2}\right\|_{\infty}$ and $S=\left\|P_{1}-P_{2}\right\|_{1}$. Then

$$
\begin{aligned}
& d_{T P D}\left(P_{1}, P_{2}\right) \geq \frac{3+\sqrt{5}}{6} M+\frac{\sqrt{5}-1}{3} M^{+}+\frac{1}{3} M^{-} \\
& d_{T P D}\left(P_{1}, P_{2}\right) \geq \frac{\sqrt{5}}{3} M+\frac{2}{3} M^{-}+\frac{\sqrt{5}-1}{6} M^{+} \\
& d_{T P D}\left(P_{1}, P_{2}\right) \geq M+\frac{\sqrt{5}-1}{6} M^{+} \\
& d_{T P D}\left(P_{1}, P_{2}\right) \geq a M+b M^{-} \\
& d_{T P D}\left(P_{1}, P_{2}\right) \geq c S
\end{aligned}
$$

where $a=\sqrt{3}\left(\frac{5 \sqrt{5}-13}{33}\right)+\frac{3+9 \sqrt{5}}{22}, b=\sqrt{3}\left(\frac{14 \sqrt{5}-32}{33}\right)+\frac{6 \sqrt{5}-9}{11}$ and $c=$ $\sqrt{6}\left(\frac{53-25 \sqrt{5}}{13}\right)+\frac{155 \sqrt{3}-177 \sqrt{2}-243}{26}+\sqrt{5}\left(\frac{123+83 \sqrt{2}-73 \sqrt{3}}{26}\right)$
Proof. Proof would be obtained trivially by the definition of maximum function.

Theorem 2.1. The distance function $d_{T P D}$ is a metric. Furthermore according to $d_{T P D}$, the unit sphere is a truncated pentakis dodecahedron in $\mathbb{R}^{3}$.

Proof. To prove $d_{T P D}$ is a metric it must be shown that metric axioms are satisfied by $d_{T P D}$. By using the definition of the distance function $d_{T P D}$, properties of absolute value metric and Lemma 2.1 it would easily seen that $d_{T P D}$ is a metric.

Finally, the set of all points $X=(x, y, z) \in \mathbb{R}^{3}$ that truncated pentakis dodecahedron distance is 1 from $O=(0,0,0)$ is

$$
S_{T P D}=\left\{(x, y, z): \max \left\{\begin{array}{c}
\frac{3+\sqrt{5}}{6} M+\frac{\sqrt{5}-1}{3} M^{+}+\frac{1}{3} M^{-}, \\
\frac{\sqrt{5}}{3} M+\frac{2}{3} M^{-}+\frac{\sqrt{5}-1}{6} M^{+}, \\
M+\frac{\sqrt{5}-1}{6} M^{+}, a M+b M^{-}, c S
\end{array}\right\}=1\right\}
$$

Thus the graph of $S_{T P D}$, the unit sphere in terms of $d_{T P D}$ is as in the Figure 3 :


Figure 3: The $S_{T P D}$ : Truncated pentakis dodecahedron
Corollary 2.1. A sphere of the truncated pentakis dodecahedron space with center $X_{0}=\left(x_{0}, y_{0}, z_{0}\right)$ and radius $r$ is

$$
\max \left\{\begin{array}{c}
\frac{3+\sqrt{5}}{6} M_{0}+\frac{\sqrt{5}-1}{3} M_{0}^{+}+\frac{1}{3} M_{0}^{-}, \frac{\sqrt{5}}{3} M_{0}+\frac{2}{3} M_{0}^{-}+\frac{\sqrt{5}-1}{6} M_{0}^{+}, \\
M_{0}+\frac{\sqrt{5}-1}{6} M_{0}^{+}, a M_{0}+b M_{0}^{-}, c S_{0}
\end{array}\right\}=r
$$

which is a polyhedron with 80 faces, 132 vertices and 210 edges, where $a=\sqrt{3}\left(\frac{5 \sqrt{5}-13}{33}\right)+\frac{3+9 \sqrt{5}}{22}, b=\sqrt{3}\left(\frac{14 \sqrt{5}-32}{33}\right)+\frac{6 \sqrt{5}-9}{11}, c=\sqrt{6}\left(\frac{53-25 \sqrt{5}}{13}\right)+$ $\frac{155 \sqrt{3}-177 \sqrt{2}-243}{26}+\sqrt{5}\left(\frac{123+83 \sqrt{2}-73 \sqrt{3}}{26}\right), M_{0}=\left\|X-X_{0}\right\|_{\infty}, S_{0}=\left\|X-X_{0}\right\|_{1}$ and $X=(x, y, z)$. Coordinates of the vertices are the translation of all vertices of $S_{T P D}$ to $\left(x_{0}, y_{0}, z_{0}\right)$ with all cyclic permutation of the three axis components and all possible $+/-$ sign changes of each axis component of $\left(0, C_{1} r, r\right),\left(C_{2} r, C_{3} r, C_{4} r\right),\left(C_{5} r, 0, C_{6} r\right),\left(C_{2} r, C_{7} r, C_{8} r\right),\left(0, C_{9} r, C_{10} r\right)$, $\left(C_{11} r, C_{12} r, C_{13} r\right)$ and ( $\left.C_{14} r, C_{15} r, C_{16} r\right)$, where $C_{1}=\frac{10 \sqrt{15-8 \sqrt{3}}}{109}-\frac{21+\sqrt{5}}{218}$, $C_{2}=\frac{30 \sqrt{5}-24}{109}-\frac{21 \sqrt{3}+\sqrt{15}}{218}, C_{3}=\frac{49 \sqrt{15}-61 \sqrt{3}}{654}+\frac{21+\sqrt{5}}{218}, C_{4}=\frac{5 \sqrt{15}-4 \sqrt{3}}{327}+$ $\frac{80+9 \sqrt{5}}{109}, C_{5}=\frac{6 \sqrt{5}-3}{19}, C_{6}=\frac{27+3 \sqrt{5}}{38}, C_{7}=\frac{9 \sqrt{15}-29 \sqrt{3}}{218}+\frac{105+5 \sqrt{5}}{218}, C_{8}=$ $\frac{22-4 \sqrt{3}+27 \sqrt{5}+5 \sqrt{15}}{109}, C_{9}=\frac{8 \sqrt{15}-50 \sqrt{3}}{327}+\frac{147+7 \sqrt{5}}{218}, C_{10}=\frac{20 \sqrt{15}-16 \sqrt{3}}{327}+\frac{36 \sqrt{5}-7}{109}$, $C_{11}=\frac{10 \sqrt{30}-8 \sqrt{6}+21 \sqrt{3}+\sqrt{15}}{109}-\frac{\sqrt{10}+21 \sqrt{2}+11 \sqrt{5}+13}{218}, C_{12}=\frac{113+29 \sqrt{3}-9 \sqrt{15}-5 \sqrt{5}}{218}$, $C_{13}=\frac{4 \sqrt{3}-5 \sqrt{15}}{109}+\frac{65+55 \sqrt{5}}{218}, C_{14}=\frac{29 \sqrt{3}}{654}+\frac{71 \sqrt{5}-3 \sqrt{15}-35}{218}, C_{15}=\frac{46 \sqrt{3}-3 \sqrt{15}}{327}+$ $\frac{13+11 \sqrt{5}}{218}$ and $C_{16}=\frac{53 \sqrt{3}}{654}+\frac{17 \sqrt{5}-13 \sqrt{15}+139}{218}$.
Lemma 2.2. Denote by $l$ the Euclidean line passing through the points $P_{1}=\left(x_{1}, y_{1}, z_{1}\right)$ and $P_{2}=\left(x_{2}, y_{2}, z_{2}\right)$ in the analytical 3 -dimensional space
and by $d_{E}$ the Euclidean metric. If direction vector of $l$ is $(p, q, r)$, then

$$
d_{T P D}\left(P_{1}, P_{2}\right)=\mu\left(P_{1} P_{2}\right) d_{E}\left(P_{1}, P_{2}\right)
$$

where
$\mu\left(P_{1} P_{2}\right)=\frac{\max \left\{\begin{array}{c}\frac{3+\sqrt{5}}{6} M_{d}+\frac{\sqrt{5}-1}{3} M_{d}^{+}+\frac{1}{3} M_{d}^{-}, \frac{\sqrt{5}}{3} M_{d}+\frac{2}{3} M_{d}^{-}+\frac{\sqrt{5}-1}{6} M_{d}^{+}, \\ M_{d}+\frac{\sqrt{5}-1}{6} M_{d}^{+}, a M_{d}+b M_{d}^{-}, c S_{d}\end{array}\right\}}{\sqrt{p^{2}+q^{2}+r^{2}}}$,
$a=\sqrt{3}\left(\frac{5 \sqrt{5}-13}{33}\right)+\frac{3+9 \sqrt{5}}{22}, b=\sqrt{3}\left(\frac{14 \sqrt{5}-32}{33}\right)+\frac{6 \sqrt{5}-9}{11}, c=\sqrt{6}\left(\frac{53-25 \sqrt{5}}{13}\right)+$ $\frac{155 \sqrt{3}-177 \sqrt{2}-243}{26}+\sqrt{5}\left(\frac{123+83 \sqrt{2}-73 \sqrt{3}}{26}\right), M_{d}=\|P\|_{\infty}, S_{d}=\|P\|_{1}, P=$ $(p, q, r)$ and $O=(0,0,0)$.

Proof. Equation of $l$ gives $x_{1}-x_{2}=\lambda p, y_{1}-y_{2}=\lambda q, z_{1}-z_{2}=\lambda r, \lambda \in \mathbb{R}$. Thus,

$$
d_{T P D}\left(P_{1}, P_{2}\right)=|\lambda| \max \left\{\begin{array}{c}
\frac{3+\sqrt{5}}{6} M_{d}+\frac{\sqrt{5}-1}{3} M_{d}^{+}+\frac{1}{3} M_{d}^{-}, \\
\frac{\sqrt{5}}{3} M_{d}+\frac{2}{3} M_{d}^{-}+\frac{\sqrt{5}-1}{6} M_{d}^{+}, \\
M_{d}+\frac{\sqrt{5}-1}{6} M_{d}^{+}, a M_{d}+b M_{d}^{-}, c S_{d}
\end{array}\right\}
$$

where $a=\sqrt{3}\left(\frac{5 \sqrt{5}-13}{33}\right)+\frac{3+9 \sqrt{5}}{22}, b=\sqrt{3}\left(\frac{14 \sqrt{5}-32}{33}\right)+\frac{6 \sqrt{5}-9}{11}, c=\sqrt{6}\left(\frac{53-25 \sqrt{5}}{13}\right)+$ $\frac{155 \sqrt{3}-177 \sqrt{2}-243}{26}+\sqrt{5}\left(\frac{123+83 \sqrt{2}-73 \sqrt{3}}{26}\right), U_{d}=\|P\|_{\infty}, S_{d}=\|P\|_{1}$, and $d_{E}\left(P_{1}, P_{2}\right)=$ $|\lambda| \sqrt{p^{2}+q^{2}+r^{2}}$. By proportioning the resulting equations the required consequence is obtained.

The following corollaries are immediate consequences of the lemma above:
Corollary 2.2. If $P_{1}, P_{2}$ and $X$ are any three collinear points in $\mathbb{R}^{3}$, then $d_{E}\left(P_{1}, X\right)=d_{E}\left(P_{2}, X\right)$ if and only if $d_{T P D}\left(P_{1}, X\right)=d_{T P D}\left(P_{2}, X\right)$.

Corollary 2.3. If $P_{1}, P_{2}$ and $X$ are any three collinear points in $\mathbb{R}^{3}$, then

$$
d_{T P D}\left(X, P_{1}\right) / d_{T P D}\left(X, P_{2}\right)=d_{E}\left(X, P_{1}\right) / d_{E}\left(X, P_{2}\right)
$$

That is, the ratios of the Euclidean and $d_{T P D}$-distances along a line are the same.

## 3. TRUNCATED TRIAKIS ICOSAHEDRON METRIC AND SOME PROPERTIES

Truncated triakis icosahedron is a Truncated Catalan solid obtained by truncation operation from triakis icosahedron. The triakis icosahedron has 60 isosceles triangular faces and the truncated triakis icosahedron consists of 60 mirror-symmetric pentagonal and 12 regular decagonal faces, 140 vertices and 210 edges.


Figure 4: Triakis icosahedron, Truncated triakis icosahedron
The notations $M, M^{+}$and $M^{-}$will be used as defined in the previous section. The metric for which the unit sphere is the truncated triakis icosahedron is defined as follows:

Definition 3.1. The distance function $d_{T T I}: \mathbb{R}^{3} \times \mathbb{R}^{3} \longrightarrow[0, \infty)$ which is defined by
$d_{T T I}\left(P_{1}, P_{2}\right)=\max \left\{\begin{array}{c}\frac{2 \sqrt{5}}{5} M+\frac{3 \sqrt{5}-5}{10} M^{+}+\frac{\sqrt{5}}{5} M^{-}, \frac{5+\sqrt{5}}{10} M+\frac{5-\sqrt{5}}{10} M^{-}+\frac{\sqrt{5}}{5} M^{+}, \\ M+\frac{3 \sqrt{5}-5}{10} M^{-}, a M+b M^{+}\end{array}\right\}$
where $a=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{15-8 \sqrt{5}}{95}\right)+\frac{15+11 \sqrt{5}}{38}$ and $b=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{23 \sqrt{5}-55}{190}\right)+$ $\frac{10+\sqrt{5}}{19}$ is called the truncated triakis icosahedron distance between $P_{1}$ and $P_{2}$ that $P_{1}=\left(x_{1}, y_{1}, z_{1}\right)$ and $P_{2}=\left(x_{2}, y_{2}, z_{2}\right)$ are two points in $\mathbb{R}^{3}$.

There are four different paths from $P_{1}$ to $P_{2}$ with the same length with respect to the truncated triakis icosahedron distance. These paths are
$i)$ union of two line segments one of which is parallel to a coordinate axis and other line segment makes arctan ( $\frac{1}{2}$ ) angle with another coordinate axis,
ii) union of two line segments one of which is parallel to a coordinate axis and other line segment makes $\arctan \left(\frac{15+6 \sqrt{5}}{10}\right)$ angle with another coordinate axis,
iii) union of three line segments one of which is parallel to a coordinate axis, one of which makes $\arctan \left(\frac{3}{4}\right)$ angle with another coordinate axis and the last line segment makes $\arctan \left(\frac{9+5 \sqrt{5}}{8}\right)$ angle with the last coordinate axis.
$i v)$ union of three line segments one of which is parallel to a coordinate axis, one of which makes arctan $\left(\frac{1}{2}\right)$ angle with another coordinate axis and the last line segment makes arctan $\left(\frac{\sqrt{5}}{2}\right)$ angle with the last coordinate axis.

Thus the truncated tetrakis hexahedron distance between $P_{1}$ and $P_{2}$ is for (i) $a$ times of the sum of Euclidean lengths of the three line segments, for (ii) the sum of Euclidean lengths of the two line segments, for (iii) $\frac{2 \sqrt{5}}{5}$ times of the sum of Euclidean lengths of the three line segments and for (iv) $\frac{5+\sqrt{5}}{10}$ times of the sum of Euclidean lengths of the three line segments, where $a=$ $\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{15-8 \sqrt{5}}{95}\right)+\frac{15+11 \sqrt{5}}{38}$. Figure 2 illustrates the truncated tetrakis hexahedron path from $P_{1}$ to $P_{2}$ if maximum value is $a\left|y_{1}-y_{2}\right|+b\left|z_{1}-z_{2}\right|$, $\left|y_{1}-y_{2}\right|+\frac{3 \sqrt{5}-5}{10}\left|x_{1}-x_{2}\right|, \frac{2 \sqrt{5}}{5}\left|y_{1}-y_{2}\right|+\frac{\sqrt{5}}{5}\left|x_{1}-x_{2}\right|+\frac{3 \sqrt{5}-5}{10}\left|z_{1}-z_{2}\right|$ or $\frac{5+\sqrt{5}}{10}\left|y_{1}-y_{2}\right|+\frac{5-\sqrt{5}}{10}\left|x_{1}-x_{2}\right|+\frac{\sqrt{5}}{5}\left|z_{1}-z_{2}\right|$.


Figure 5: Some TTI ways from $P_{1}$ to $P_{2}$

Lemma 3.1. Let $P_{1}=\left(x_{1}, y_{1}, z_{1}\right)$ and $P_{2}=\left(x_{2}, y_{2}, z_{2}\right)$ be distinct two points in $\mathbb{R}^{3}, M=\left\|P_{1}-P_{2}\right\|_{\infty}, a=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{15-8 \sqrt{5}}{95}\right)+\frac{15+11 \sqrt{5}}{38}$ and $b=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{23 \sqrt{5}-55}{190}\right)+\frac{10+\sqrt{5}}{19}$. Then

$$
\begin{aligned}
& d_{T T I}\left(P_{1}, P_{2}\right) \geq \frac{2 \sqrt{5}}{5} M+\frac{3 \sqrt{5}-5}{10} M^{+}+\frac{\sqrt{5}}{5} M^{-} \\
& d_{T T I}\left(P_{1}, P_{2}\right) \geq \frac{5+\sqrt{5}}{10} M+\frac{5-\sqrt{5}}{10} M^{-}+\frac{\sqrt{5}}{5} M^{+} \\
& d_{T T I}\left(P_{1}, P_{2}\right) \geq M+\frac{3 \sqrt{5}-5}{10} M^{-} \\
& d_{T T I}\left(P_{1}, P_{2}\right) \geq a M+b M^{+}
\end{aligned}
$$

where $a=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{15-8 \sqrt{5}}{95}\right)+\frac{15+11 \sqrt{5}}{38}$ and $b=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{23 \sqrt{5}-55}{190}\right)+$ $\frac{10+\sqrt{5}}{19}$

Proof. Proof is trivial by the definition of maximum function.

Theorem 3.1. The distance function $d_{T T I}$ is a metric. Furthermore according to $d_{T T I}$, the unit sphere is a truncated triakis icosahedron in $\mathbb{R}^{3}$.

Proof. By using the definition of the distance function $d_{T T I}$, properties of absolute value metric and Lemma 3.1 proof would be done easily.

Finally, the set of points for which the truncated triakis icosahedron distance from the origin is 1 (the unit sphere with respect to the truncated triakis icosahedron distance) is

$$
S_{T T I}=\left\{(x, y, z): \max \left\{\begin{array}{c}
\frac{2 \sqrt{5}}{5} M+\frac{3 \sqrt{5}-5}{10} M^{+}+\frac{\sqrt{5}}{5} M^{-} \\
\frac{5+\sqrt{5}}{10} M+\frac{5-\sqrt{5}}{10} M^{-}+\frac{\sqrt{5}}{5} M^{+}, \\
M+\frac{3 \sqrt{5}-5}{10} M^{-}, a M+b M^{+}
\end{array}\right\}=1\right\}
$$

where $a=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{15-8 \sqrt{5}}{95}\right)+\frac{15+11 \sqrt{5}}{38}$ and $b=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{23 \sqrt{5}-55}{190}\right)+$ $\frac{10+\sqrt{5}}{19}$. Thus the graph of $S_{T T I}$, the unit sphere in terms of $d_{T T I}$ is as in the Figure 6:


Figure 6: The $S_{T T I}$ : Truncated triakis icosahedron
A sphere of the truncated triakis icosahedron space with center $X_{0}=\left(x_{0}, y_{0}, z_{0}\right)$ and radius $r$ is
$\max \left\{\begin{array}{c}\frac{2 \sqrt{5}}{5} M_{0}+\frac{3 \sqrt{5}-5}{10} M_{0}^{+}+\frac{\sqrt{5}}{5} M_{0}^{-}, \frac{5+\sqrt{5}}{10} M_{0}+\frac{5-\sqrt{5}}{10} M_{0}^{-}+\frac{\sqrt{5}}{5} M_{0}^{+}, \\ M_{0}+\frac{3 \sqrt{5}-5}{10} M_{0}^{-}, a M_{0}+b M_{0}^{+}\end{array}\right\}=r$
which is a polyhedron with 72 faces, 140 vertices and 210 edges, where $M_{0}=$ $\left\|X-X_{0}\right\|_{\infty}, X=(x, y, z), a=\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{15-8 \sqrt{5}}{95}\right)+\frac{15+11 \sqrt{5}}{38}$ and $b=$ $\sqrt{2} \sqrt{5+\sqrt{5}}\left(\frac{23 \sqrt{5}-55}{190}\right)+\frac{10+\sqrt{5}}{19}$. Coordinates of the vertices are the translation of all vertices of $S_{T T I}$ to ( $x_{0}, y_{0}, z_{0}$ ) with all cyclic permutation of the three axis components and all possible $+/-$ sign changes of each axis component of $\left(0, C_{2} r, C_{13} r\right),\left(0, C_{5} r, C_{11} r\right),\left(C_{0} r, C_{1} r, C_{14} r\right),\left(C_{1} r, C_{8} r, C_{10} r\right)$, $\left(C_{3} r, C_{4} r, C_{12} r\right),\left(C_{6} r, C_{6} r, C_{6} r\right),\left(C_{7} r, C_{4} r, C_{9} r\right)$ and $\left(C_{15} r, 0, r\right)$, where $C_{0}=\sqrt{2 \sqrt{5}+5}\left(\frac{120-47 \sqrt{5}}{305}\right)+\frac{11 \sqrt{5}-19}{122}, C_{1}=\sqrt{2} \sqrt{\sqrt{5}+5}\left(\frac{19-11 \sqrt{5}}{244}\right)+$ $\frac{18 \sqrt{5}-20}{61}, \quad C_{2}=\frac{4 \sqrt{5}-5}{11}, \quad C_{3}=\sqrt{2} \sqrt{\sqrt{5}+5}\left(\frac{11 \sqrt{5}-19}{244}\right)+\frac{25 \sqrt{5}-21}{122}$, $C_{4}=\sqrt{2 \sqrt{5}+5}\left(\frac{19-11 \sqrt{5}}{122}\right)+\frac{35-\sqrt{5}}{61}, C_{5}=\sqrt{2 \sqrt{5}+5}\left(\frac{74 \sqrt{5}-150}{305}\right)+\frac{16 \sqrt{5}-11}{61}$, $C_{6}=\frac{15-\sqrt{5}}{22}, C_{7}=\sqrt{2 \sqrt{5}+5}\left(\frac{335-149}{610} \sqrt{5}\right)+\frac{51+9 \sqrt{5}}{122}, C_{8}=\sqrt{2 \sqrt{5}+5}\left(\frac{11 \sqrt{5}-19}{122}\right)+$ $\frac{26+\sqrt{5}}{61}, C_{9}=\sqrt{2 \sqrt{5}+5}\left(\frac{9 \sqrt{5}-10}{305}\right)+\frac{44-3 \sqrt{5}}{61}, C_{10}=\sqrt{2 \sqrt{5}+5}\left(\frac{28-13 \sqrt{5}}{61}\right)+$ $\frac{49+23 \sqrt{5}}{122}, C_{11}=\sqrt{2 \sqrt{5}+5}\left(\frac{75-37 \sqrt{5}}{305}\right)+\frac{11+45 \sqrt{5}}{122}, C_{12}=\sqrt{2 \sqrt{5}+5}\left(\frac{13 \sqrt{5}-28}{61}\right)+$ $\frac{6+19 \sqrt{5}}{61}, C_{13}=\frac{5+7 \sqrt{5}}{22}, C_{14}=\sqrt{2 \sqrt{5}+5}\left(\frac{93 \sqrt{5}-205}{610}\right)+\frac{24+15 \sqrt{5}}{61}$ and $C_{15}=$ $\sqrt{2 \sqrt{5}+5}\left(\frac{37-15 \sqrt{5}}{61}\right)+\frac{19-11 \sqrt{5}}{122}$.

Lemma 3.2. The line through the points $P_{1}=\left(x_{1}, y_{1}, z_{1}\right)$ and $P_{2}=\left(x_{2}, y_{2}, z_{2}\right)$ in the analytical 3-dimensional space be $l$ and $d_{E}$ denotes the Euclidean metric. If direction vector of $l$ is $(p, q, r)$, then

$$
d_{T T I}\left(P_{1}, P_{2}\right)=\mu\left(P_{1} P_{2}\right) d_{E}\left(P_{1}, P_{2}\right)
$$

where

$$
\mu\left(P_{1} P_{2}\right)=\frac{\max \left\{\begin{array}{l}
\frac{2 \sqrt{5}}{5} M_{d}+\frac{3 \sqrt{5}-5}{10} M_{d}^{+}+\frac{\sqrt{5}}{5} M_{d}^{-}, \\
\frac{5+\sqrt{5}}{10} M_{d}+\frac{5-\sqrt{5}}{10} M_{d}^{-}+\frac{\sqrt{5}}{5} M_{d}^{+} \\
M_{d}+\frac{3 \sqrt{5}-5}{10} M_{d}^{-}, a M_{d}+b M_{d}^{+}
\end{array}\right\}}{\sqrt{p^{2}+q^{2}+r^{2}}}
$$

$M_{d}=\|P\|_{\infty}$ and $P=(p, q, r)$.
Proof. By using the equations $x_{1}-x_{2}=\lambda p, y_{1}-y_{2}=\lambda q, z_{1}-z_{2}=\lambda r$, $\lambda \in \mathbb{R}$ obtained by $l$ and definitions of $d_{T T I}\left(P_{1}, P_{2}\right)$ and $d_{E}\left(P_{1}, P_{2}\right)$ the denoted result is found.

The lemma above states that $d_{T T I}$-distance along a line is a particular positive constant multiple of Euclidean distance along the same line, thus the following corollaries are immediate consequences of this statement:

Corollary 3.1. If $P_{1}, P_{2}$ and $X$ are any three collinear points in $\mathbb{R}^{3}$, then $d_{E}\left(P_{1}, X\right)=d_{E}\left(P_{2}, X\right)$ if and only if $d_{T T I}\left(P_{1}, X\right)=d_{T T I}\left(P_{2}, X\right)$.

Corollary 3.2. If $P_{1}, P_{2}$ and $X$ are any three collinear points in $\mathbb{R}^{3}$, then

$$
d_{T T I}\left(X, P_{1}\right) / d_{T T I}\left(X, P_{2}\right)=d_{E}\left(X, P_{1}\right) / d_{E}\left(X, P_{2}\right)
$$

That is, the ratios of the Euclidean and $d_{T T I}$-distances along a line are the same.

## 4. ISOMETRY GROUPS OF TRUNCATED PENTAKIS DODECAHEDRON AND TRUNCATED TRIAKIS ICOSAHEDRON

There are three essential methods in geometric investigations; synthetic, metric and group approach. The group approach treats isometry groups of a geometry and convex sets play a substantial role in indication of the group isometries of geometries. These properties are invariant under the group of motions and geometry studies those properties. Since Minkowski geometries have the same underlying sets of points and lines with the Euclidean geometry an interesting problem is to find the group of isometries. For some studies about group of isometries some metric spaces see [10], [11], [12], [13], [16], [18]. In [1] the author gives the following theorem:

Theorem 4.1. If the unit ball $C$ of $(V,\| \|)$ does not intersect a two-plane in an ellipse, then the group $I(3)$ of isometries of $(V,\| \|)$ is isomorphic to the semi-direct product of the translation group $T(3)$ of $\mathbb{R}^{3}$ with a finite subgroup of the group of linear transformations with determinant $\pm 1$.

By this theorem, there only left to determine what the relevant subgroup is.

To show that the group of isometries of the 3 -dimensional space covered by $T P D$ - metric and $T T I$-metric are the semi-direct product of $I_{h}$ and $T(3)$, where icosahedral group $I_{h}$ is the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3 -dimensional space we first give the following definition. In the rest of the article we take $\Delta_{1}=T P D$ and $\Delta_{2}=T T I$. That is, $\Delta_{i} \in\left\{\Delta_{1}, \Delta_{2}\right\}, i=1,2$.

Definition 4.1. Let $R, S$ be two points in $\mathbb{R}_{\Delta_{i}}^{3}, i=1,2$. The minimum distance set of $P, Q$ is defined by

$$
\left\{X: d_{\Delta_{i}}(R, X)+d_{\Delta_{i}}(S, X)=d_{\Delta_{i}}(R, S), i=1,2\right\}
$$

and denoted by $[R S]_{\Delta_{i}}, i=1,2$.
$[R S]_{T P D}$ stands for a hexagonal dipyramid in $\mathbb{R}_{T P D}^{3}$ and $[R S]_{T T I}$ stands for a decagonal dipyramid in $\mathbb{R}_{T T I}^{3}$ as shown in Figure(7a) and Figure(7b).


Figure 7(a)


Figure 7(b)

Proposition 4.1. Let $\phi: \mathbb{R}_{\Delta_{i}}^{3} \longrightarrow \mathbb{R}_{\Delta_{i}}^{3}, i=1,2$, be an isometry and let $[R S]_{\Delta_{i}}$ be the minimum distance set of $R$, $S$ where $i=1,2$. Then $\phi\left([R S]_{\Delta_{i}}\right)=[\phi(R) \phi(S)]_{\Delta_{i}}, i=1,2$.

Proof. Let $Y \in \phi\left([R S]_{\Delta_{i}}\right), i=1,2$. Then, there exists $X \in[R S]_{\Delta_{i}}$ such that $Y=\phi(X) . d_{\Delta_{i}}(R, X)+d_{\Delta_{i}}(S, X)=d_{\Delta_{i}}(R, S), i=1,2$, since $X \in$ $[R S]_{\Delta_{i}}$. Thus $d_{\Delta_{i}}(\phi(R), \phi(X))+d_{\Delta_{i}}(\phi(S), \phi(X))=d_{\Delta_{i}}(\phi(R), \phi(S))$, $i=1,2$ which means $Y=\phi(X) \in[\phi(R) \phi(S)]_{\Delta_{i}}, i=1,2$. By similar way it is easy to prove that $[\phi(R) \phi(S)]_{\Delta_{i}} \subset \phi\left([R S]_{\Delta_{i}}\right), i=1,2$. So $\phi\left([R S]_{\Delta_{i}}\right)=[\phi(R) \phi(S)]_{\Delta_{i}}, i=1,2$, is obtained.

Corollary 4.1. Let $\phi: \mathbb{R}_{\Delta_{i}}^{3} \longrightarrow \mathbb{R}_{\Delta_{i}}^{3}$, $i=1,2$, be an isometry and let $[R S]_{\Delta_{i}}$ be the minimum distance set of $R$ and $S$ where $i=1,2$. Then $\phi$ maps vertices to vertices and preserves the lengths of the edges of $[R S]_{\Delta_{i}}$, $i=1,2$.

Proposition 4.2. Let $\phi: \mathbb{R}_{\Delta_{i}}^{3} \longrightarrow \mathbb{R}_{\Delta_{i}}^{3}$ be an isometry such that $\phi(O)=O$, where $i=1,2$. Then $\phi \in I_{h}$.

Proof. Let $\Delta_{i}=\Delta_{1}=T P D$. Consider seven points $V_{1}=\left(C_{11}, C_{12}, C_{13}\right)$, $V_{2}=\left(C_{13}, C_{11}, C_{12}\right), V_{3}=\left(C_{12}, C_{13}, C_{11}\right), V_{4}=\left(C_{14}, C_{15}, C_{16}\right), V_{5}=$ $\left(C_{16}, C_{14}, C_{15}\right), V_{6}=\left(C_{15}, C_{16}, C_{14}\right)$ and $W=\left(C_{0}, C_{0}, C_{0}\right)$ in $\mathbb{R}_{T P D}^{3}$ where the values of $C_{i}$ for $i=11,12, . ., 16$ are as given in Corollary 2.1 and $C_{0}=\frac{39+33 \sqrt{5}}{109}+\frac{58 \sqrt{3}-18 \sqrt{15}}{327}$. Thus $[O W]_{T P D}$ is the hexagonal dipyramid as seen in Figure 8(a).


Besides points $V_{i}(i=1,2, . ., 6)$ lie on minimum distance set $[O W]_{T P D}$ and unit sphere centered at origin. Furthermore these six points are the corner points of a truncated pentakis dodecahedron's hexagonal face. $\phi$ maps points $V_{i}(i=1,2, . .6)$ to the vertices of a truncated pentakis dodecahedron by Corollary 17. Since $\phi$ preserves the lenghts of the edges and truncated pentakis dodecahedron has 20 hexagonal faces and for each face there are 6 possibilities to the points $V_{i}(i=1,2, . ., 6)$ which they can map to, the total number of possibilities is 120 . By dealing with each possibility it would seen that the elements of desired subgroup are obtained.

If $\Delta_{i}=\Delta_{2}=T T I$, then let the choosen eleven points be $V_{1}=\left(C_{15}, 0,1\right)$, $V_{2}=\left(C_{0},-C_{1}, C_{14}\right), \quad V_{3}=\left(C_{3},-C_{4}, C_{12}\right), \quad V_{4}=\left(C_{7},-C_{4}, C_{9}\right)$, $V_{5}=\left(C_{10},-C_{1}, C_{8}\right), V_{6}=\left(C_{11}, 0, C_{5}\right), V_{7}=\left(C_{10}, C_{1}, C_{8}\right), V_{8}=\left(C_{7}, C_{4}, C_{9}\right)$, $V_{9}=\left(C_{3}, C_{4}, C_{12}\right), V_{10}=\left(C_{0}, C_{1}, C_{14}\right)$ and $W=\left(C_{16}, 0, C_{17}\right)$ in $\mathbb{R}_{T T I}^{3}$ where the values of $C_{i}$ for $i=0,1, . ., 15$ are as given in Theorem 3.1 and

$$
C_{16}=\frac{(260-112 \sqrt{5}) \sqrt{5+2 \sqrt{5}}}{305}+\frac{15+17 \sqrt{5}}{61}
$$

and

$$
C_{17}=\frac{52+2 \sqrt{5}}{61}+\frac{(11 \sqrt{5}-19-6 \sqrt{5} \sqrt{25+10 \sqrt{2}}) \sqrt{5+2 \sqrt{5}}}{61}
$$

Consider $[O W]_{T T I}$ which is the decagonal dipyramid as seen in Figure $8(\mathrm{~b})$.Also points $V_{i}(i=1,2, . ., 10)$ lie on minimum distance set $[O W]_{T T I}$ and the unit sphere centered at origin. Furthermore these eight points are the corner points of a truncated triakis icosahedron's decagonal face. $\phi$ maps points $V_{i}(i=1,2, . ., 10)$ to the vertices of a truncated triakis icosahedron by Corollary 17. Since $\phi$ preserves the lenghts of the edges and truncated triakis icosahedron has 12 decagonal faces and for each face there are 10 possibilities to the points which they can map to, the total number of possibilities is 120 . By dealing with each possibility it would seen that the elements of desired subgroup are obtained.

Theorem 4.2. Let $\phi: \mathbb{R}_{\Delta_{i}}^{3} \longrightarrow \mathbb{R}_{\Delta_{i}}^{3}, i=1,2$, be an isometry. Then there exists a unique $T_{A} \in T(3)$ and $\psi \in O_{h}$ where $\phi=T_{A} \circ \psi$.

Proof. Let $\phi(O)=A$ such that $A=\left(a_{1}, a_{2}, a_{3}\right)$. Define $\psi=T_{-A} \circ \phi$. We know that $\psi(O)=O$ and $\psi$ is an isometry. Thereby, $\psi \in O_{h}$ and $\phi=T_{A} \circ \psi$ by Proposition 18. The proof of uniqueness is trivial.

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