INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 12 (2023), No. 3, 56 - 68

ISOTHERMALLY ASYMPTOTIC CENTROAFFINE MINIMAL SURFACES OF COHOMOGENEITY ONE

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Abstract. We study indefinite centroaffine surfaces and classify indefinite centroaffine minimal surfaces which are isothermally asymptotic and of cohomogeneity one.

1. Introduction

In centroaffine differential geometry, we consider hypersurfaces in the vector space whose position vector field is transversal to the tangent space at each point. Then centroaffine minimal hypersurfaces are defined to be extremals for the volume with respect to the volume form of the centroaffine metric when hypersurfaces are nondegenerate. Since there are plenty of hypersurfaces which are centroaffine minimal, we often study such hypersurfaces under some geometric assumptions [3, 5, 6, 7, 8, 9, 14]. In particular, in the previous paper [5], we classify centroaffine minimal surfaces of cohomogeneity one which have centroaffine metrics of constant curvature.

On the other hand, in affine differential geometry, cubic forms are fundamental invariants for affine hypersurfaces. Moreover, we can consider the traceless part of the cubic form. In particular, for indefinite non-ruled affine surfaces, asymptotic curves and null curves of the traceless part of the cubic form define three families of curves, which is called a 3-web in web geometry [1]. Then as in the theory of projective surfaces, we can define isothermally asymptotic surfaces for indefinite non-ruled affine surfaces. In this paper, we study indefinite centroaffine surfaces and classify indefinite centroaffine minimal surfaces which are isothermally asymptotic and of cohomogeneity one.

Keywords and phrases: Isothermally asymptotic, Centroaffine minimal surfaces
(2020)Mathematics Subject Classification: 53A15, 53A05
Received: 19.12.2022. In revised form: 13.04.2023. Accepted: 22.02.2023.

2. CENTROAFFINE HYPERSURFACES

A centroaffine hypersurface in the $(n+1)$-dimensional vector space $\mathbf{R}^{n+1}$ is a hypersurface whose position vector field is transversal to the tangent space at each point. Such a hypersurface is given by an immersion $f : M \to \mathbf{R}^{n+1}$ from an n-dimensional manifold M into $\mathbf{R}^{n+1}$ with a transversal vector field ξ defined by

$$\xi = -\left.\left(\sum_{i=1}^{n+1} x_i \partial_{x_i}\right)\right|_f,$$

where $(x_1, \ldots, x_{n+1})$ are affine coordinates on $\mathbf{R}^{n+1}$ and $\left.\left(\sum_{i=1}^{n+1} x_i \partial_{x_i}\right)\right|_f$ is the restriction of the radial vector field to f. Then the Gauss-Weingarten formulas for f are given by

$$D_X f_* Y = f_* \nabla_X Y + h(X,Y)\xi,\ D_X \xi = -f_* X$$

for $X, Y \in \mathfrak{X}(M)$, where D is the standard flat connection on $\mathbf{R}^{n+1}$ and $\mathfrak{X}(M)$ is the set of vector fields on M. Note that ∇ defines a torsion-free affine connection on M, called the induced connection, and h defines a symmetric $(0,2)$-tensor field on M. We call f to be nondegenerate if h is nondegenerate at each point. In the same way, we call f to be positive definite, negative definite, and indefinite if h is positive definite, negative definite, and indefinite at each point, respectively. Moreover, we call f to be definite if f is positive definite or negative definite.

If f is nondegenerate, we call h the centroaffine metric, and denote the Levi-Civita connection for h by $\hat{\nabla}$. Then we have a $(1,2)$-tensor field K, a vector field T and a $(1,1)$-tensor field $\mathcal{T}$ on M defined by

$$K = \nabla - \hat{\nabla},\ T = \frac{1}{n}\mathrm{tr}_h C,\ \mathcal{T} = \hat{\nabla} T,$$

called the difference tensor field, the Tchebychev vector field and the Tchebychev operator, respectively. It is known that $T = 0$ if and only if f is a proper affine hypersphere centered at the origin, i.e., the Blaschke normals of f meet at the origin regarding f as a Blaschke hypersurface in equiaffine differential geometry ([14, Example 1]). We also have a $(0,3)$-tensor field C, a $(1,2)$-tensor field $\tilde{K}$ and a 1-form $\hat{T}$ on M defined by

$$C(X,Y,Z) = h(X, K(Y,Z)),$$
$$\tilde{K}(X,Y) = K(X,Y) - \frac{n}{n+2}(h(T,X)Y + h(T,Y)X + h(X,Y)T),$$
$$\hat{T}(X) = h(T,X)$$

for $X, Y, Z \in \mathfrak{X}(M)$, called the cubic form, the traceless part of K and the Tchebychev form, respectively. Note that C is totally symmetric by the Codazzi equation:

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z).$$

Moreover, we have a $(0,3)$-tensor field $\tilde{C}$ on M defined by

$$\tilde{C}(X,Y,Z) = C(X,Y,Z) - \frac{n}{n+2}(\hat{T}(X)h(Y,Z) + \hat{T}(Y)h(Z,X) + \hat{T}(Z)h(X,Y))$$

for $X, Y, Z \in \mathfrak{X}(M)$, called the traceless part of C, and a function $\tilde{J}$ defined by

$$\tilde{J} = \frac{1}{n(n-1)}\|\tilde{K}\|_h^2,$$

called the generalized Pick function.

A nondegenerate centroaffine hypersurface $f : M \to \mathbf{R}^{n+1}$ is called to be centroaffine minimal if it is an extremal for the volume with respect to the volume form of the centroaffine metric h, which is equivalent to the condition that

$$\operatorname{tr} \mathcal{T} = 0 \tag{1}$$

([14, Theorem 2]). In particular, proper affine hyperspheres centered at the origin are centroaffine minimal.

3. ISOTHERMALLY ASYMPTOTIC CENTROAFFINE SURFACES

In the following, we study indefinite centroaffine surfaces using local expressions of the Gauss formula and its integrability conditions. For an indefinite centroaffine surface $f : M \to \mathbf{R}^3$ with the centroaffine metric h, we can take asymptotic line coordinates (u, v) and the Gauss formula becomes as follows (cf. [12, Theorem 1], [3]):

$$f_{uu} = \left(\frac{\varphi_u}{\varphi} + \rho_u\right) f_u + \frac{a}{\varphi} f_v, \ f_{vv} = \left(\frac{\varphi_v}{\varphi} + \rho_v\right) f_v + \frac{b}{\varphi} f_u, \tag{2}$$

$$f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v \tag{3}$$

with the integrability conditions:

$$(\log|\varphi|)_{uv} = -\varphi - \frac{ab}{\varphi^2} + \rho_u\rho_v, \ a_v + \rho_u\varphi_u = \rho_{uu}\varphi, \ b_u + \rho_v\varphi_v = \rho_{vv}\varphi, \tag{4}$$

where

$$\varphi = h(\partial_u, \partial_v) = -\det\begin{pmatrix} f_{uv} \\ f_u \\ f_v \end{pmatrix} \Big/ \det\begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix} \neq 0, \tag{5}$$

$$a = \varphi \det\begin{pmatrix} f \\ f_u \\ f_{uu} \end{pmatrix} \Big/ \det\begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}, \tag{6}$$

$$b = \varphi \det\begin{pmatrix} f \\ f_v \\ f_{vv} \end{pmatrix} \Big/ \det\begin{pmatrix} f \\ f_v \\ f_u \end{pmatrix} \tag{7}$$

and

$$\rho = -\frac{1}{4}\log\left(\det\begin{pmatrix} f_{uv} \\ f_u \\ f_v \end{pmatrix}^2 \Big/ \det\begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}^4\right).$$

Then, the difference tensor K, the Tchebychev vector field T and the Tchebychev operator $\mathcal{T}$ are computed as follows:

$$K(\partial_u,\partial_u)=\rho_u\partial_u+\frac{a}{\varphi}\partial_v,\ K(\partial_v,\partial_v)=\frac{b}{\varphi}\partial_u+\rho_v\partial_v,\ K(\partial_u,\partial_v)=\rho_v\partial_u+\rho_u\partial_v,$$

(8) $$T=\frac{\rho_v}{\varphi}\partial_u+\frac{\rho_u}{\varphi}\partial_v,$$

(9) $$\mathcal{T}(\partial_u)=\frac{\rho_{uv}}{\varphi}\partial_u+\frac{a_v}{\varphi^2}\partial_v,\ \mathcal{T}(\partial_v)=\frac{b_u}{\varphi^2}\partial_u+\frac{\rho_{uv}}{\varphi}\partial_v.$$

In particular, from (1) and (9), f is centroaffine minimal if and only if

(10) $$\rho_{uv}=0.$$

Moreover, the cubic form C, the traceless part of the difference tensor $\tilde{K}$ and the Tchebychev form $\hat{T}$ are computed as follows:

$$C(\partial_u,\partial_u,\partial_u)=-2a,\ C(\partial_u,\partial_u,\partial_v)=-2\rho_u\varphi,$$
$$C(\partial_u,\partial_v,\partial_v)=-2\rho_v\varphi,\ C(\partial_v,\partial_v,\partial_v)=-2b,$$
$$\tilde{K}(\partial_u,\partial_u)=\frac{a}{\varphi}\partial_v,\ \tilde{K}(\partial_u,\partial_v)=0,\ \tilde{K}(\partial_v,\partial_v)=\frac{b}{\varphi}\partial_u,$$
$$\hat{T}(\partial_u)=-2\rho_u,\ \hat{T}(\partial_v)=-2\rho_v.$$

Hence the traceless part of the cubic form $\tilde{C}$ and the generalized Pick function $\tilde{J}$ are computed as follows:

$$\tilde{C}(\partial_u,\partial_u,\partial_u)=-2a,\ \tilde{C}(\partial_u,\partial_u,\partial_v)=\tilde{C}(\partial_u,\partial_v,\partial_v)=0,\ \tilde{C}(\partial_v,\partial_v,\partial_v)=-2b,$$

(11) $$\tilde{J}=\frac{ab}{\varphi^3}.$$

In particular, $\tilde{C}$ is written as

$$\tilde{C}=-2adu^3-2bdv^3.$$

We also note that the centroaffine curvature κ, which is the curvature of h, is given by

(12) $$\kappa=-\frac{(\log|\varphi|)_{uv}}{\varphi}.$$

Example 3.1. *Let $f : M \to \mathbf{R}^3$ be an indefinite proper affine sphere centered at the origin. If f is flat, i.e., $\kappa = 0$, then it is known that $f = (X, Y, Z)$ is a piece of a surface given by*

(13) $$(X^2+Y^2)Z=1$$

up to centroaffine congruence [10, Theorem 3].

Example 3.2. *Let $f : M \to \mathbf{R}^3$ be an indefinite centroaffine surface. It is known that f is ruled if and only if the generalized Pick function vanishes, i.e., $a = 0$ or $b = 0$ from* (11) *(cf.* [4, Proposition 6.1]*). If f is ruled and the Tchebychev operator vanishes, i.e., $\mathcal{T} = 0$, we can see that the centroaffine curvature $\kappa = 0, 1$.*

In the case of $\kappa = 0$, then f is given by

$$f(x,y)=(e^x,\psi_1(x)e^y,\psi_2(x)e^y),$$

where ψ_1 *and* ψ_2 *are linearly independent solutions to the differential equation:*

$$\psi'' - \psi' - \chi\psi = 0$$

for an arbitrary function $\chi = \chi(x)$ *(cf.* [8, Theorem 3.2], [9, Theorem 4.2]*).*

In the case of $\kappa = 1$*, then* f *is a proper affine sphere centered at the origin given by*

$$f(x, y) = \lambda'(x) + y\lambda(x),$$

where λ *is an* $\mathbf{R}^3$*-valued function such that* $\det \begin{pmatrix} \lambda \\ \lambda' \\ \lambda'' \end{pmatrix}$ *is a non-zero constant (cf.* [13, Theorem 4.3.1]*).*

We call an indefinite centroaffine surface to be non-ruled, if the generalized Pick function never vanishes. Let $f : M \to \mathbf{R}^3$ be a non-ruled indefinite centroaffine surface, i.e., $a, b \neq 0$. Then asymptotic curves and null curves of the traceless part of the cubic form $\tilde{C}$ define three families of curves, which is called a 3-web in web geometry [1]. In this paper, we call the above 3-web the characteristic 3-web. If a 3-web is given by 1-forms $\omega_1, \omega_2, \omega_3$, i.e., each curve is a null curve of one of ω_i's, we can normalize ω_i's as

$$\omega_1 + \omega_2 + \omega_3 = 0. \tag{14}$$

Then there exists a 1-form θ, called the Chern connection form, satisfying the web structure equations:

$$d\omega_1 = \omega_1 \wedge \theta, \ d\omega_2 = \omega_2 \wedge \theta, \ d\omega_3 = \omega_3 \wedge \theta.$$

Note that a 3-web is called to be hexagonal if all three families of curves are diffeomorphic to three families of parallel lines. If we define the web curvature K by

$$d\gamma = K\omega_1 \wedge \omega_2,$$

the above 3-web is hexagonal if and only if $K = 0$, which is equivalent to the existence of functions g, u_1, u_2, u_3 locally such that

$$g\omega_i = du_i \quad (i = 1, 2, 3)$$

and $u_1 + u_2 + u_3$ is constant.

Proposition 3.1. *Let* $f : M \to \mathbf{R}^3$ *be a non-ruled indefinite centroaffine surface. Then the characteristic* 3*-web for* f *is hexagonal if and only if*

$$\left(\log \frac{a}{b}\right)_{uv} = 0. \tag{15}$$

Proof. If we put

$$\omega_1 = -a^{\frac{1}{3}} du, \ \omega_2 = -b^{\frac{1}{3}} dv, \ \omega_3 = a^{\frac{1}{3}} du + b^{\frac{1}{3}} dv,$$

then ω_i's are 1-forms satisfying (14) which define the characteristic 3-web for f. Moreover, the Chern connection form θ and the web curvature K are computed as

$$\theta = -\frac{1}{3}\left(\frac{b_u}{b} du + \frac{a_v}{a} dv\right), \ K = -\frac{1}{3}(ab)^{-\frac{1}{3}} \left(\log \frac{a}{b}\right)_{uv}.$$

□

Definition 3.1. *Let $f : M \to \mathbf{R}^3$ be a non-ruled idefinite centroaffine surface. We call f to be isothemally asymptotic if the characteristic* 3*-web for f is hexagonal, which is equivalent to* (15)*.*

Remark 3.1. *If we regard a non-ruled idefinite centroaffine surface $f : M \to \mathbf{R}^3$ as a suface $\tilde{f}$ in the* 3*-dimensional projective space by setting*

$$\tilde{f} = [f_1 : f_2 : f_3 : 1],$$

we can see that the conformal class of the traceless part of the cubic form $\tilde{C}$ is equal to the Darboux form of $\tilde{f}$ [6]*. Hence isothermally asymptotic centroaffine surfaces can be regarded as isothermally asymptotic projective surfaces* [2, 11]*.*

Example 3.3. *Let $f : M \to \mathbf{R}^3$ be a non-ruled idefinite centroaffine surface with vanishing Tchebychev operator. Then from* (9)*, we have $a_v = b_u = 0$, which satisfies* (15)*. Hence f is isothermally asymptotic.*

If the Tchebychev vector vanishes, i.e., $T = 0$, then f is a proper affine sphere centered at the origin.

If $T \neq 0$, then we can see that f is flat, and $f = (X, Y, Z)$ is a piece of the following surfaces up to centroaffine congruence [9, Theorem 4.2] *(see also* [7, Example 2.9]*):*

(i) $X^p Y^q Z^r = 1$ $(p, q, r \in \mathbf{R}, pqr(p + q + r) < 0, (p + q)(q + r)(r + p) \neq 0)$,

(ii) $\left\{\exp\left(-p \tan^{-1} \dfrac{X}{Y}\right)\right\} (X^2 + Y^2)^q Z^r = 1$ $(p, q, r \in \mathbf{R}, (p, q) \neq (0, r)$, $r(2q + r)(p^2 + q^2) > 0)$,

(iii) $Z = -X(p \log X + q \log Y)$ $(p, q \in \mathbf{R}, q(p + q) > 0)$,

(iv) $Z = -X \log X + \dfrac{Y^2}{X}$.

4. CENTROAFFINE SURFACES OF COHOMOGENEITY ONE

A centroaffine surface of cohomogeneity one is given by the form

$$f(x, y) = \gamma(x) e^{yA}, \tag{16}$$

where γ is a curve in $\mathbf{R}^3$ and $\{e^{yA}\}_{y \in \mathbf{R}}$ is a one-parameter subgroup of the centroaffine transformation group $\mathrm{GL}(3, \mathbf{R})$ for $A \in \mathfrak{gl}(3, \mathbf{R})$.

Example 4.1. *Let $f : M \to \mathbf{R}^3$ be a flat indefinite proper affine sphere centered at the origin, which is given by* (13) *in Example 3.1. Then f is of cohomogeneity one. Indeed, we can choose γ and A in* (16) *up to centroaffine congruence as*

$$\gamma(x) = (0, x, x^{-2}), \ A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 4.2. *Let f be a non-ruled indefinite centroaffine surface with vanishing Tchebychev operator. If the Tchebychev vector of f does not vanish, then $f = (X, Y, Z)$ is given by* (i) $\sim$ (iv) *in Example 3.3. Moreover, f is* *of cohomogeneity one. Indeed, depending on the cases* (i) $\sim$ (iv)*, we can choose* γ *and* A *in* (16) *up to centroaffine congruence as follows:*

$$\text{(i) } \gamma(x) = (e^x, 1, e^{-\frac{p}{r}x}),\ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{q}{r} \end{pmatrix},$$

$$\text{(ii) } \gamma(x) = (0, x, x^{-\frac{2q}{r}}),\ A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{p}{r} \end{pmatrix},$$

$$\text{(iii) } \gamma(x) = \left(e^x, \frac{q}{p+q}x, 1\right),\ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\text{(iv) } \gamma(x) = (x, -x^2, 1),\ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let $f : M \to \mathbf{R}^3$ be an indefinite centroaffine surface of cohomogeneity one. Then, by changing γ and A in (16), if necessary, we have a normal parametrization for f:

$$f(u, v) = \gamma(u + v)e^{(u-v)A} \tag{17}$$

for some asymptotic line coordinates (u, v) such that

$$\det \begin{pmatrix} \gamma \\ \gamma A \\ \gamma' \end{pmatrix} \neq 0 \tag{18}$$

[5, Lemma 3.3]. Moreover, the functions φ, a and b as in §3 depend only on $u + v$, while

$$\rho(u, v) = \alpha(u + v) + \frac{1}{2}(u - v)\mathrm{tr}\, A, \tag{19}$$

where α is a function of $u + v$ only [5, Proposition 4.1].

Example 4.3. *Let* $f : M \to \mathbf{R}^3$ *be a non-ruled, non-flat, indefinite proper affine sphere of cohomogeneity one. Then we can take* γ *and* A *as in the normal parametrization* (17)*, and* φ*,* a *and* b *as in* (5) $\sim$ (7)*. By* [5, Theorem 5.1]*, we have the following:*
(a) *a and b are non-zero constants.*
(b) *φ satisfies*

$$(\varphi'^2 = -2\varphi^3 + c\varphi^2 + ab$$

for some $c \in \mathbf{R}$*, and*

$$2\varphi' + a + b \neq 0.$$

(c) *The minimal polynomial of* A *is*

$$t^3 - \frac{c}{4}t + \frac{a-b}{8}.$$

(d) *γ satisfies*

$$(2\varphi' + a + b)\gamma'^2 + (a - b)A - 2\varphi^2 E\}$$

with

$$\det\begin{pmatrix}\gamma\\ \gamma A\\ \gamma A^2\end{pmatrix}\neq 0.$$

Example 4.4. *Let $f: M\to \mathbf{R}^3$ be a non-ruled, non-flat, indefinite centroaffine minimal surface of cohomogeneity one with constant centroaffine curvature. Then we can take γ and A as in the normal parametrization* (17)*, and φ, a, b, κ and ρ as in* (5) $\sim$ (7)*,* (12) *and* (19)*. By* [5, Lemma 6.1]*, we have*

$$a=-c_1\varphi,\ b=-c_2\varphi,\ \kappa=1,\ \rho=c_1u+c_2v+c_3,$$

where $c_1, c_2\in\mathbf{R}\setminus\{0\}$, $c_3\in\mathbf{R}$. In particular, from (15)*, f is isothermally asymptotic. Moreover, by* [5, Theorem 6.7]*, if we define a function μ for $c\in\mathbf{R}$ by*

$$\mu(s)=\begin{cases}c_1+c_2+\sqrt{c}\tanh\frac{\sqrt{c}}{2}s\ (c>0),\\ c_1+c_2+\frac{2}{s}\ (c=0),\\ c_1+c_2-\sqrt{-c}\tan\frac{\sqrt{-c}}{2}s\ (c<0),\end{cases}$$

we can choose γ and A up to centroaffine congruence as follows:
(i) $c>4c_1c_2$, $c\neq(c_1+c_2)^2$,

$$\gamma(s)=\begin{cases}\left(\mu(s), \frac{e^{\frac{c_1+c_2}{2}s}}{\cosh\frac{\sqrt{c}}{2}s}, \frac{e^{\frac{c_1+c_2}{2}s}}{\cosh\frac{\sqrt{c}}{2}s}\right)\ (c>0),\\ \left(\mu(s), \frac{e^{\frac{c_1+c_2}{2}s}}{s}, \frac{e^{\frac{c_1+c_2}{2}s}}{s}\right)\ (c=0),\\ \left(\mu(s), \frac{e^{\frac{c_1+c_2}{2}s}}{\cos\frac{\sqrt{-c}}{2}s}, \frac{e^{\frac{c_1+c_2}{2}s}}{\cos\frac{\sqrt{-c}}{2}s}\right)\ (c<0),\end{cases}$$

$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{c_1-c_2+\sqrt{c-4c_1c_2}}{2} & 0\\ 0 & 0 & \frac{c_1-c_2-\sqrt{c-4c_1c_2}}{2}\end{pmatrix},$$

(ii) $c<4c_1c_2$, *γ is defined as in the case* (i)*, and*

$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{c_1-c_2}{2} & -\frac{\sqrt{4c_1c_2-c}}{2}\\ 0 & \frac{\sqrt{4c_1c_2-c}}{2} & \frac{c_1-c_2}{2}\end{pmatrix},$$

(iii)$_1$ $c=4c_1c_2$, $c_1\neq c_2$, *γ is defined as in the case* (i)*, and*

$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{c_1-c_2}{2} & 0\\ 0 & 1 & \frac{c_1-c_2}{2}\end{pmatrix},$$

(iii)$_2$ $c=(c_1+c_2)^2$, $c_1\neq c_2$,

$$\gamma(s)=\begin{cases}\mu(s)\left(\frac{c_1^2-c_2^2}{c}\left(s+\frac{1}{\sqrt{c}}\sinh\sqrt{c}s\right)-\frac{2(c_1-c_2)}{c}\cosh^2\frac{\sqrt{c}}{2}s, 1, 1\right)\ (c>0),\\ \mu(s)\left(\frac{c_1-c_2}{2}s^2, 1, 1\right)\ (c=0),\end{cases}$$

$$A=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & c_1-c_2\end{pmatrix},$$

(iv) $c=(c_1+c_2)^2,\ c_1=c_2,$

$$\gamma(s)=\begin{cases}\mu(s)\left(\frac{2}{c}\cosh^2\frac{\sqrt{c}}{2}s-\frac{2c_1}{c}\left(s+\frac{1}{\sqrt{c}}\sinh\sqrt{c}s\right),1,1\right) & (c>0),\\ \mu(s)\left(-\frac{1}{2}s^2,1,1\right) & (c=0),\end{cases}$$

$$A=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}.$$

5. MAIN RESULT

As a main result of this paper, we shall classify indefinite centroaffine minimal surfaces which are isothermally asymptotic and of cohomogeneity one. We use the same notation as in the previous sections.

Lemma 5.1. *Let $f: M\to\mathbf{R}^3$ be an indefinite centroaffine minimal surface which is isothermally asymptotic and of cohomogeneity one. Then there exist $c_1\sim c_5\in\mathbf{R}$ and $c_6\in\mathbf{R}\setminus\{0\}$ such that*

$$\rho(u,v)=c_1u+c_2v+c_3,\ a=-c_1\varphi+c_4,\ b=-c_2\varphi+c_5,\ \frac{a}{b}=c_6. \tag{20}$$

Proof. Since f is of cohomogeneity one, we may assume that f is given by a normal parametrization (17). From (10) and (19), there exist $\tilde{c},c_3\in\mathbf{R}$ such that

$$\alpha(s)=\tilde{c}s+c_3,$$

where $s=u+v$. Putting

$$c_1=\tilde{c}+\frac{1}{2}\mathrm{tr}\,A,\ c_2=\tilde{c}-\frac{1}{2}\mathrm{tr}\,A,$$

we have $(20)_1$. Since φ, a and b depend only on s, from $(4)_2$, $(4)_3$ and $(20)_1$, there exist $c_4,c_5\in\mathbf{R}$ such that $(20)_2$ and $(20)_3$ hold. Then from (15), there exist $c_6\in\mathbf{R}\setminus\{0\}$ and $\hat{c}\in\mathbf{R}$ such that

$$\frac{a(s)}{b(s)}=\frac{c_1\varphi(s)-c_4}{c_2\varphi(s)-c_5}=c_6e^{\hat{c}s}.$$

In order to prove that $\hat{c}=0$ by contradiction, we assume that $\hat{c}\neq0$. By changing coorinates (u,v), if necessary, we may assume that $c_6=\pm1$ and $\hat{c}=1$, i.e.,

$$\frac{c_1\varphi(s)-c_4}{c_2\varphi(s)-c_5}=\pm e^s, \tag{21}$$

which is equivalent to

$$\varphi(s)=\frac{\pm c_5e^s-c_4}{\pm c_2e^s-c_1}. \tag{22}$$

From $(4)_1$, $(20)_1\sim(20)_3$ and (22), we have

$$\frac{\mp c_4c_5e^s}{(\pm c_5e^s-c_4)^2}-\frac{\mp c_1c_2e^s}{(\pm c_2e^s-c_1)^2}$$
$$=-\frac{\pm c_5e^s-c_4}{\pm c_2e^s-c_1}+(c_1c_5+c_2c_4)\frac{\pm c_2e^s-c_1}{\pm c_5e^s-c_4}-c_4c_5\frac{(\pm c_2e^s-c_1)^2}{(\pm c_5e^s-c_4)^2},$$

i.e.,

$$\mp c_4c_5e^s(\pm c_2e^s - c_1)^2 \pm c_1c_2e^s(\pm c_5e^s - c_4)^2 \tag{23}$$
$$= -(\pm c_2e^s - c_1)(\pm c_5e^s - c_4)^3$$
$$+ (c_1c_5 + c_2c_4)(\pm c_2e^s - c_1)^3(\pm c_5e^s - c_4) - c_4c_5(\pm c_2e^s - c_1)^4.$$

A direct computation shows that (23) is equivalent to

$$c_2c_5^2(c_5 - c_1c_2^2)e^{4s} \pm (3c_1^2c_2^2c_5^2 + c_2^4c_4^2 + c_1c_2c_5^2 - c_1c_5^3 - c_2^2c_4c_5 - 3c_2c_4c_5^2)e^{3s}$$
$$- 3(c_1^3c_2c_5^2 + c_1c_2^3c_4^2 - c_1c_4c_5^2 - c_2c_4^2c_5)e^{2s}$$
$$\pm (c_1^4c_5^2 + 3c_1^2c_2^2c_4^2 - c_1^2c_4c_5 + c_1c_2c_4^2 - 3c_1c_4^2c_5 - c_2c_4^3)e^s + c_1c_4^2(c_4 - c_1^2c_2)$$
$$= 0,$$

i.e.,

$$c_2c_5^2(c_5 - c_1c_2^2) = 0,\ 3c_1^2c_2^2c_5^2 + c_2^4c_4^2 + c_1c_2c_5^2 - c_1c_5^3 - c_2^2c_4c_5 - 3c_2c_4c_5^2 = 0, \tag{24}$$
$$c_1^3c_2c_5^2 + c_1c_2^3c_4^2 - c_1c_4c_5^2 - c_2c_4^2c_5 = 0,$$

$$c_1^4c_5^2 + 3c_1^2c_2^2c_4^2 - c_1^2c_4c_5 + c_1c_2c_4^2 - 3c_1c_4^2c_5 - c_2c_4^3 = 0,\ c_1c_4^2(c_4 - c_1^2c_2) = 0. \tag{25}$$

From $(24)_1$, we have one of the following: (i) $c_2 = 0$, (ii) $c_2 \neq 0$, $c_5 = 0$, (iii) $c_2, c_5 \neq 0$, $c_5 = c_1c_2^2$. From $(25)_3$, we have one of the following: (iv) $c_1 = 0$, (v) $c_1 \neq 0$, $c_4 = 0$, (vi) $c_1, c_4 \neq 0$, $c_4 = c_1^2c_2$.

If $c_2 = 0$, from $(24)_2$, we have $c_1c_5 = 0$, i.e., $c_1 = 0$ or $c_5 = 0$. If $c_1 = 0$, from (21), we have

$$\frac{c_4}{c_5} = \pm e^s,$$

which is a contradiction. If $c_5 = 0$, from $(20)_3$, we have $b = 0$, which is a contradiction. Similary, if $c_1 = 0$, we have a contradiction.

If $c_2 \neq 0$, $c_5 = 0$, from $(24)_2$, we have $c_4 = 0$. Then from (21), we have

$$\frac{c_1}{c_2} = \pm e^s,$$

which is a contradiction. Similary, if $c_1 \neq 0$, $c_4 = 0$, we have a contradiction.

If $c_1, c_2, c_4, c_5 \neq 0$, $c_5 = c_1c_2^2$, $c_4 = c_1^2c_2$, from (22), we have $\varphi = c_1c_2$. Then from $(20)_2$, we have $a = 0$, which is a contradiction. □

Theorem 5.1. *Let $f : M \to \mathbf{R}^3$ be an indefinite centroaffine minimal surface which is isothermally asymptotic and of cohomogeneity one. Then f is a surface as in Examples 4.1∼4.4, or by taking γ and A as in the normal parametrization* (17)*, and φ, a and b as in* (5) ∼ (7)*, we have the following:*
(a) *There exist $c_1, c_2, c_4 \sim c_6 \in \mathbf{R} \setminus \{0\}$ and $c_3 \in \mathbf{R}$ which satisfy* (20) *and*

$$c_1 = c_2c_6,\ c_4 = c_5c_6. \tag{26}$$

(b) *φ satisfies*

$$(\varphi'^2 = -2\varphi^3 + c_7\varphi^2 - 4c_2c_5c_6\varphi + c_5^2c_6 \tag{27}$$

for some $c_7 \in \mathbf{R}$, and

$$2\varphi' - 2(c_1 + c_2)\varphi + c_4 + c_5 \neq 0. \tag{28}$$

(c) *The minimal polynomial of A is*

$$t^3 - (c_1 - c_2)t^2 + \frac{1}{4}\{(c_1 + c_2)^2 - c_7\}t + \frac{c_4 - c_5}{8}. \tag{29}$$

(d) γ *satisfies*

$$\begin{aligned}\{2\varphi' - 2(c_1 + c_2)\varphi + c_4 + c_5\}\gamma' \\ = \gamma[4\varphi A^2 - \{4(c_1 - c_2)\varphi - c_4 + c_5\}A - 2\varphi^2 E]\end{aligned} \tag{30}$$

with

$$\det \begin{pmatrix} \gamma \\ \gamma A \\ \gamma A^2 \end{pmatrix} \neq 0. \tag{31}$$

Proof. We use the same notation as in Lemma 5.1. From $(20)_2 \sim (20)_4$, we have

$$(c_1 - c_2 c_6)\varphi = c_4 - c_5 c_6.$$

If $c_1 - c_2 c_6 \neq 0$, then φ is a constant, which implies that a and b are constants from $(20)_2$ and $(20)_3$. Hence from (9), the Tchebychev operator vanishes. Moreover, since $(c_1, c_2) \neq (0, 0)$, from (8), the Tchebychev vector does not vanish. Therefore f is a surface as in Example 4.2.

If $c_1 - c_2 c_6 = 0$, then $c_4 - c_5 c_6 = 0$. From these equations and $(20)_1 \sim (20)_3$, $(4)_1$ becomes

$$(\log|\varphi|)'' = -\varphi + c_5 c_6 \left(\frac{2c_2}{\varphi} - \frac{c_5}{\varphi^2} \right). \tag{32}$$

Moreover, if $c_2 = 0$, then $c_1 = 0$, which implies that f is a proper affine sphere. Hence f is a surface as in Examples 4.1, 4.3.

If $c_2 \neq 0$, $c_5 = 0$, then (32) becomes

$$(\log|\varphi|)'' = -\varphi,$$

which implies that the centroaffine curvature of f is 1. Hence f is a surface as in Example 4.4.

In the following, we consider the remaining case given by (26). First, from (32), there exists $c_7 \in \mathbf{R}$ which satisfies (27). On the other hand, from $(20)_1 \sim (20)_3$, the Gauss formula (2), (3) becomes

$$f_{uu} = \left(\frac{\varphi'}{\varphi} + c_1 \right) f_u + \left(\frac{c_4}{\varphi} - c_1 \right) f_v, \tag{33}$$

$$f_{vv} = \left(\frac{\varphi'}{\varphi} + c_2 \right) f_v + \left(\frac{c_5}{\varphi} - c_2 \right) f_u, \tag{34}$$

$$f_{uv} = -\varphi f + c_1 f_u + c_2 f_v. \tag{35}$$

Substituting (17) into (33) $\sim$ (35), we have

$$\gamma'' + 2\gamma'^2 = \left(\frac{\varphi'}{\varphi} + c_1 \right) (\gamma' + \gamma A) + \left(\frac{c_4}{\varphi} - c_1 \right) (\gamma' - \gamma A),$$

$$\gamma'' - 2\gamma'^2 = \left(\frac{\varphi'}{\varphi} + c_2 \right) (\gamma' - \gamma A) + \left(\frac{c_5}{\varphi} - c_2 \right) (\gamma' + \gamma A),$$

$$\gamma''^2 = -\varphi\gamma + c_2(\gamma' + \gamma A) + c_1(\gamma' - \gamma A),$$

which is equivalent to (30) and

$$\gamma' \{4\varphi A - (c_4 - c_5)E\} = \{2\varphi' + 2(c_1 + c_2)\varphi - (c_4 + c_5)\}\gamma A, \tag{36}$$

(37) $$\gamma'' - (c_1 + c_2)\gamma'^2 - (c_1 - c_2)A - \varphi E\}.$$

From (30) and (36), we have

$$\begin{aligned}&\gamma\{4\varphi A - (c_4 - c_5)E\}[4\varphi A^2 - \{4(c_1 - c_2)\varphi - c_4 + c_5\}A - 2\varphi^2 E]\\ &\quad = \{2\varphi' - 2(c_1 + c_2)\varphi + c_4 + c_5\}\{2\varphi' + 2(c_1 + c_2)\varphi - (c_4 + c_5)\}\gamma A,\end{aligned}$$

which is equivalent to

(38) $$8A^3 - 8(c_1 - c_2)A^2 + 2\{(c_1 + c_2)^2 - c_7\}A + (c_4 - c_5)E = O$$

from (18) and (27). Since φ is not a constant, we have (28). Note that from (18), (28) and (30), we have (31). Hence (29) is the minimal polynomial of A.

We also note that (36) can be deduced from (30) under the condition (38). Moreover, from (30) we have

(39)

$$\begin{aligned}&\gamma'' - (c_1 + c_2)\gamma'\\ &= -\frac{2\varphi'' - 2(c_1 + c_2)\varphi'}{\{2\varphi' - 2(c_1 + c_2)\varphi + c_4 + c_5\}^2}\gamma[4\varphi A^2 - \{4(c_1 - c_2)\varphi - c_4 + c_5\}A - 2\varphi^2 E]\\ &+ \frac{1}{\{2\varphi' - 2(c_1 + c_2)\varphi + c_4 + c_5\}^2}\gamma[4\varphi A^2 - \{4(c_1 - c_2)\varphi - c_4 + c_5\}A - 2\varphi^2 E]^2\\ &+ \frac{1}{2\varphi' - 2(c_1 + c_2)\varphi + c_4 + c_5}\gamma\{4\varphi'^2 - 4(c_1 - c_2)\varphi' A - 4\varphi\varphi' E\}\\ &- \frac{c_1 + c_2}{2\varphi' - 2(c_1 + c_2)\varphi + c_4 + c_5}\gamma[4\varphi A^2 - \{4(c_1 - c_2)\varphi - c_4 + c_5\}A - 2\varphi^2 E].\end{aligned}$$

On the other hand, from (27) we have

(40) $$\varphi''^2 + c_7\varphi - 2c_2c_5c_6.$$

From (27), (38) and (40), a direct computation shows that (39) is equivalent to (37). □

Acknowledgements. This work was supported by the Kansai University Grant-in-Aid for progress of research in graduate course, 2022.

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