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CENTROIDS OF HYPERBOLIC SIMPLICES

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Abstract. In this paper, we represent the integral of vectors of a hyperbolic simplex by volumes of facets and unit vectors pseudo-perpendicular to them. By using it, we can define a new hyperbolic centroid of a hyperbolic simplex.

1. Introduction

Let S be a simplex of the unit sphere $\mathbb{S}^{n-1}$ with facets $S_0, \ldots, S_{n-1}$. Then, Schläfli represented the derivation of the volume of S by volumes of $(n-3)$-dimensional faces $S_{i,j} = S_i \cap S_j$ and derivations of dihedral angles $\langle i, j\rangle$ between facets S_i and S_j of S (See [8]. See also [2], [3], and [4] (the errata of [2] was revised in [6])). It was extend to a hyperbolic simplex $S \subseteq \mathbb{H}^{n-1}$ (see Appendix for another proof):

Theorem 1.1 (The hyperbolic version of Schläfli's differential formula ([1, Chap. 7, §2.2] and [5])). *For a hyperbolic simplex S, it holds that:*

$$\mathrm{d}|S| = -\frac{1}{n-2}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}|S_{i,j}|\mathrm{d}\langle i, j\rangle,$$

where $|S|$ and $|S_{i,j}|$ are $(n-1)$-dimensional and $(n-3)$-dimensional Lebesgue measures of S and $S_{i,j}$, respectively.

In this paper, we consider the representation of the integration of vectors of a hyperbolic simplex by volumes of its facets and spacelike unit vectors pseudo-perpendicular to facets. Precisely, let $\mathbb{H}^{n-1}$ be Minkowski model of $(n-1)$-dimensional hyperbolic space and $\langle\cdot|\cdot\rangle$ be the pseudo-inner product, that is,

$$\mathbb{H}^{n-1} = \{\mathbf{x} = \begin{pmatrix}\underline{\mathbf{x}}\\ x_{n-1}\end{pmatrix} \in \mathbb{R}^{n-1}\times\mathbb{R} : -\langle\mathbf{x}|\mathbf{x}\rangle = 1,\, x_{n-1} > 0\}$$

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and $\langle \mathbf{x}|\mathbf{y}\rangle = x_0y_0 + \cdots + x_{n-2}y_{n-2} - x_{n-1}y_{n-1}$ for $\mathbf{x} = (x_0, \ldots, x_{n-2}; x_{n-1})^T$ and $\mathbf{y} = (y_0, \ldots, y_{n-2}; y_{n-1})^T$, in other words, let $\mathbb{H}^{n-1}$ be the set of all future timelike unit vectors of Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$. And let S be a hyperbolic simplex determined by linearly independent spacelike vectors $\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}$ of $\mathbb{R}^{n-1} \times \mathbb{R}$ (i.e., $\langle \mathbf{p}_i|\mathbf{p}_i\rangle = 1$), and let $S_0, \ldots, S_{n-1}$ be corresponding facets, that is,

$$S = \{\mathbf{x} \in \mathbb{H}^{n-1} : \langle \mathbf{x}|\mathbf{p}_i\rangle \geq 0 \text{ for } {}^{\forall} i = 0, \ldots, n-1\},$$

and

$$S_k = \{\mathbf{x} \in S : \langle \mathbf{x}|\mathbf{p}_k\rangle = 0\},$$

for each $k = 0, \ldots, n-1$. Then, we have the following equation, which is the main result of this paper:

Theorem 1.2. *For a hyperbolic simplex S of $\mathbb{H}^{n-1}$, it holds that:*

$$\int_{\mathbf{x}\in S} \mathbf{x} dS = -\frac{1}{n-1} \sum_{k=0}^{n-1} |S_k|\mathbf{p}_k,$$

where $|S_k|$ is $(n-2)$-dimensional Lebesgue measure of the facet S_k.

Remark 1.1. *The hyperbolic version of Theorem 1.1 (resp. 1.2) differs from the spherical version of Theorem 1.1 (resp. 1.2) only in the sign of the right-hand side.*

Remark 1.2. *The spherical version of Theorem 1.2 is easy because the integral of the unit outward normal over the boundary of a convex body equals zero.*

We define the centroids of a hyperbolic simplex by the following:

Definition 1.1. *Let*

$$\tilde{\mathbf{g}} = \frac{1}{|S|} \int_{\mathbf{x}\in S} \mathbf{x} dS = -\frac{1}{(n-1)|S|} \sum_{k=0}^{n-1} |S_k|\mathbf{p}_k,$$

and

$$\mathbf{g} = \frac{\tilde{\mathbf{g}}}{\sqrt{-\langle \tilde{\mathbf{g}}|\tilde{\mathbf{g}}\rangle}} = \frac{\int_{\mathbf{x}\in S} \mathbf{x} dS}{\sqrt{-\langle \int_{\mathbf{x}\in S} \mathbf{x} dS | \int_{\mathbf{x}\in S} \mathbf{x} dS\rangle}} =$$

$$= -\frac{\sum_{k=0}^{n-1} |S_k|\mathbf{p}_k}{\sqrt{-\langle \sum_{k=0}^{n-1} |S_k|\mathbf{p}_k \,|\, \sum_{k=0}^{n-1} |S_k|\mathbf{p}_k\rangle}},$$

where the last equalities of both come from Theorem 1.2. Then vectors $\tilde{\mathbf{g}} \in \mathbb{R}^n$ and $\mathbf{g} \in S$ denotes Minkowski centroid *(*Minkowski barycenter*,* Minkowski center of gravity*, or* Minkowski center of mass*) and* the hyperbolic centroid *(*the hyperbolic barycenter*,* the hyperbolic center of gravity*, or* the hyperbolic center of mass*) of S, respectively.*

The definition of Minkowski centroid is quite natural, but there exists another definition of the hyperbolic centroid.

Remark 1.3. *In general, the hyperbolic centroid above,* $\mathbf{g}$*, does not coincide with another hyperbolic centroid,*

$$\frac{\frac{1}{n}(\mathbf{p}_0^* + \cdots + \mathbf{p}_{n-1}^*)}{\sqrt{-\langle \frac{1}{n}(\mathbf{p}_0^* + \cdots + \mathbf{p}_{n-1}^*) | \frac{1}{n}(\mathbf{p}_0^* + \cdots + \mathbf{p}_{n-1}^*)\rangle}},$$

which is the normalization of the centroid $\frac{1}{n}(\mathbf{p}_0^* + \cdots + \mathbf{p}_{n-1}^*)$ *of Euclidean* $(n-1)$*-simplex whose vertices* $\mathbf{p}_0^*,\ldots, \mathbf{p}_{n-1}^*$ *are vertices of* S*. See Remark 4.1.*

2. Preliminaries

For $j = 0,\ldots, n-1$, let $\mathbf{p}_j^* \in \mathbb{R}^n$ be the vector such that

$$\langle \mathbf{p}_i | \mathbf{p}_j^* \rangle = 0 \text{ for } {}^\forall i = 0,\ldots \widehat{j} \ldots, n-1, \ \ \langle \mathbf{p}_j | \mathbf{p}_j^* \rangle > 0, \text{ and } -\langle \mathbf{p}_j^* | \mathbf{p}_j^* \rangle = 1,$$

where the circumflex indicates that the term below it has been omitted. Then, $\mathbf{p}_0^*,\ldots, \mathbf{p}_{n-1}^*$ are in $\mathbb{H}^{n-1}$ and they are vertices of S (see Remark 1.3), $\mathbf{p}_0^*,\ldots \widehat{\mathbf{p}_k^*} \ldots, \mathbf{p}_{n-1}^*$ are vertices of S_k for each index k, and $\mathbf{p}_0^*,\ldots \widehat{\mathbf{p}_k^*} \ldots \widehat{\mathbf{p}_\ell^*} \ldots, \mathbf{p}_{n-1}^*$ are vertices of $S_{k,\ell}$ for distinct indices k and ℓ, that is,

$$S = (\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*) \cap \mathbb{H}^{n-1},$$

$$S_k = (\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \widehat{\mathbb{R}^+ \cdot \mathbf{p}_k^*} + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*) \cap \mathbb{H}^{n-1},$$

and

$$S_{k,\ell} = (\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \widehat{\mathbb{R}^+ \cdot \mathbf{p}_k^*} + \cdots + \widehat{\mathbb{R}^+ \cdot \mathbf{p}_\ell^*} + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*) \cap \mathbb{H}^{n-1},$$

where $\mathbb{R}^+$ is the set of all non-negative real numbers. If $n \geq 3$, for distinct indices k and i, let

$$\mathbf{p}_{\{k\},i} = \frac{\mathbf{p}_i - \langle \mathbf{p}_i | \mathbf{p}_k \rangle \mathbf{p}_k}{\sqrt{1 - \langle \mathbf{p}_i | \mathbf{p}_k \rangle^2}},$$

where $1 - \langle \mathbf{p}_i | \mathbf{p}_k \rangle^2 > 0$ from (31) of [7]. Then, unit vectors $\mathbf{p}_{\{k\},0},\ldots \widehat{\mathbf{p}_{\{k\},k}} \ldots, \mathbf{p}_{\{k\},n-1}$ construct the facet S_k, that is,

$$S_k = \{\mathbf{x} \in (\sum_i^{i\neq k} \mathbb{R} \cdot \mathbf{p}_{\{k\},i}) \cap \mathbb{H}^{n-1} : \langle \mathbf{x} | \mathbf{p}_{\{k\},i} \rangle \geq 0 \text{ for } {}^\forall i \neq k\}.$$

Moreover, if $n \geq 4$, for pairwise distinct indices k, ℓ, and i, let

(1)

$$\mathbf{p}_{\{k,\ell\},i} = \frac{\mathbf{p}_{\{\ell\},i} - \langle \mathbf{p}_{\{\ell\},i} | \mathbf{p}_{\{\ell\},k} \rangle \mathbf{p}_{\{\ell\},k}}{\sqrt{1 - \langle \mathbf{p}_{\{\ell\},i} | \mathbf{p}_{\{\ell\},k} \rangle^2}} =$$

$$= \frac{\sqrt{1 - \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle^2}}{\sqrt{1 - \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle^2 - \langle \mathbf{p}_k | \mathbf{p}_i \rangle^2 - \langle \mathbf{p}_\ell | \mathbf{p}_i \rangle^2 + 2\langle \mathbf{p}_k | \mathbf{p}_\ell \rangle \langle \mathbf{p}_k | \mathbf{p}_i \rangle \langle \mathbf{p}_\ell | \mathbf{p}_i \rangle}} \cdot$$

$$\cdot (\mathbf{p}_i - \frac{\langle \mathbf{p}_k | \mathbf{p}_i \rangle - \langle \mathbf{p}_\ell | \mathbf{p}_i \rangle \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle}{1 - \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle^2} \mathbf{p}_k - \frac{\langle \mathbf{p}_\ell | \mathbf{p}_i \rangle - \langle \mathbf{p}_k | \mathbf{p}_i \rangle \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle}{1 - \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle^2} \mathbf{p}_\ell) =$$

$$= \frac{\sqrt{1 - \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle^2} \mathbf{p}_i - \langle \mathbf{p}_i | \mathbf{p}_{\{\ell\},k} \rangle \mathbf{p}_k - \langle \mathbf{p}_i | \mathbf{p}_{\{k\},\ell} \rangle \mathbf{p}_\ell}{\sqrt{1 - \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle^2 - \langle \mathbf{p}_k | \mathbf{p}_i \rangle^2 - \langle \mathbf{p}_\ell | \mathbf{p}_i \rangle^2 + 2\langle \mathbf{p}_k | \mathbf{p}_\ell \rangle \langle \mathbf{p}_k | \mathbf{p}_i \rangle \langle \mathbf{p}_\ell | \mathbf{p}_i \rangle}},$$

where the formula in the square root of the denominator of the rightmost side is positive from (31) of [7], and the second equality comes from a direct calculation similar to Prop. 4.3 of [6]. Then, unit vectors $\mathbf{p}_{\{k,\ell\},0},\cdots\widehat{\mathbf{p}_{\{k,\ell\},k}}\cdots\widehat{\mathbf{p}_{\{k,\ell\},\ell}}\cdots$, $\mathbf{p}_{\{k,\ell\},n-1}$ construct the face $S_{k,\ell}$, that is,

$$S_{k,\ell}=\{\mathbf{x}\in(\sum_{i}^{i\neq k,\ell}\mathbb{R}\cdot\mathbf{p}_{\{k,\ell\},i})\cap\mathbb{H}^{n-1}:\langle\mathbf{x}|\mathbf{p}_{\{k,\ell\},i}\rangle\geq 0\ \text{ for }\ ^{\forall}i\neq k,\ell\}.$$

As mentioned above, $\langle i,j\rangle\in]0,\pi[$ denotes the dihedral angle between facets S_i and S_j of S, and $\langle i,j\rangle_k\in]0,\pi[$ denotes the dihedral angle between facets $S_{k,i}$ and $S_{k,j}$ of S_k, that is,

$$-\cos\langle i,j\rangle=\langle\mathbf{p}_i|\mathbf{p}_j\rangle,\quad -\cos\langle i,j\rangle_k=\langle\mathbf{p}_{\{k\},i}|\mathbf{p}_{\{k\},j}\rangle.$$

For the notations above, see [7]. Without loss of generality, we can represent $\mathbf{p}_0,\ldots,\mathbf{p}_{n-1}$, and $\mathbf{p}^*_0,\ldots,\mathbf{p}^*_{n-1}$ by the following:

$$\mathbf{p}_k=\begin{pmatrix}p_{k,0}\\ \vdots\\ \vdots\\ p_{k,k}\\ 0\\ \vdots\\ 0\end{pmatrix}\quad\text{and}\quad \mathbf{p}^*_k=\begin{pmatrix}0\\ \vdots\\ 0\\ p^*_{k,k}\\ \vdots\\ \vdots\\ p^*_{k,n-1}\end{pmatrix}\tag{2}$$

with $p_{k,k}>0$ for $0\leq k\leq n-2$, $p_{n-1,n-1}<0$, and $p^*_{k,k}>0$ for $0\leq k\leq n-1$.

3. Perturbation

Let S' be an infinitesimal perturbation of S determined by

$$\mathrm{d}\mathbf{p}_0=-\varepsilon\mathbf{p}^*_{n-1}\ \text{ and }\ \mathrm{d}\mathbf{p}_1=\cdots=\mathrm{d}\mathbf{p}_{n-1}=\mathbf{0},\tag{3}$$

that is,

$$S'=\{\mathbf{x}\in\mathbb{H}^{n-1}:\langle\mathbf{x}|\mathbf{p}'_i\rangle\geq 0\ \text{ for }\ ^{\forall}i=0,\ldots,n-1\},$$

where $\mathbf{p}'_i=\mathbf{p}_i+\mathrm{d}\mathbf{p}_i$. We ignore the second order of infinitesimal values, that is, we do not ignore ε but ε^2. Notice that $\langle\mathbf{p}'_0|\mathbf{p}'_0\rangle=1$ from $\langle\mathbf{p}_0|\mathbf{p}^*_{n-1}\rangle=0$.

Lemma 3.1. *If $n\geq 3$, for dihedral angles $\langle i,j\rangle'=\langle i,j\rangle+\mathrm{d}\langle i,j\rangle$, we have*

$$\mathrm{d}\langle 0,n-1\rangle=-\varepsilon\langle\mathbf{p}_{\{0\},n-1}|\mathbf{p}^*_{n-1}\rangle,\tag{4}$$

and

$$\mathrm{d}\langle i,j\rangle=0,\tag{5}$$

otherwise. Moreover, if $n\geq 4$, for $\langle i,j\rangle'_k=\langle i,j\rangle_k+\mathrm{d}\langle i,j\rangle_k$, we also have

$$\mathrm{d}\langle 0,n-1\rangle_k=\frac{\sqrt{1-\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle^2}\,\mathrm{d}\langle 0,n-1\rangle}{\sqrt{1-\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle^2-\langle\mathbf{p}_0|\mathbf{p}_k\rangle^2-\langle\mathbf{p}_{n-1}|\mathbf{p}_k\rangle^2+2\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle\langle\mathbf{p}_0|\mathbf{p}_k\rangle\langle\mathbf{p}_{n-1}|\mathbf{p}_k\rangle}},\tag{6}$$

for $k=1,\ldots,n-2$,

$$\mathrm{d}\langle 0,j\rangle_{n-1}=\frac{-\langle\mathbf{p}_j|\mathbf{p}_{\{0\},n-1}\rangle\mathrm{d}\langle 0,n-1\rangle}{\sqrt{1-\langle\mathbf{p}_0|\mathbf{p}_j\rangle^2-\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle^2-\langle\mathbf{p}_j|\mathbf{p}_{n-1}\rangle^2+2\langle\mathbf{p}_0|\mathbf{p}_j\rangle\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle\langle\mathbf{p}_j|\mathbf{p}_{n-1}\rangle}},\tag{7}$$

for $j=1,\ldots,n-2$,

$$\text{(8)}\quad \mathrm{d}\langle i,n-1\rangle_0=\frac{-\langle \mathbf{p}_i|\mathbf{p}_{\{n-1\},0}\rangle\mathrm{d}\langle 0,n-1\rangle}{\sqrt{1-\langle \mathbf{p}_i|\mathbf{p}_{n-1}\rangle^2-\langle \mathbf{p}_i|\mathbf{p}_0\rangle^2-\langle \mathbf{p}_{n-1}|\mathbf{p}_0\rangle^2+2\langle \mathbf{p}_i|\mathbf{p}_{n-1}\rangle\langle \mathbf{p}_i|\mathbf{p}_0\rangle\langle \mathbf{p}_{n-1}|\mathbf{p}_0\rangle}},$$

for $i=1,\ldots,n-2$, *and*

$$\text{(9)}\qquad \mathrm{d}\langle i,j\rangle_k=0,$$

otherwise.

Proof. If $n\geq 3$, for distinct indices i and j, we have

$$\langle \mathbf{p}'_i|\mathbf{p}'_j\rangle=-\cos\langle i,j\rangle'=-\cos\langle i,j\rangle+\sin\langle i,j\rangle\mathrm{d}\langle i,j\rangle=$$
$$=\langle \mathbf{p}_i|\mathbf{p}_j\rangle+\sqrt{1-\langle \mathbf{p}_i|\mathbf{p}_j\rangle^2}\mathrm{d}\langle i,j\rangle,$$

and

$$\text{(10)}\qquad \langle \mathbf{p}'_i|\mathbf{p}'_j\rangle=\langle \mathbf{p}_i|\mathbf{p}_j\rangle-\varepsilon\delta^{\{0,n-1\}}_{\{i,j\}}\langle \mathbf{p}_{n-1}|\mathbf{p}^*_{n-1}\rangle,$$

where (10) comes from $\mathbf{p}'_i=\mathbf{p}_i-\varepsilon\delta^0_i\mathbf{p}^*_{n-1}$ (notice that δ^k_i is Kronecker's delta and $\delta^{\{0,n-1\}}_{\{i,j\}}=\delta^0_i\delta^{n-1}_j+\delta^0_j\delta^{n-1}_i$). Comparing two equations above, it holds that

$$\mathrm{d}\langle i,j\rangle=-\frac{\varepsilon\delta^{\{0,n-1\}}_{\{i,j\}}\langle \mathbf{p}_{n-1}|\mathbf{p}^*_{n-1}\rangle}{\sqrt{1-\langle \mathbf{p}_i|\mathbf{p}_j\rangle^2}},$$

which means (4) and (5). Moreover, if $n\geq 4$, for pairwise distinct indices i, j, and k, we have

$$\langle \mathbf{p}'_{\{k\},i}|\mathbf{p}'_{\{k\},j}\rangle=-\cos\langle i,j\rangle'_k=-\cos\langle i,j\rangle_k+\sin\langle i,j\rangle_k\mathrm{d}\langle i,j\rangle_k=\langle \mathbf{p}_{\{k\},i}|\mathbf{p}_{\{k\},j}\rangle+$$
$$+\frac{\sqrt{1-\langle \mathbf{p}_i|\mathbf{p}_j\rangle^2-\langle \mathbf{p}_i|\mathbf{p}_k\rangle^2-\langle \mathbf{p}_j|\mathbf{p}_k\rangle^2+2\langle \mathbf{p}_i|\mathbf{p}_j\rangle\langle \mathbf{p}_i|\mathbf{p}_k\rangle\langle \mathbf{p}_j|\mathbf{p}_k\rangle}}{\sqrt{1-\langle \mathbf{p}_i|\mathbf{p}_k\rangle^2}\sqrt{1-\langle \mathbf{p}_j|\mathbf{p}_k\rangle^2}}\mathrm{d}\langle i,j\rangle_k,$$

and

$$\langle \mathbf{p}'_{\{k\},i}|\mathbf{p}'_{\{k\},j}\rangle=\frac{1}{\sqrt{1-\langle \mathbf{p}'_i|\mathbf{p}'_k\rangle^2}\sqrt{1-\langle \mathbf{p}'_j|\mathbf{p}'_k\rangle^2}}(\langle \mathbf{p}'_i|\mathbf{p}'_j\rangle-\langle \mathbf{p}'_i|\mathbf{p}'_k\rangle\langle \mathbf{p}'_j|\mathbf{p}'_k\rangle)=$$
$$=\frac{1-\varepsilon\langle \mathbf{p}_{n-1}|\mathbf{p}^*_{n-1}\rangle(\delta^{\{0,n-1\}}_{\{i,k\}}\frac{\langle \mathbf{p}_i|\mathbf{p}_k\rangle}{1-\langle \mathbf{p}_i|\mathbf{p}_k\rangle^2}+\delta^{\{0,n-1\}}_{\{j,k\}}\frac{\langle \mathbf{p}_j|\mathbf{p}_k\rangle}{1-\langle \mathbf{p}_j|\mathbf{p}_k\rangle^2})}{\sqrt{1-\langle \mathbf{p}_i|\mathbf{p}_k\rangle^2}\sqrt{1-\langle \mathbf{p}_j|\mathbf{p}_k\rangle^2}}\cdot$$
$$\cdot((\langle \mathbf{p}_i|\mathbf{p}_j\rangle-\langle \mathbf{p}_i|\mathbf{p}_k\rangle\langle \mathbf{p}_j|\mathbf{p}_k\rangle-$$
$$-\varepsilon\langle \mathbf{p}_{n-1}|\mathbf{p}^*_{n-1}\rangle(\delta^{\{0,n-1\}}_{\{i,j\}}-\delta^{\{0,n-1\}}_{\{i,k\}}\langle \mathbf{p}_j|\mathbf{p}_k\rangle-\delta^{\{0,n-1\}}_{\{j,k\}}\langle \mathbf{p}_i|\mathbf{p}_k\rangle))=$$
$$=\frac{1}{\sqrt{1-\langle \mathbf{p}_i|\mathbf{p}_k\rangle^2}\sqrt{1-\langle \mathbf{p}_j|\mathbf{p}_k\rangle^2}}((\langle \mathbf{p}_i|\mathbf{p}_j\rangle-\langle \mathbf{p}_i|\mathbf{p}_k\rangle\langle \mathbf{p}_j|\mathbf{p}_k\rangle-\varepsilon\langle \mathbf{p}_{n-1}|\mathbf{p}^*_{n-1}\rangle\cdot$$
$$\cdot(\delta^{\{0,n-1\}}_{\{i,j\}}+\delta^{\{0,n-1\}}_{\{i,k\}}(((\langle \mathbf{p}_i|\mathbf{p}_j\rangle-\langle \mathbf{p}_i|\mathbf{p}_k\rangle\langle \mathbf{p}_j|\mathbf{p}_k\rangle)\frac{\langle \mathbf{p}_i|\mathbf{p}_k\rangle}{1-\langle \mathbf{p}_i|\mathbf{p}_k\rangle^2}-\langle \mathbf{p}_j|\mathbf{p}_k\rangle)+$$
$$+\delta^{\{0,n-1\}}_{\{j,k\}}(((\langle \mathbf{p}_i|\mathbf{p}_j\rangle-\langle \mathbf{p}_i|\mathbf{p}_k\rangle\langle \mathbf{p}_j|\mathbf{p}_k\rangle)\frac{\langle \mathbf{p}_j|\mathbf{p}_k\rangle}{1-\langle \mathbf{p}_j|\mathbf{p}_k\rangle^2}-\langle \mathbf{p}_i|\mathbf{p}_k\rangle))))=$$

$$= \langle \mathbf{p}_{\{k\},i} | \mathbf{p}_{\{k\},j} \rangle + \frac{\sqrt{1 - \langle \mathbf{p}_0 | \mathbf{p}_{n-1} \rangle^2} \mathrm{d}\langle 0, n-1 \rangle}{\sqrt{1 - \langle \mathbf{p}_i | \mathbf{p}_k \rangle^2} \sqrt{1 - \langle \mathbf{p}_j | \mathbf{p}_k \rangle^2}} \cdot$$

$$\cdot \, (\delta^{\{0,n-1\}}_{\{i,j\}} - \delta^{\{0,n-1\}}_{\{i,k\}} \frac{\langle \mathbf{p}_j | \mathbf{p}_{\{i\},k} \rangle}{\sqrt{1 - \langle \mathbf{p}_i | \mathbf{p}_k \rangle^2}} - \delta^{\{0,n-1\}}_{\{j,k\}} \frac{\langle \mathbf{p}_i | \mathbf{p}_{\{j\},k} \rangle}{\sqrt{1 - \langle \mathbf{p}_j | \mathbf{p}_k \rangle^2}}),$$

where the second equality of the latter calculation comes from (10). Comparing two equations above, it holds that

$$\mathrm{d}\langle i, j \rangle_k = \frac{\sqrt{1 - \langle \mathbf{p}_0 | \mathbf{p}_{n-1} \rangle^2} \mathrm{d}\langle 0, n-1 \rangle}{\sqrt{1 - \langle \mathbf{p}_i | \mathbf{p}_j \rangle^2 - \langle \mathbf{p}_i | \mathbf{p}_k \rangle^2 - \langle \mathbf{p}_j | \mathbf{p}_k \rangle^2 + 2 \langle \mathbf{p}_i | \mathbf{p}_j \rangle \langle \mathbf{p}_i | \mathbf{p}_k \rangle \langle \mathbf{p}_j | \mathbf{p}_k \rangle}} \cdot$$

$$\cdot \, (\delta^{\{0,n-1\}}_{\{i,j\}} - \delta^{\{0,n-1\}}_{\{i,k\}} \frac{\langle \mathbf{p}_j | \mathbf{p}_{\{i\},k} \rangle}{\sqrt{1 - \langle \mathbf{p}_i | \mathbf{p}_k \rangle^2}} - \delta^{\{0,n-1\}}_{\{j,k\}} \frac{\langle \mathbf{p}_i | \mathbf{p}_{\{j\},k} \rangle}{\sqrt{1 - \langle \mathbf{p}_j | \mathbf{p}_k \rangle^2}}),$$

which means (6), (7), (8), and (9). □

Remark 3.1. *The perturbation above moves only the angle $\langle 0, n-1 \rangle$, so we can treat easily an arbitrary perturbation (in particular, for moving only the angle $\langle k, \ell \rangle$, we can use the perturbation $\mathrm{d}\mathbf{p}_i = -\varepsilon \delta^k_i \mathbf{p}^*_\ell$ for $i = 0, \ldots, n-1$).*

4. Proof

We prove the main result by division into three cases: an arc ($n = 2$), a hyperbolic triangle ($n = 3$), a simplex with $n \geq 4$.

Proof of Theorem 1.2 for an arc ($n = 2$). For an arc S with length A, from (2), we can put

$$\mathbf{p}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{p}_1 = \begin{pmatrix} -\cosh A \\ -\sinh A \end{pmatrix}, \qquad \mathbf{p}^*_0 = \begin{pmatrix} \sinh A \\ \cosh A \end{pmatrix}, \qquad \mathbf{p}^*_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we can get the conclusion by an easy calculation:

$$\int_{\mathbf{x} \in S} \mathbf{x} dS = \int_0^A \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} dt = -(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\cosh A \\ -\sinh A \end{pmatrix}) = -\sum_{k=0}^{1} |S_k| \mathbf{p}_k,$$

where the first equality comes from the fact that $(\sinh t, \cosh t)^T$ is a unit speed geodesic (see also (13)) and the last equality comes from $|S_k| = \mu_0(\{\mathbf{p}^*_{1-k}\}) = 1$ (see Figure 1). □

For a hyperbolic triangle S, let

$$\alpha = \langle 1, 2 \rangle, \quad \beta = \langle 2, 0 \rangle, \quad \gamma = \langle 0, 1 \rangle,$$
$$A = |S_0|, \ B = |S_1|, \quad \text{and} \quad C = |S_2|.$$

Then, according to (2), we can represent

$$\mathbf{p}_0 = \begin{pmatrix} p_{0,0} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{p}^*_0 = \begin{pmatrix} p^*_{0,0} \\ p^*_{0,1} \\ p^*_{0,2} \end{pmatrix} = \begin{pmatrix} \sin\gamma \sinh B \\ \cos\gamma \sinh B \\ \cosh B \end{pmatrix},$$

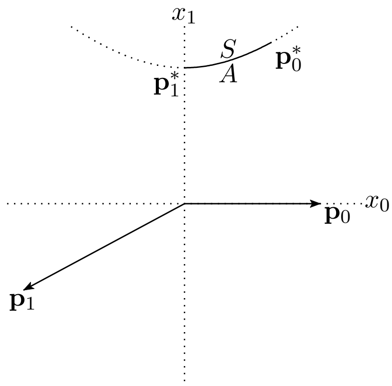

FIGURE 1. Illustrating the proof of Theorem 1.2 for an arc ($n = 2$)

$$\mathbf{p}_1 = \begin{pmatrix} p_{1,0} \\ p_{1,1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\gamma \\ \sin\gamma \\ 0 \end{pmatrix}, \qquad \mathbf{p}_1^* = \begin{pmatrix} 0 \\ p_{1,1}^* \\ p_{1,2}^* \end{pmatrix} = \begin{pmatrix} 0 \\ \sinh A \\ \cosh A \end{pmatrix},$$

and

$$\mathbf{p}_2 = \begin{pmatrix} p_{2,0} \\ p_{2,1} \\ p_{2,2} \end{pmatrix} = \begin{pmatrix} -\cos\beta \\ -\sin\beta\cosh A \\ -\sin\beta\sinh A \end{pmatrix}, \quad \mathbf{p}_2^* = \begin{pmatrix} 0 \\ 0 \\ p_{2,2}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The sum of angles $\alpha + \beta + \gamma$ is less than π, so, without loss of generality, we can assume that $\gamma < \frac{\pi}{2}$. So all coordinates $p_{k,i}^*$ of 3 vertices are positive (see Figure 2).

Proof of Theorem 1.2 for a hyperbolic triangle ($n = 3$). Let $\mathbf{x} \in S$ be represented by

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sinh s \\ \cosh s \sinh t \\ \cosh s \cosh t \end{pmatrix},$$

where $s,\ t \in]-\infty, \infty[$. Then the domain of integration is

$$\{(s,t) : 0 \le x_0 \le p_{0,0}^*, \langle \mathbf{x}|\mathbf{p}_1\rangle \ge 0, \langle \mathbf{x}|\mathbf{p}_2\rangle \ge 0\} = \{(s,t) : 0 \le \sinh s \le \sin\gamma \sinh B,$$
$$-\cos\gamma\sinh s + \sin\gamma\cosh s\sinh t \ge 0, -\cos\beta\sinh s + \sin\beta\cosh s\sinh(A-t) \ge 0\}.$$

We can prove the theorem by the following direct calculation (notice that if $\beta > \frac{\pi}{2}$ (resp. $\beta = \frac{\pi}{2}$) then $A - \operatorname{arsinh}\frac{\tanh s}{\tan\beta}$ means $A + \operatorname{arsinh}\frac{\tanh s}{-\tan\beta}$ (resp. A)):

$$\int_{\mathbf{x}\in S} \mathbf{x} dS = \int_0^{\operatorname{arsinh}(\sin\gamma\sinh B)} \int_{\operatorname{arsinh}\frac{\tanh s}{\tan\gamma}}^{A-\operatorname{arsinh}\frac{\tanh s}{\tan\beta}} \begin{pmatrix} \sinh s \\ \cosh s\sinh t \\ \cosh s\cosh t \end{pmatrix} dt \cosh s ds =$$

$$= \int_0^{\operatorname{arsinh}(\sin\gamma\sinh B)} \left[\begin{pmatrix} \sinh s \cdot t \\ \cosh s\cosh t \\ \cosh s\sinh t \end{pmatrix} \right]_{t=\operatorname{arsinh}\frac{\tanh s}{\tan\gamma}}^{t=A-\operatorname{arsinh}\frac{\tanh s}{\tan\beta}} \cosh s ds =$$

$$= \int_0^{\operatorname{arsinh}(\sin\gamma\sinh B)} (\begin{pmatrix} \sinh s(A - \operatorname{arsinh}\frac{\tanh s}{\tan\beta}) \\ \cosh s(\cosh A\sqrt{1+(\frac{\tanh s}{\tan\beta})^2} - \sinh A\frac{\tanh s}{\tan\beta}) \\ \cosh s(\sinh A\sqrt{1+(\frac{\tanh s}{\tan\beta})^2} - \cosh A\frac{\tanh s}{\tan\beta}) \end{pmatrix} -$$

$$- \begin{pmatrix} \sinh s \operatorname{arsinh}\frac{\tanh s}{\tan\gamma} \\ \cosh s\sqrt{1+(\frac{\tanh s}{\tan\gamma})^2} \\ \cosh s\frac{\tanh s}{\tan\gamma} \end{pmatrix}) \cosh s ds =$$

$$= \frac{1}{2}[\begin{pmatrix} \cosh^2 s(A - \operatorname{arsinh}\frac{\tanh s}{\tan\beta}) + \cos\beta \operatorname{arsinh}\frac{\sinh s}{\sin\beta} \\ \cosh A\sin\beta(\operatorname{arsinh}\frac{\sinh s}{\sin\beta} + \frac{\sinh s}{\sin\beta}\sqrt{1+(\frac{\sinh s}{\sin\beta})^2}) - \sinh A\frac{\sinh^2 s}{\tan\beta} \\ \sinh A\sin\beta(\operatorname{arsinh}\frac{\sinh s}{\sin\beta} + \frac{\sinh s}{\sin\beta}\sqrt{1+(\frac{\sinh s}{\sin\beta})^2}) - \cosh A\frac{\sinh^2 s}{\tan\beta} \end{pmatrix} -$$

$$- \begin{pmatrix} -\cos\gamma \operatorname{arsinh}\frac{\sinh s}{\sin\gamma} + \cosh^2 s \operatorname{arsinh}\frac{\tanh s}{\tan\gamma} \\ \sin\gamma(\operatorname{arsinh}\frac{\sinh s}{\sin\gamma} + \frac{\sinh s}{\sin\gamma}\sqrt{1+(\frac{\sinh s}{\sin\gamma})^2}) \\ \frac{\sinh^2 s}{\tan\gamma} \end{pmatrix}]_0^{\operatorname{arsinh}(\sin\gamma\sinh B)} =$$

$$= \frac{1}{2}(\begin{pmatrix} (1+(\sin\gamma\sinh B)^2)(A - \operatorname{artanh}(\cos\beta\tanh C)) + \cos\beta\cdot C - A \\ \cosh A\sin\beta(C + \sinh C\cosh C) - \sinh A\cos\beta\sin\beta\sinh^2 C \\ \sinh A\sin\beta(C + \sinh C\cosh C) - \cosh A\cos\beta\sin\beta\sinh^2 C \end{pmatrix} -$$

$$- \begin{pmatrix} -\cos\gamma\cdot B + (1+(\sin\gamma\sinh B)^2)\operatorname{artanh}(\cos\gamma\tanh B) \\ \sin\gamma(B + \sinh B\cosh B) \\ \cos\gamma\sin\gamma\sinh^2 B \end{pmatrix}) =$$

$$= \frac{1}{2}(\begin{pmatrix} (1+(\sin\gamma\sinh B)^2)(A - \operatorname{artanh}(\cos\beta\tanh C) - \operatorname{artanh}(\cos\gamma\tanh B)) - A \\ \sin\gamma\sinh B(\cosh C\cosh A - \cos\beta\sinh C\sinh A - \cosh B) \\ \sin\gamma\sinh B(\cosh C\sinh A - \cos\beta\sinh C\cosh A - \cos\gamma\sinh B) \end{pmatrix} +$$

$$+ \begin{pmatrix} B\cos\gamma \\ -B\sin\gamma \\ 0 \end{pmatrix} + \begin{pmatrix} C\cos\beta \\ C\sin\beta\cosh A \\ C\sin\beta\sinh A \end{pmatrix}) =$$

$$= -\frac{1}{2}(A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + B\begin{pmatrix} -\cos\gamma \\ \sin\gamma \\ 0 \end{pmatrix} + C\begin{pmatrix} -\cos\beta \\ -\sin\beta\cosh A \\ -\sin\beta\sinh A \end{pmatrix}) = -\frac{1}{2}\sum_{k=0}^{2}|S_k|\mathbf{p}_k,$$

where the first equality comes from the fact that the surface element of $\mathbb{H}^2$ is

$$\sqrt{\det\begin{pmatrix} \langle\frac{\partial\mathbf{x}}{\partial s}|\frac{\partial\mathbf{x}}{\partial s}\rangle & \langle\frac{\partial\mathbf{x}}{\partial s}|\frac{\partial\mathbf{x}}{\partial t}\rangle \\ \langle\frac{\partial\mathbf{x}}{\partial t}|\frac{\partial\mathbf{x}}{\partial s}\rangle & \langle\frac{\partial\mathbf{x}}{\partial t}|\frac{\partial\mathbf{x}}{\partial t}\rangle \end{pmatrix}} dsdt = \sqrt{\det\begin{pmatrix} 1 & 0 \\ 0 & \cosh^2 s \end{pmatrix}} dsdt = \cosh s dsdt,$$

the fifth equality comes from $\sin\gamma\sinh B = \sin\beta\sinh C$ and

$$\operatorname{arsinh}\frac{\tanh(\operatorname{arsinh}(\sin\beta\sinh C))}{\tan\beta} = \operatorname{arsinh}\frac{\sin\beta\sinh C/\sqrt{1+(\sin\beta\sinh C)^2}}{\tan\beta} =$$

$$= \operatorname{arsinh}\frac{\cos\beta\tanh C}{\sqrt{1-(\cos\beta\tanh C)^2}} = \operatorname{artanh}(\cos\beta\tanh C)$$

(and $\operatorname{arsinh}\frac{\tanh(\operatorname{arsinh}(\sin\gamma\sinh B))}{\tan\gamma} = \operatorname{artanh}(\cos\gamma\tanh B)$), the sixth equality comes from $\sin\gamma\sinh B = \sin\beta\sinh C$ again, and the seventh equality

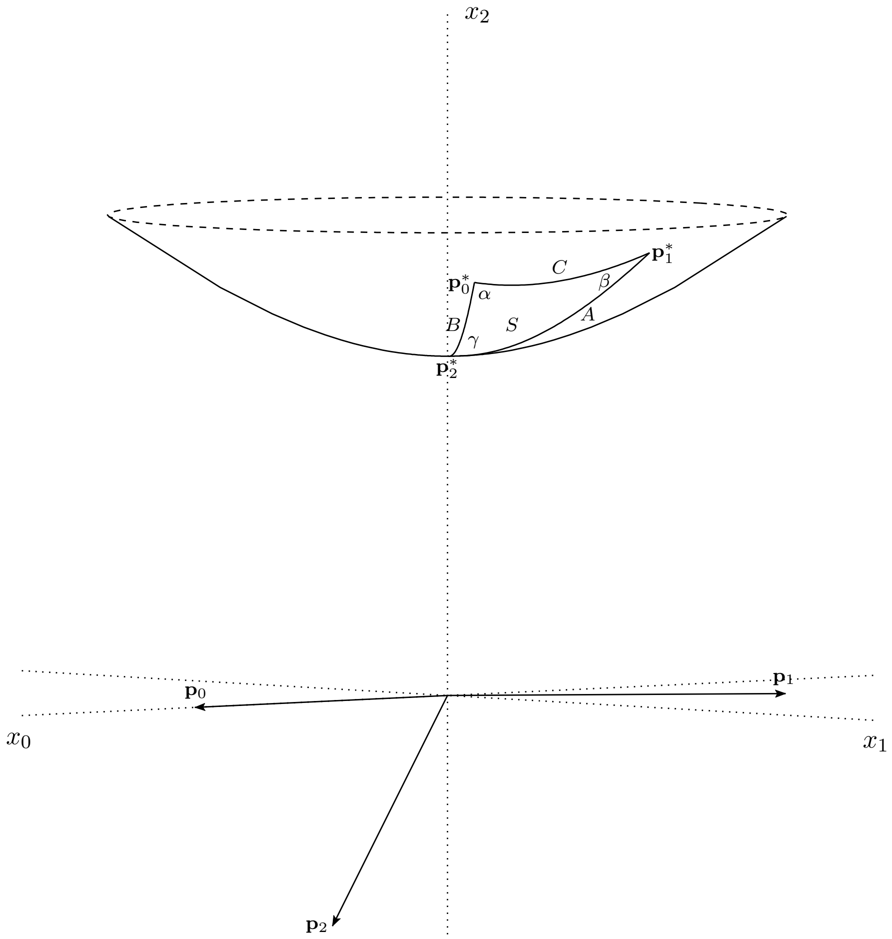

FIGURE 2. Illustrating the proof of Theorem 1.2 for a hyperbolic triangle ($n = 3$)

comes from $\operatorname{artanh}(\cos\gamma\tanh B) + \operatorname{artanh}(\cos\beta\tanh C) = A$ (notice that if $\beta > \frac{\pi}{2}$ (resp. $\beta = \frac{\pi}{2}$) then it means $\operatorname{artanh}(\cos\gamma\tanh B) - \operatorname{artanh}(\cos(\pi - \beta)\tanh C) = A$ (resp. $\operatorname{artanh}(\cos\gamma\tanh B) = A$)). $\square$

Remark 4.1. *For a hyperbolic triangle S, in general,* $\mathbf{g}$ *does not coincide with another hyperbolic centroid, the intersection point of medians, that is,*

$$\mathbf{g} = \frac{\frac{1}{2}\begin{pmatrix} -A + B\cos\gamma + C\cos\beta \\ -B\sin\gamma + C\sin\beta\cosh A \\ C\sin\beta\sinh A \end{pmatrix}}{\sqrt{-\langle \frac{1}{2}\begin{pmatrix} -A + B\cos\gamma + C\cos\beta \\ -B\sin\gamma + C\sin\beta\cosh A \\ C\sin\beta\sinh A \end{pmatrix} | \frac{1}{2}\begin{pmatrix} -A + B\cos\gamma + C\cos\beta \\ -B\sin\gamma + C\sin\beta\cosh A \\ C\sin\beta\sinh A \end{pmatrix}\rangle}} \neq$$

$$\neq \frac{\frac{1}{3}\begin{pmatrix}\sin\gamma\sinh B\\ \cos\gamma\sinh B+\sinh A\\ \cosh B+\cosh A+1\end{pmatrix}}{\sqrt{-\langle\frac{1}{3}\begin{pmatrix}\sin\gamma\sinh B\\ \cos\gamma\sinh B+\sinh A\\ \cosh B+\cosh A+1\end{pmatrix}|\frac{1}{3}\begin{pmatrix}\sin\gamma\sinh B\\ \cos\gamma\sinh B+\sinh A\\ \cosh B+\cosh A+1\end{pmatrix}\rangle}} =$$

$$= \frac{\frac{1}{3}(\mathbf{p}_0^*+\mathbf{p}_1^*+\mathbf{p}_2^*)}{\sqrt{-\langle\frac{1}{3}(\mathbf{p}_0^*+\mathbf{p}_1^*+\mathbf{p}_2^*)|\frac{1}{3}(\mathbf{p}_0^*+\mathbf{p}_1^*+\mathbf{p}_2^*)\rangle}}.$$

For higher dimensional case, the proof is slightly complicated. The point $\mathbf{x}\in S$ can be represented by

$$\mathbf{x}=\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\\vdots\\x_{n-3}\\x_{n-2}\\x_{n-1}\end{pmatrix}=\begin{pmatrix}\sin\psi_0\cdots\sin\psi_{n-3}\sinh\psi_{n-2}\\ \cos\psi_0\sin\psi_1\cdots\sin\psi_{n-3}\sinh\psi_{n-2}\\ \cos\psi_1\sin\psi_2\cdots\sin\psi_{n-3}\sinh\psi_{n-2}\\ \cos\psi_2\sin\psi_3\cdots\sin\psi_{n-3}\sinh\psi_{n-2}\\ \vdots\\ \cos\psi_{n-4}\sin\psi_{n-3}\sinh\psi_{n-2}\\ \cos\psi_{n-3}\sinh\psi_{n-2}\\ \cosh\psi_{n-2}\end{pmatrix},$$

where $\psi_0\in[-\pi,\pi[$, $\psi_1,\ldots,\ \psi_{n-3}\in[0,\pi]$, and $\psi_{n-2}\in[0,\infty[$. For the proof for general case, we need the following lemma with the following remark.

Remark 4.2. *Notice that the symbols* d *and* d *mean volume elements and infinitesimal values, respectively, that is,* dS_0, $dS_{0,\overline{n-1}}$, *and* $dS_{0,n-1}$ *mean volume elements of* S_0, $S_{0,\overline{n-1}}$, *and* $S_{0,n-1}$, *respectively, and* $\mathrm{d}\mathbf{p}_0$ *and* $\mathrm{d}\langle i,j\rangle$ *(and so on) mean infinitesimal values.*

Lemma 4.1. *If*

$$\mathbf{y}=\begin{pmatrix}0\\y_1\\y_2\\y_3\\\vdots\\y_{n-3}\\y_{n-2}\\y_{n-1}\end{pmatrix}=\begin{pmatrix}0\\ \sin\psi_1\cdots\sin\psi_{n-3}\sinh\psi_{n-2}\\ \cos\psi_1\sin\psi_2\cdots\sin\psi_{n-3}\sinh\psi_{n-2}\\ \cos\psi_2\sin\psi_3\cdots\sin\psi_{n-3}\sinh\psi_{n-2}\\ \vdots\\ \cos\psi_{n-4}\sin\psi_{n-3}\sinh\psi_{n-2}\\ \cos\psi_{n-3}\sinh\psi_{n-2}\\ \cosh\psi_{n-2}\end{pmatrix}$$

moves on $S_0\setminus\{\mathbf{p}_{n-1}^*\}$, *then*

$$\mathbf{w}=\begin{pmatrix}0\\w_1\\w_2\\w_3\\\vdots\\w_{n-3}\\w_{n-2}\\0\end{pmatrix}=\frac{1}{\sqrt{y_{n-1}^2-1}}\begin{pmatrix}0\\y_1\\y_2\\y_3\\\vdots\\y_{n-3}\\y_{n-2}\\0\end{pmatrix}=\begin{pmatrix}0\\ \sin\psi_1\cdots\sin\psi_{n-3}\\ \cos\psi_1\sin\psi_2\cdots\sin\psi_{n-3}\\ \cos\psi_2\sin\psi_3\cdots\sin\psi_{n-3}\\ \vdots\\ \cos\psi_{n-4}\sin\psi_{n-3}\\ \cos\psi_{n-3}\\ 0\end{pmatrix}$$

and

$$\mathbf{z} = \begin{pmatrix} 0 \\ z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-3} \\ z_{n-2} \\ z_{n-1} \end{pmatrix} = \frac{1}{\sqrt{\langle \mathbf{w} | \mathbf{p}_{n-1} \rangle^2 - p_{n-1,n-1}^2}} \begin{pmatrix} 0 \\ -p_{n-1,n-1} w_1 \\ -p_{n-1,n-1} w_2 \\ -p_{n-1,n-1} w_3 \\ \vdots \\ -p_{n-1,n-1} w_{n-3} \\ -p_{n-1,n-1} w_{n-2} \\ -\langle \mathbf{w} | \mathbf{p}_{n-1} \rangle \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ \sin\psi_1 \cdots \sin\psi_{n-3} \sinh\phi \\ \cos\psi_1 \sin\psi_2 \cdots \sin\psi_{n-3} \sinh\phi \\ \cos\psi_2 \sin\psi_3 \cdots \sin\psi_{n-3} \sinh\phi \\ \vdots \\ \cos\psi_{n-4} \sin\psi_{n-3} \sinh\phi \\ \cos\psi_{n-3} \sinh\phi \\ \cosh\phi \end{pmatrix} \quad \textit{with} \quad \phi = \operatorname{artanh} \frac{-p_{n-1,n-1}}{-\langle \mathbf{w} | \mathbf{p}_{n-1} \rangle}$$

move on

$$S_{0,\overline{n-1}} = \{\mathbf{w}' \in \mathbb{S}^{n-1} : \langle \mathbf{w}' | \mathbf{p}_i \rangle \geq 0 \ \textit{for}\ 1 \leq^{\forall} i \leq n-2,\ \langle \mathbf{w}' | \mathbf{p}_0 \rangle = \langle \mathbf{w}' | \mathbf{p}^*_{n-1} \rangle = 0\}$$

and $S_{0,n-1}$*, respectively (notice that* ϕ *is determined by the angles* $\psi_1, \ldots, \psi_{n-3}$*). Moreover, we have the following equations of volume elements:*

$$dS_0 = \sinh^{n-3}\psi_{n-2}\, d\psi_{n-2}\, dS_{0,\overline{n-1}} \tag{11}$$

and

$$\frac{\sinh^{n-2}\phi}{\langle \mathbf{p}_{\{0\},n-1} | \mathbf{p}^*_{n-1} \rangle} dS_{0,\overline{n-1}} = dS_{0,n-1}. \tag{12}$$

Proof. For $\mathbf{w} = (\mathbf{y} + \langle \mathbf{y} | \mathbf{p}^*_{n-1} \rangle \mathbf{p}^*_{n-1}) / \sqrt{\langle \mathbf{y} | \mathbf{p}^*_{n-1} \rangle^2 - 1}$, it is easy to check that

$$\langle \mathbf{w} | \mathbf{p}_i \rangle = \frac{\langle \mathbf{y} | \mathbf{p}_i \rangle}{\sqrt{\langle \mathbf{y} | \mathbf{p}^*_{n-1} \rangle^2 - 1}} \begin{cases} = 0 & (i = 0), \\ \geq 0 & (1 \leq i \leq n-2), \end{cases}$$

and $\langle \mathbf{w} | \mathbf{p}^*_{n-1} \rangle = 0$, which also shows $\|\mathbf{w}\|^2 = \langle \mathbf{w} | \mathbf{w} \rangle = 1$. For

$$\tilde{\mathbf{z}} = (0, \tilde{z}_1, \ldots, \tilde{z}_{n-2}; \tilde{z}_{n-1})^T = -p_{n-1,n-1}\mathbf{w} - \langle \mathbf{w} | \mathbf{p}_{n-1} \rangle \mathbf{p}^*_{n-1},$$

we have $\tilde{\mathbf{z}} \neq \mathbf{0}$, $\langle \tilde{\mathbf{z}} | \mathbf{p}_{n-1} \rangle = 0$, and

$$\langle \tilde{\mathbf{z}} | \mathbf{p}_i \rangle = -p_{n-1,n-1} \langle \mathbf{w} | \mathbf{p}_i \rangle \begin{cases} = 0 & (i = 0), \\ \geq 0 & (1 \leq i \leq n-2). \end{cases}$$

So we also have

$$-\langle \tilde{\mathbf{z}} | \tilde{\mathbf{z}} \rangle > 0, \quad \tilde{z}_{n-1} > 0, \quad \text{and} \quad \mathbf{z} = \frac{\tilde{\mathbf{z}}}{\sqrt{-\langle \tilde{\mathbf{z}} | \tilde{\mathbf{z}} \rangle}} \in S_{0,n-1}.$$

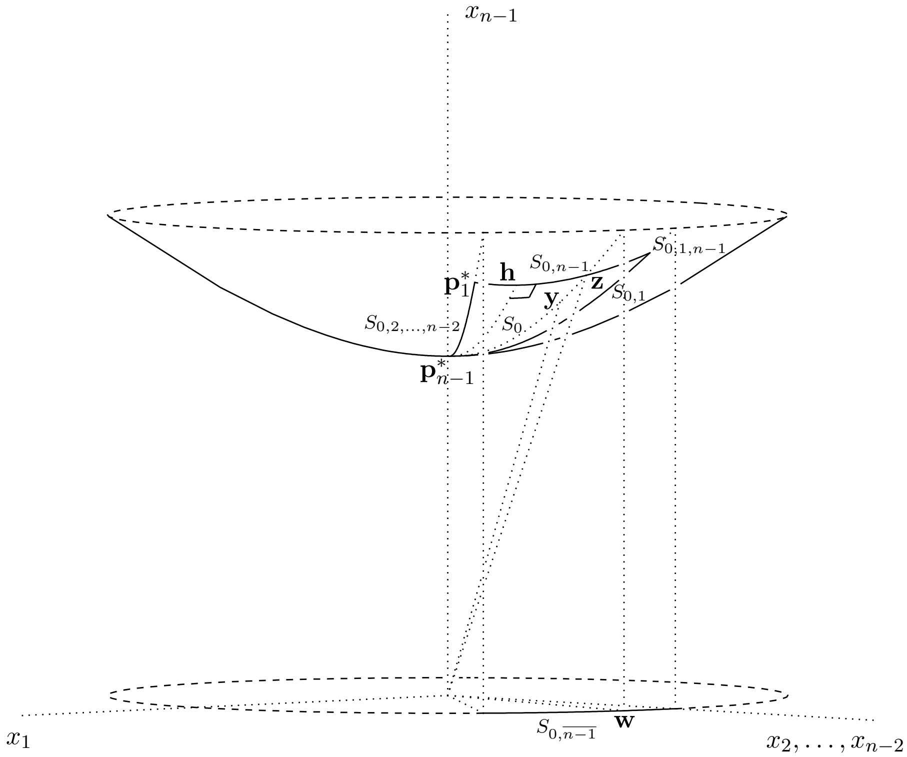

FIGURE 3. Illustrating the proof of Lemma 4.1

For the representations $\mathbf{w}$ and $\mathbf{z}$, see Figure 3 of $(n-1)$-dimensional Minkowski space $\{0\} \times \mathbb{R}^{n-1}$. For (11), see Figure 4. The equation (12) comes from

$$\frac{\sinh^{n-2}\phi}{\langle \mathbf{p}_{\{0\},n-1} | \mathbf{p}_{n-1}^* \rangle} dS_{0,\overline{n-1}} = \frac{\sinh^{n-2}\phi}{\sinh h} dS_{0,\overline{n-1}} = \frac{\sinh^{n-3}\phi}{\sin\theta} dS_{0,\overline{n-1}} = dS_{0,n-1},$$

where the first equality holds because $\mathbf{h}$ is the foot of the perpendicular great circle arc from $\mathbf{p}_0^*$ to $S_{0,n-1} = \{\mathbf{z} \in S_0 : \langle \mathbf{z} | \mathbf{p}_{\{0\},n-1} \rangle = 0\}$ (see Figures 3 and 5), the second equality comes from

$$1 - \sin^2\theta = \cos^2\theta = \left(\frac{\cosh\gamma\cosh\phi - \cosh h}{\sinh\gamma\sinh\phi}\right)^2 = \left(\frac{\frac{\cosh\phi}{\cosh h}\cosh\phi - \cosh h}{\sqrt{(\frac{\cosh\phi}{\cosh h})^2 - 1}\sinh\phi}\right)^2 =$$

$$= \frac{\cosh^2 h((\frac{\cosh\phi}{\cosh h})^2 - 1)^2}{((\frac{\cosh\phi}{\cosh h})^2 - 1)\sinh^2\phi} = \frac{\cosh^2\phi - \cosh^2 h}{\sinh^2\phi} = \frac{\sinh^2\phi - \sinh^2 h}{\sinh^2\phi}$$

(see Figure 5 again), and the last equality comes from Figure 4. □

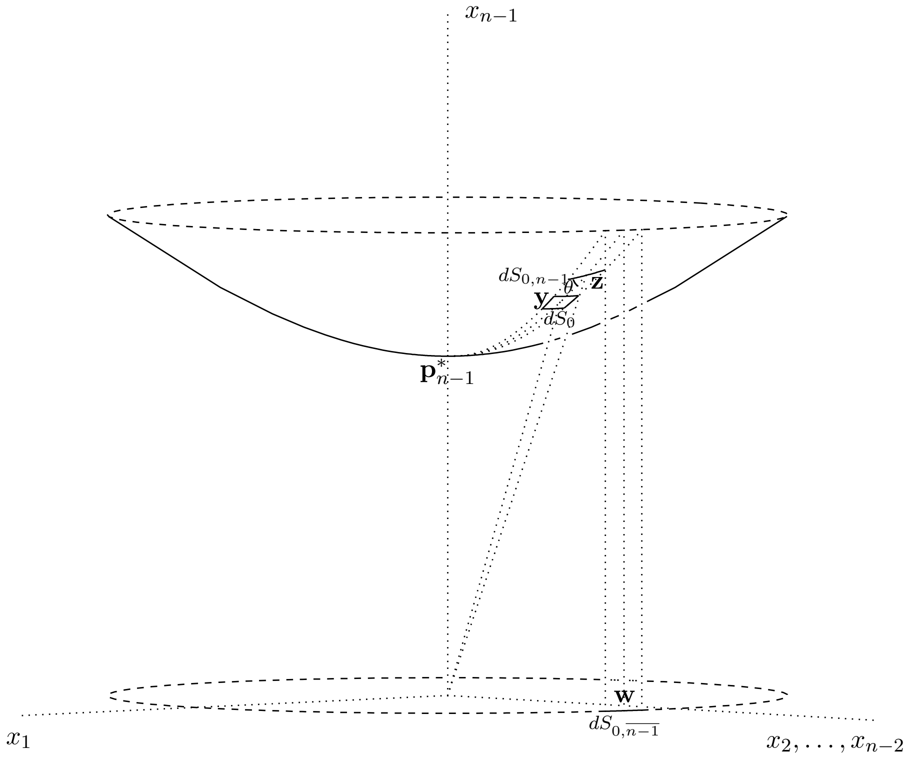

FIGURE 4. Illustrating the proof of Lemma 4.1

Another proof of (11) and (12). These equations can be proved by the following direct calculations:

$$g_{\mathbf{x}} = \begin{pmatrix} \langle \frac{\partial \mathbf{x}}{\partial \psi_0} | \frac{\partial \mathbf{x}}{\partial \psi_0} \rangle & \cdots & \langle \frac{\partial \mathbf{x}}{\partial \psi_0} | \frac{\partial \mathbf{x}}{\partial \psi_{n-2}} \rangle \\ \vdots & & \vdots \\ \langle \frac{\partial \mathbf{x}}{\partial \psi_{n-2}} | \frac{\partial \mathbf{x}}{\partial \psi_0} \rangle & \cdots & \langle \frac{\partial \mathbf{x}}{\partial \psi_{n-2}} | \frac{\partial \mathbf{x}}{\partial \psi_{n-2}} \rangle \end{pmatrix} =$$
$$= \mathrm{diag}(\sin^2 \psi_1 \sin^2 \psi_2 \sin^2 \psi_3 \cdots \sin^2 \psi_{n-3} \sinh^2 \psi_{n-2},$$
$$\sin^2 \psi_2 \sin^2 \psi_3 \cdots \sin^2 \psi_{n-3} \sinh^2 \psi_{n-2}, \cdots$$
$$\ldots, \sin^2 \psi_{n-3} \sinh^2 \psi_{n-2}, \sinh^2 \psi_{n-2}, 1),$$

$$\begin{aligned} dS &= \sqrt{\det g_{\mathbf{x}}} d\psi_0 d\psi_1 d\psi_2 \cdots d\psi_{n-3} d\psi_{n-2} = \\ &= \sin \psi_1 \sin^2 \psi_2 \sin^3 \psi_3 \cdots \sin^{n-3} \psi_{n-3} \sinh^{n-2} \psi_{n-2} \cdot \\ &\cdot d\psi_0 d\psi_1 d\psi_2 \cdots d\psi_{n-3} d\psi_{n-2}, \end{aligned} \tag{13}$$

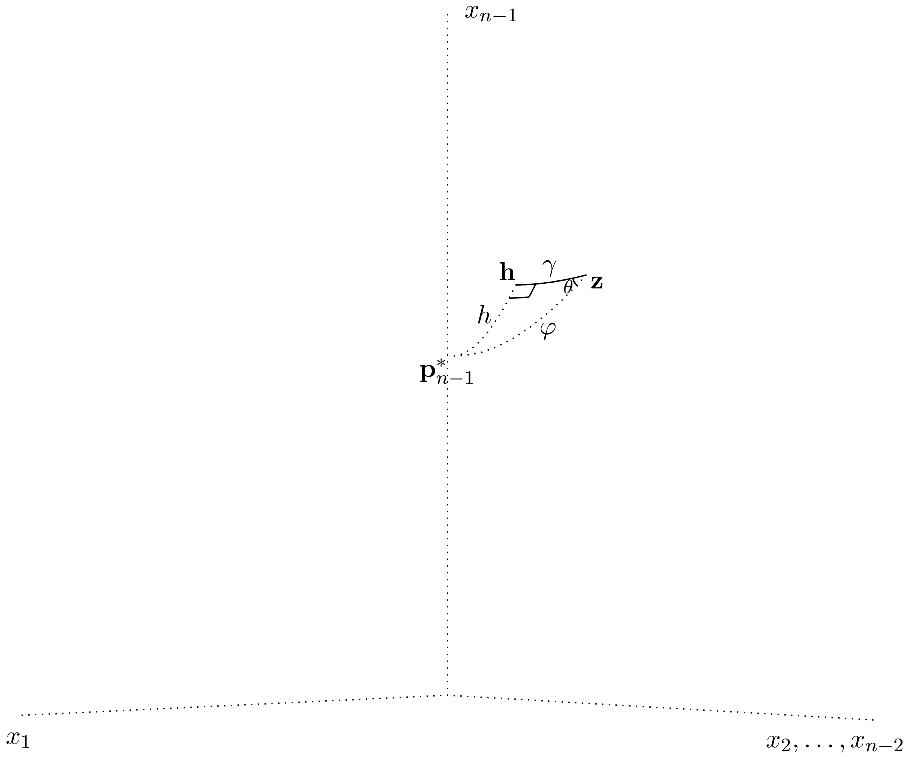

FIGURE 5. Illustrating the proof of Lemma 4.1

$$
\begin{aligned}
g_{\mathbf{y}} = &\begin{pmatrix} \langle \frac{\partial \mathbf{y}}{\partial \psi_1} | \frac{\partial \mathbf{y}}{\partial \psi_1} \rangle & \cdots & \langle \frac{\partial \mathbf{y}}{\partial \psi_1} | \frac{\partial \mathbf{y}}{\partial \psi_{n-2}} \rangle \\ \vdots & & \vdots \\ \langle \frac{\partial \mathbf{y}}{\partial \psi_{n-2}} | \frac{\partial \mathbf{y}}{\partial \psi_1} \rangle & \cdots & \langle \frac{\partial \mathbf{y}}{\partial \psi_{n-2}} | \frac{\partial \mathbf{y}}{\partial \psi_{n-2}} \rangle \end{pmatrix} = \\
&= \operatorname{diag}(\sin^2 \psi_2 \sin^2 \psi_3 \cdots \sin^2 \psi_{n-3} \sinh^2 \psi_{n-2}, \\
&\quad \sin^2 \psi_3 \cdots \sin^2 \psi_{n-3} \sinh^2 \psi_{n-2}, \ldots \\
&\quad \ldots, \sin^2 \psi_{n-3} \sinh^2 \psi_{n-2}, \sinh^2 \psi_{n-2}, 1),
\end{aligned}
$$

(14)
$$
\begin{aligned}
dS_0 &= \sqrt{\det g_{\mathbf{y}}} d\psi_1 d\psi_2 \cdots d\psi_{n-3} d\psi_{n-2} = \\
&= \sin \psi_2 \sin^2 \psi_3 \cdots \sin^{n-4} \psi_{n-3} \sinh^{n-3} \psi_{n-2} d\psi_1 d\psi_2 \cdots d\psi_{n-3} d\psi_{n-2},
\end{aligned}
$$

$$
\begin{aligned}
g_{\mathbf{w}} = &\begin{pmatrix} \langle \frac{\partial \mathbf{w}}{\partial \psi_1} | \frac{\partial \mathbf{w}}{\partial \psi_1} \rangle & \cdots & \langle \frac{\partial \mathbf{w}}{\partial \psi_1} | \frac{\partial \mathbf{w}}{\partial \psi_{n-3}} \rangle \\ \vdots & & \vdots \\ \langle \frac{\partial \mathbf{w}}{\partial \psi_{n-3}} | \frac{\partial \mathbf{w}}{\partial \psi_1} \rangle & \cdots & \langle \frac{\partial \mathbf{w}}{\partial \psi_{n-3}} | \frac{\partial \mathbf{w}}{\partial \psi_{n-3}} \rangle \end{pmatrix} = \\
&= \operatorname{diag}(\sin^2 \psi_2 \sin^2 \psi_3 \cdots \sin^2 \psi_{n-3}, \sin^2 \psi_3 \cdots \sin^2 \psi_{n-3}, \ldots, \sin^2 \psi_{n-3}, 1),
\end{aligned}
$$

(15)
$$dS_{0,\overline{n-1}} = \sqrt{\det g_{\mathbf{w}}}d\psi_1 d\psi_2 \cdots d\psi_{n-3} = \\ = \sin\psi_2 \sin^2\psi_3 \cdots \sin^{n-4}\psi_{n-3} d\psi_1 d\psi_2 \cdots d\psi_{n-3},$$

$$g_{\mathbf{z}} = \begin{pmatrix} \langle \frac{\partial \mathbf{z}}{\partial \psi_1} | \frac{\partial \mathbf{z}}{\partial \psi_1} \rangle & \cdots & \langle \frac{\partial \mathbf{z}}{\partial \psi_1} | \frac{\partial \mathbf{z}}{\partial \psi_{n-3}} \rangle \\ \vdots & & \vdots \\ \langle \frac{\partial \mathbf{z}}{\partial \psi_{n-3}} | \frac{\partial \mathbf{z}}{\partial \psi_1} \rangle & \cdots & \langle \frac{\partial \mathbf{z}}{\partial \psi_{n-3}} | \frac{\partial \mathbf{z}}{\partial \psi_{n-3}} \rangle \end{pmatrix} = \\ = \sinh^2\phi \,\mathrm{diag}(\sin^2\psi_2 \sin^2\psi_3 \cdots \sin^2\psi_{n-3}, \\ \sin^2\psi_3 \cdots \sin^2\psi_{n-3}, \ldots, \sin^2\psi_{n-3}, 1) + \\ + \sinh^4\phi \begin{pmatrix} \frac{\partial(\coth\phi)}{\partial\psi_1} \\ \vdots \\ \frac{\partial(\coth\phi)}{\partial\psi_{n-3}} \end{pmatrix} \begin{pmatrix} \frac{\partial(\coth\phi)}{\partial\psi_1} & \cdots & \frac{\partial(\coth\phi)}{\partial\psi_{n-3}} \end{pmatrix},$$

and
(16)
$$dS_{0,n-1} = \sqrt{\det g_{\mathbf{z}}} d\psi_1 d\psi_2 \cdots d\psi_{n-3} d\psi_{n-2} = \\ = \sin\psi_2 \sin^2\psi_3 \cdots \sin^{n-4}\psi_{n-3} \sinh^{n-2}\phi \cdot \\ \cdot \Big(\frac{1}{\sinh^2\phi} + \frac{(\frac{\partial(\coth\phi)}{\partial\psi_1})^2}{\sin^2\psi_2 \sin^2\psi_3 \cdots \sin^2\psi_{n-3}} + \frac{(\frac{\partial(\coth\phi)}{\partial\psi_2})^2}{\sin^2\psi_3 \cdots \sin^2\psi_{n-3}} + \cdots \\ \cdots + \frac{(\frac{\partial(\coth\phi)}{\partial\psi_{n-4}})^2}{\sin^2\psi_{n-3}} + \frac{(\frac{\partial(\coth\phi)}{\partial\psi_{n-3}})^2}{1} \Big)^{\frac{1}{2}} d\psi_1 d\psi_2 \cdots d\psi_{n-3} = \\ = \sin\psi_2 \sin^2\psi_3 \cdots \sin^{n-4}\psi_{n-3} \sinh^{n-2}\phi \cdot \\ \cdot \frac{1}{\langle \mathbf{p}_{\{0\},n-1} | \mathbf{p}^*_{n-1} \rangle} \cdot d\psi_1 d\psi_2 \cdots d\psi_{n-3}. \quad \square$$

Now we can conclude the last part of the proof by induction.

Proof of Theorem 1.2 for a simplex with $n \geq 4$. The goal is the equation

$$\int_{\mathbf{x}\in S} \mathbf{x} dS = -\frac{1}{n-1} \sum_{k=0}^{n-1} |S_k| \mathbf{p}_k.$$

If S is shrinked to one point, the both sides of the above obviously converge to the zero-vector. So it is enough to get the equality of the derivations of both sides:

$$\mathrm{d}(\int_{\mathbf{x}\in S} \mathbf{x} dS) = -\frac{1}{n-1} \mathrm{d}(\sum_{k=0}^{n-1} |S_k| \mathbf{p}_k).$$

We only have to consider the case of the perturbation (3), which is shown by using induction of dimension (see Remarks 3.1 and 4.2):

$$\mathrm{d}(\int_{\mathbf{x}\in S} \mathbf{x} dS) + \frac{1}{n-1} |S_0| \mathrm{d}\mathbf{p}_0 = \int_{\mathbf{y}\in S_0} \langle \mathbf{y} | \mathrm{d}\mathbf{p}_0 \rangle \mathbf{y} dS_0 - \frac{1}{n-1} |S_0| \varepsilon \mathbf{p}^*_{n-1} =$$

$$= \int_{S_0} \varepsilon(\cosh\psi_{n-2}\begin{pmatrix} 0 \\ \sin\psi_1\cdots\sin\psi_{n-3}\sinh\psi_{n-2} \\ \cos\psi_1\sin\psi_2\cdots\sin\psi_{n-3}\sinh\psi_{n-2} \\ \cos\psi_2\sin\psi_3\cdots\sin\psi_{n-3}\sinh\psi_{n-2} \\ \vdots \\ \cos\psi_{n-4}\sin\psi_{n-3}\sinh\psi_{n-2} \\ \cos\psi_{n-3}\sinh\psi_{n-2} \\ \cosh\psi_{n-2}\end{pmatrix} - \frac{1}{n-1}\begin{pmatrix}0\\0\\0\\0\\\vdots\\0\\0\\1\end{pmatrix})dS_0 =$$

$$= \varepsilon\int_{S_{0,\overline{n-1}}}\int_0^{\phi}\begin{pmatrix} 0 \\ \sin\psi_1\cdots\sin\psi_{n-3}\sinh^{n-2}\psi_{n-2}\cosh\psi_{n-2} \\ \cos\psi_1\sin\psi_2\cdots\sin\psi_{n-3}\sinh^{n-2}\psi_{n-2}\cosh\psi_{n-2} \\ \cos\psi_2\sin\psi_3\cdots\sin\psi_{n-3}\sinh^{n-2}\psi_{n-2}\cosh\psi_{n-2} \\ \vdots \\ \cos\psi_{n-4}\sin\psi_{n-3}\sinh^{n-2}\psi_{n-2}\cosh\psi_{n-2} \\ \cos\psi_{n-3}\sinh^{n-2}\psi_{n-2}\cosh\psi_{n-2} \\ \sinh^{n-3}\psi_{n-2}\cosh^2\psi_{n-2} - \frac{1}{n-1}\sinh^{n-3}\psi_{n-2}\end{pmatrix} d\psi_{n-2}dS_{0,\overline{n-1}} =$$

$$= -\frac{\mathrm{d}\langle 0,n-1\rangle}{\langle \mathbf{p}_{\{0\},n-1}|\mathbf{p}^*_{n-1}\rangle}\int_{S_{0,\overline{n-1}}}\begin{pmatrix} 0 \\ \sin\psi_1\cdots\sin\psi_{n-3}\sinh^{n-1}\phi/(n-1) \\ \cos\psi_1\sin\psi_2\cdots\sin\psi_{n-3}\sinh^{n-1}\phi/(n-1) \\ \cos\psi_2\sin\psi_3\cdots\sin\psi_{n-3}\sinh^{n-1}\phi/(n-1) \\ \vdots \\ \cos\psi_{n-4}\sin\psi_{n-3}\sinh^{n-1}\phi/(n-1) \\ \cos\psi_{n-3}\sinh^{n-1}\phi/(n-1) \\ \sinh^{n-2}\phi\cosh\phi/(n-1)\end{pmatrix} dS_{0,\overline{n-1}} =$$

$$= -\frac{\mathrm{d}\langle 0,n-1\rangle}{n-1}\int_{S_{0,n-1}}\begin{pmatrix} 0 \\ \sin\psi_1\cdots\sin\psi_{n-3}\sinh\phi \\ \cos\psi_1\sin\psi_2\cdots\sin\psi_{n-3}\sinh\phi \\ \cos\psi_2\sin\psi_3\cdots\sin\psi_{n-3}\sinh\phi \\ \vdots \\ \cos\psi_{n-4}\sin\psi_{n-3}\sinh\phi \\ \cos\psi_{n-3}\sinh\phi \\ \cosh\phi\end{pmatrix} dS_{0,n-1} =$$

$$= -\frac{\mathrm{d}\langle 0,n-1\rangle}{n-1}\int_{\mathbf{z}\in S_{0,n-1}}\mathbf{z}dS_{0,n-1} =$$

$$= \frac{\mathrm{d}\langle 0,n-1\rangle}{n-1}\cdot\frac{1}{n-3}\sum_{k=1}^{n-2}|S_{0,n-1,k}|\cdot\mathbf{p}_{\{0,n-1\},k} =$$

$$= \frac{\mathrm{d}\langle 0,n-1\rangle}{(n-1)(n-3)}\sum_{k=1}^{n-2}|S_{0,n-1,k}|\cdot$$

$$\cdot\frac{\sqrt{1-\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle^2}\mathbf{p}_k - \langle\mathbf{p}_k|\mathbf{p}_{\{n-1\},0}\rangle\mathbf{p}_0 - \langle\mathbf{p}_k|\mathbf{p}_{\{0\},n-1}\rangle\mathbf{p}_{n-1}}{\sqrt{1-\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle^2-\langle\mathbf{p}_0|\mathbf{p}_k\rangle^2-\langle\mathbf{p}_{n-1}|\mathbf{p}_k\rangle^2+2\langle\mathbf{p}_0|\mathbf{p}_{n-1}\rangle\langle\mathbf{p}_0|\mathbf{p}_k\rangle\langle\mathbf{p}_{n-1}|\mathbf{p}_k\rangle}} =$$

$$= \frac{\sum\limits_{k=1}^{n-2}|S_{k,0,n-1}|\mathrm{d}\langle 0,n-1\rangle_k\cdot\mathbf{p}_k + \sum\limits_{i=1}^{n-2}|S_{0,i,n-1}|\mathrm{d}\langle i,n-1\rangle_0\cdot\mathbf{p}_0 + \sum\limits_{j=1}^{n-2}|S_{n-1,0,j}|\mathrm{d}\langle 0,j\rangle_{n-1}\cdot\mathbf{p}_{n-1}}{(n-1)(n-3)} =$$

$$= \frac{1}{n-1}\sum_{k=0}^{n-1}\frac{1}{n-3}\sum_{\substack{i=0\\i\neq k}}^{n-1}\sum_{\substack{j=i+1\\j\neq k}}^{n-1}|S_{k,i,j}|\mathrm{d}\langle i,j\rangle_k\cdot\mathbf{p}_k = -\frac{1}{n-1}\sum_{k=0}^{n-1}\mathrm{d}|S_k|\cdot\mathbf{p}_k =$$
$$= -\frac{1}{n-1}\mathrm{d}(\sum_{k=0}^{n-1}|S_k|\mathbf{p}_k) + \frac{1}{n-1}\sum_{k=0}^{n-1}|S_k|\mathrm{d}\mathbf{p}_k =$$
$$= -\frac{1}{n-1}\mathrm{d}(\sum_{k=0}^{n-1}|S_k|\mathbf{p}_k) + \frac{1}{n-1}|S_0|\mathrm{d}\mathbf{p}_0,$$

where $S_{k,i,j} = S_k \cap S_i \cap S_j$, the first equality comes from

$$\begin{aligned}(17)\qquad S'\setminus S &= \{\mathbf{x}=(x_0,0,\ldots,0)^T+\mathbf{y} : \langle\mathbf{x}|\mathbf{p}_0\rangle < 0 \le \langle\mathbf{x}|\mathbf{p}_0+\mathrm{d}\mathbf{p}_0\rangle\} =\\ &= \{\mathbf{x}=(x_0,0,\ldots,0)^T+\mathbf{y} : x_0 < 0 \le x_0 + \langle\mathbf{y}|\mathrm{d}\mathbf{p}_0\rangle\} =\\ &= \{\mathbf{x}=(x_0,0,\ldots,0)^T+\mathbf{y} : 0 < -x_0 \le \langle\mathbf{y}|\mathrm{d}\mathbf{p}_0\rangle\}\end{aligned}$$

(see Figure 6 and notice that S'_i overlaps with S_i for $i \neq 0$ from $\mathbf{p}'_i = \mathbf{p}_i$), the third equality comes from (11), the fourth equality comes from (4), the fifth equality comes from (12), the seventh equality comes from the hypothesis of induction, the eighth equality comes from (1), the nineth equality comes from (6), (7), and (8), the tenth equality comes from (9), and the eleventh equality comes from Schläfli's differential formula. □

5. Appendix.

We have another short proof of the hyperbolic version of Schläfli's differential formula.

Proof of Theorem 1.1. The following calculation is enough (see Remarks 3.1 and 4.2 again):

$$\begin{aligned}\mathrm{d}|S| &= \int_{\mathbf{y}\in S_0}\langle\mathbf{y}|\mathrm{d}\mathbf{p}_0\rangle dS_0 = \int_{S_0}\varepsilon\cosh\psi_{n-2}dS_0 =\\ &= \varepsilon\int_{S_{0,\overline{n-1}}}\int_0^{\phi}\cosh\psi_{n-2}\sinh^{n-3}\psi_{n-2}d\psi_{n-2}dS_{0,\overline{n-1}} =\\ &= -\frac{\mathrm{d}\langle 0,n-1\rangle}{\langle\mathbf{p}_{\{0\},n-1}|\mathbf{p}^*_{n-1}\rangle}\int_{S_{0,\overline{n-1}}}\frac{\sinh^{n-2}\phi}{n-2}dS_{0,\overline{n-1}} =\\ &= -\frac{\mathrm{d}\langle 0,n-1\rangle}{n-2}\int_{S_{0,n-1}}dS_{0,n-1} = -\frac{\mathrm{d}\langle 0,n-1\rangle}{n-2}|S_{0,n-1}|,\end{aligned}$$

where the first, third, fourth, and fifth equalities come from (17), (11), (4), and (12), respectively. □

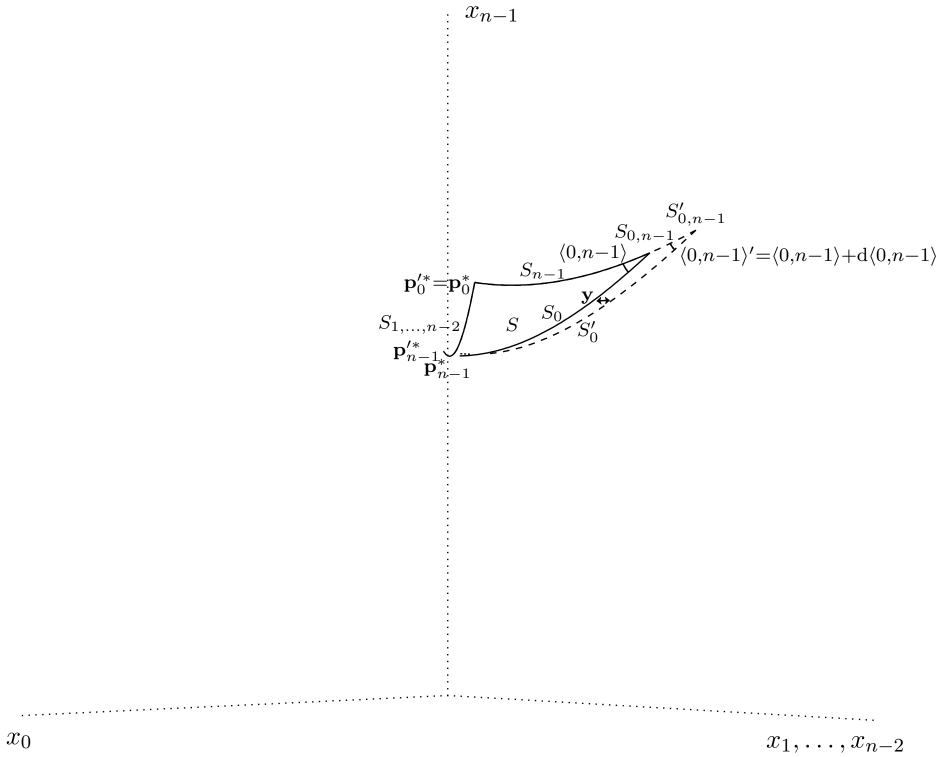

FIGURE 6. Illustrating the proof of Theorem 1.2 for a simplex with $n \geq 4$

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