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GREAT COMPILATION OF CHARACTERIZATIONS OF SQUARES

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Abstract. We study what additional conditions ten different classes of convex quadrilaterals must satisfy in order for them to be a square.

1. Introduction

Let us start this paper by discussing the difference between a property and a characterization. All mathematical objects have certain properties, also called necessary conditions, but in order for a property to be a characterization, it must also be a sufficient condition. Take for example a rhombus. A necessary condition is perpendicular diagonals, but that is not a sufficient condition since there are quadrilaterals with perpendicular diagonals that are not rhombi (called orthodiagonal quadrilaterals, see [16]). A parallelogram has bisecting diagonals, and any quadrilateral with bisecting diagonals is a parallelogram. Hence bisecting diagonals is both a necessary and sufficient condition, that is, it is a characterization of parallelograms, whereas perpendicular diagonals is not a characterization of rhombi. We can conclude that characterizations are the subset of properties that are unique to a certain object, that is, those properties that distinguishes an object from other objects. With objects we mean anything we can study in mathematics, such as numbers, polynomials, circles, polygons, and so on.

We have written many papers on characterizations of different classes of quadrilaterals (see [15, 16, 18, 21, 22, 23, 24, 25]), but never before considered to write about squares. In fact, just a couple of months before starting this paper, we knew of only about a dozen characterizations of squares, and at the English Wikipedia page there are just seven listed at the time of writing this paper. But when we started looking closer into squares we found out that it is actually the second most distinguished quadrilateral when assessing the number of published characterizations (taking this paper into account). For number one, see [24].

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In this paper we shall study all 78 characterizations of squares that we know of. A majority of these were found in various books and papers, but a few are original as far as we know since we have been unable to track any references for them. Many of the proofs are very elementary and require little more than congruence, but about a third use heavier machinery. We will study when rectangles, rhombi, parallelograms, isosceles trapezoids, kites, cyclic quadrilaterals, tangential quadrilaterals, bicentric quadrilaterals, orthodiagonal quadrilaterals, and convex quadrilaterals are squares.

This collection can by no means be considered complete, since in our experience, it has been quite easy to find new characterizations of squares just by doing a web search or picking a class of quadrilaterals and start thinking about what restrictions it shall have in order to be a square. The main merit of this paper is to collect a very large number of characterizations of squares in one place, and perhaps inspire further research on this topic.

In order to prove a characterization, we normally need to prove that it is both a necessary and sufficient condition.[1] But since a square is a very well-known type of quadrilateral with well-known properties, *we will exclude proving most of the necessary parts* and assume the reader is familiar with the following basic properties of a square $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, diagonal intersection P, area K, circumradius R, inradius r, circumcenter O and incenter I:

- $\angle A = \angle B = \angle C = \angle D = 90°$
- $a = b = c = d$
- $AC \perp BD$
- $AC = BD$
- $AP = BP = CP = DP$ (equal *diagonal parts*)
- $\angle CAB = \angle DBA = \angle DBC = \angle ACB = \angle DCA = \angle CDB = \angle BDA = \angle DAC$
- it is cyclic
- it is tangential
- $K = 2R^2 = 4r^2$
- $R = \sqrt{2}r$
- $O = I$

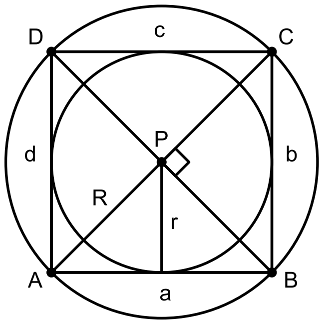

FIGURE 1. General notations

[1]However when using inequalities, those conditions can often be proved at the same time, as in the last section of this paper.

These notations are illustrated in Figure 1 and used throughout this paper. The seventh and eighth properties mean that a square has the capacity of having a circumcircle and an incircle, that is, one circle passing through all vertices and one that is internally tangent to all sides respectively. The centers of these circles are the circumcenter and incenter, and their radii are called circumradius and inradius respectively.

Another way to look at these properties is that squares are always located at the bottom of all classifications of quadrilaterals, which means that they inherit all properties (many of which are true by definition) of all other quadrilaterals, and very few of the necessary conditions are therefore required to be proved if the higher classes are studied first. Anyway, if this is unsatisfactory to the reader, we invite them to prove the necessary conditions themselves. We claim it's the sufficient conditions that are the most interesting, since they are the ones to use when tasked to prove a certain quadrilateral is a square.

Before we continue, it's about time to clarify what we mean by a square, in other words, how it is defined. There are several different ways to define a square, and in fact, any one of the 78 characterizations would be a possible definition, although some of them might seem less suitable. We shall use the following definition: *A square is a quadrilateral with 4 equal sides and 4 right angles.* This is actually the definition given by Euclid, although he put it slightly differently: a square is a quadrilateral that is both equilateral and right-angled. We will also use the definitions that a *rhombus* is a quadrilateral with 4 equal sides and a *rectangle* is a quadrilateral with 4 right angles.

2. Rectangles

In this section we study what additional properties a rectangle must have in order to be a square. There are the following ten criteria. Many high school text books use a variant of the first: they state that a square is a rectangle with four equal sides [31, p. 59], but since that contains redundant information, it is not included here. The ninth is a well-known maximization property of rectangles that is seldom considered to be a characterization of squares. We found the tenth condition in [2, p. 8].

The characterizations in this and the next four sections are so basic that the majority of them have probably been known for a long time. Thus trying to trace their original publication is an almost impossible task. It is however difficult to locate any one source that have collected more than say ten of these 40 characterizations.

A *bimedian* is a line segment connecting the midpoints of two opposite sides in a quadrilateral.

Theorem 2.1. *A rectangle is a square if and only if it satisfies any one of:*

- *(a) it has two adjacent equal sides*
- *(b) it has perpendicular diagonals*
- *(c) it has diagonals that bisect the vertex angles*
- *(d) it is tangential*
- *(e) its angle bisectors are concurrent*

(f) it has equal bimedians
(g) the diagonals divide it into four congruent triangles
(h) the diagonal intersection is equidistant to the sides
(i) it has the largest area for a given perimeter
(j) it has the largest area of all rectangles inscribed in a given circle

Proof. *(a)* True by definition.

(b) Since the diagonals in a rectangle $ABCD$ are equal and bisecting, triangles ABP and BCP are congruent (SAS), where P is the diagonal intersection (see Figure 2), so $AB = BC$. Thus it's a square according *(a)*.

(c) We have $\angle PAB = \angle PBA = \angle PBC = \angle PCB$, so $\angle APB = \angle BPC$, and $AP = BP = CP$ (see Figure 2). Then triangles ABP and BCP are congruent (SAS). Hence $AB = BC$, so $ABCD$ is a square by to *(a)*.

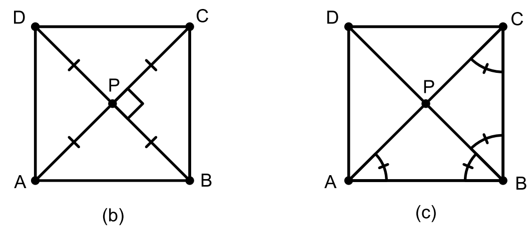

FIGURE 2. Given assumptions in Theorem 2.1 (b) and (c)

(d) Since $AB = CD$ and $BC = DA$ in a rectangle and $AB + CD = BC + DA$ when it is tangential, we get $AB = BC$, so the rectangle is a square according *(a)*.

(e) Suppose the angle bisectors intersect at a point I and let E, F, G, H be the projections of I on the sides AB, BC, CD, DA (see Figure 3). Then the eight triangles AEI, BEI, BFI, CFI, CGI, DGI, DHI, AHI are congruent (AAS), so any two adjacent sides in the rectangle are equal, making it a square according to *(a)*.

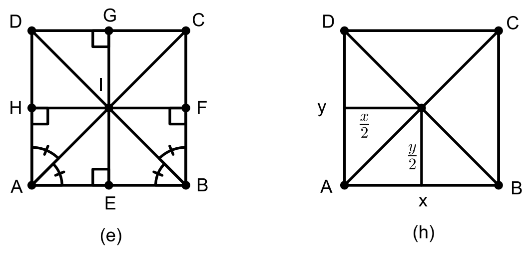

FIGURE 3. Given assumptions in Theorem 2.1 (e) and (h)

(f) The bimedians in a rectangle have the same lengths as two adjacent sides. Hence this condition is equivalent to *(a)*.

(g) This congruency directly yields that two adjacent sides are equal, so the rectangle is a square by *(a)*.

(h) If two adjacent sides in the rectangle have lengths x and y, then the diagonal intersection is at a distance $\frac{y}{2}$ from side x and $\frac{x}{2}$ from side y (see

Figure 3), so these distances are equal if and only if $x = y$. This makes it a square according to *(a)*.

(i) In a rectangle with given perimeter L, let two adjacent sides have length x and $\frac{L}{2} - x$. Then the area is

$$K = x\left(\frac{L}{2} - x\right) = -\left(\frac{L}{4} - x\right)^2 + \left(\frac{L}{4}\right)^2 \leq \left(\frac{L}{4}\right)^2$$

with equality if and only if $x = \frac{L}{4}$. Then two adjacent sides in the rectangle are both equal to $\frac{L}{4}$, making it a square.

(j) For a rectangle with sides x and y inscribed in a circle of radius R, we get

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = R^2$$

by the Pythagorean theorem (see Figure 4). Then the area satisfies

$$K = xy = \sqrt{x^2y^2} \leq \frac{x^2 + y^2}{2} = 2R^2$$

according to the AM-GM inequality, where equality holds if and only if $x = y$. In that case the rectangle is a square. □

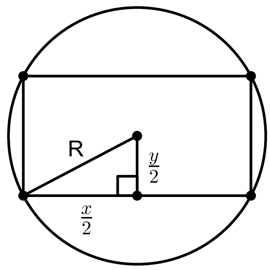

FIGURE 4. A rectangle inscribed in a circle

3. Rhombi

Here we study what additional properties a rhombus must have in order to be a square. We have seven different criteria. There is sometimes one in use that has redundant information, stating that a square is a rhombus with four equal vertex angles (see [31, p. 59]) when two is enough, so that one is excluded. A rhombus has an incircle, and the distance from a vertex to a point where the incircle is tangent to a side is called a *tangent length.*

Theorem 3.1. *A rhombus is a square if and only if it satisfies any one of:*

- *(a) it has a right vertex angle*
- *(b) it has two adjacent equal vertex angles*
- *(c) it has two adjacent equal diagonal parts*
- *(d) it has equal diagonals*
- *(e) it is cyclic*
- *(f) two adjacent tangent lengths at different vertices are equal*
- *(g) the diagonal intersection is equidistant to the vertices*

Proof. *(a), (b)* Since a rhombus is a special case of a parallelogram, any one of these two assumptions directly yield that $\angle A = \angle B = \angle C = \angle D = 90°$, so the rhombus is a square.

(c), (d) With these assumptions, the rhombus is partitioned into four congruent isosceles triangles by the diagonals, making it a square.

(e) Opposite vertex angles are equal in a rhombus and supplementary in a cyclic quadrilateral. Together this make all four vertex angles right angles. Then the rhombus is a square according to *(a)* or *(b)*.

(f) If the incircle in a rhombus $ABCD$ has center I and is tangent to side AB at E, then the condition $AE = BE$ implies that triangles AIE and BIE are congruent (SAS), see Figure 5, so half the vertex angles at A and B are equal, and since these vertex angles are supplementary, they are right angles. Then the rhombus is a square by to *(a)*.

(g) The diagonal intersection P coincide with the incenter I in a rhombus (see Figure 5), so when $AP = BP$, triangles APE and BPE are congruent (RHS). By the same argument as in *(f)*, the rhombus is a square. □

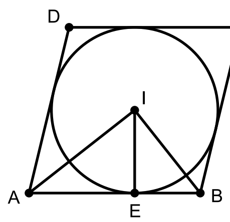

FIGURE 5. A rhombus with its incircle

4. Parallelograms

In this section we study what additional properties a parallelogram must have for it to be a square. We know of the following eight criteria. A variant of the first characterization is to state that a square is a parallelogram with four equal sides and four right angles [31, p. 59], but since that is just a reformulation, it is excluded. To prove the last of these eight was given as Problem 2.1.7 in [4, p. 70].

Theorem 4.1. *A parallelogram is a square if and only if it satisfies any one of:*

(a) it is both a rhombus and a rectangle
(b) it has one right angle and two adjacent equal sides
(c) it has equal diagonals that bisect the vertex angles
(d) it has equal and perpendicular diagonals
(e) it has two adjacent equal diagonal parts and perpendicular diagonals
(f) it is cyclic and has perpendicular diagonals
(g) it is cyclic and tangential
(h) it has the largest area for a given perimeter

Proof. *(a)* True by definition.

(b) These assumptions force the parallelogram into being both a rectangle and a rhombus, that is, a square.

(c) Equal diagonals means it has right vertex angles (due to SSS congruence), so it is a rectangle. Bisected vertex angles yield that a diagonal divide it into two isosceles triangles, so adjacent sides are equal, and it must also be a rhombus. Thus a square according to *(a)*.

(d) Equal diagonals imply it's a rectangle, and a rectangle with perpendicular diagonals is a square according to Theorem 2.1 *(b)*.

(e) A parallelogram with perpendicular diagonals is a rhombus, and with two adjacent equal diagonal parts it's a square by Theorem 3.1 *(c)*.

(f) A cyclic parallelogram is a rectangle, and with perpendicular diagonals it is a square.

(g) A parallelogram that is both cyclic and tangential is both a rectangle and a rhombus, that is, a square according to *(a)*.

(h) A parallelogram with sides a and b and angle B between them has area

$$K = ab \sin B \leq ab \leq \left(\frac{a+b}{2}\right)^2 = \left(\frac{L}{4}\right)^2$$

where L is the given perimeter and we applied the AM-GM inequality. Equality holds if and only if $\angle B = 90°$ and $a = b$, that is, only when the parallelogram is both a rectangle and a rhombus. Thus a square. □

5. Isosceles trapezoids

Now we study what additional properties an isosceles trapezoid must have in order to be a square. An isosceles trapezoid can be defined in several different ways, for instance as a convex quadrilateral with two pairs of adjacent equal angles [7, p. 30], or as a convex quadrilateral with two consecutive pairs of equal diagonal segments [8, p. 73]. Two well-known properties are that an isosceles trapezoid has a pair of opposite equal sides and equal diagonals. We have the following eight characterizations. The first four are taken from [9] and the last from [8].

Theorem 5.1. *An isosceles trapezoid $ABCD$ with diagonal intersection P is a square if and only if it satisfies any one of:*

(a) $\angle A = \angle B$, $\angle C = \angle D$, $AB = CD$ *and* $AC \perp BD$
(b) $\angle A = \angle B$, $\angle C = \angle D$, $AP = CP$ *and* $AC \perp BD$
(c) $AP = BP$, $CP = DP$, $AB = CD$ *and* $AC \perp BD$
(d) $AP = BP$, $CP = DP$, $\angle A = \angle C$ *and* $AC \perp BD$
(e) $AP = BP$, $CP = DP$, $BC = CD$ *and* $\angle A = \angle D$
(f) $\angle C = \angle D$, $BC = CD = DA$ *and* $AC \perp BD$
(g) $\angle A = \angle B$, $\angle C = \angle D$, $BC = CD$ *and* $AP = DP$
(h) $\angle A = \angle B$, $\angle C = \angle D$ *and* $AB = BC = CD$

Proof. *(a)* These conditions directly imply that we have a rectangle with perpendicular diagonals, so a square (see Figure 6).

(b) In this isosceles trapezoid, $AD = BC$ and $DC \parallel AB$ (see Figure 6). Now triangles APD and CPB are congruent (RHS), so $\angle ADB = \angle CBD$ and thus $AD \parallel BC$. Also, triangles DAB and CBA are congruent (SAS), so $BD = AC$. Hence we have a parallelogram with equal and perpendicular diagonals, that is, a square according to Theorem 4.1 *(d)*.

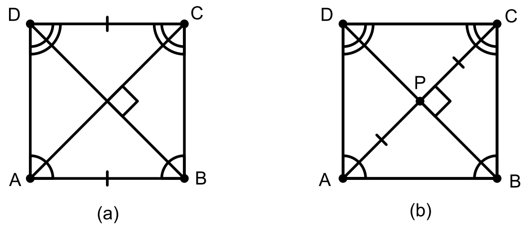

FIGURE 6. Given assumptions in Theorem 5.1 (a) and (b)

(c) Since it's an isosceles trapezoid, $AD = BC$, and with $AB = CD$ given, we have a parallelogram (see Figure 7). The diagonals in an isosceles trapezoid are equal (implying a rectangle here) and since they are also perpendicular, $ABCD$ is a square.

(d) Triangles APD and BPC are congruent (SAS), so $AD = BC$, and $\angle DAC = \angle DBC$ implies that $ABCD$ is cyclic (see Figure 7). Then $\angle A = \angle C = 90°$, so $AB = CD$ since triangles ABD and CDB are congruent (RHS). A cyclic parallelogram with perpendicular diagonals is a square according to Theorem 4.1 *(f)*.

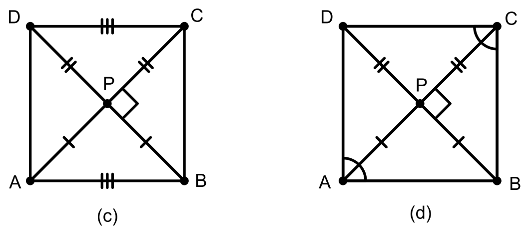

FIGURE 7. Given assumptions in Theorem 5.1 (c) and (d)

(e) In this isosceles trapezoid, $\angle A = \angle D$ implies that all four vertex angles are right angles, so it is a rectangle (see Figure 8). Together with two adjacent equal sides, we have a square.

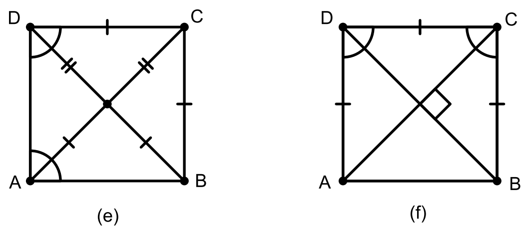

FIGURE 8. Given assumptions in Theorem 5.1 (e) and (f)

(f) According to the isosceles triangle theorem, $\angle CAD = \angle ACD = \angle BDC = \angle DBC$, and since $AC \perp BD$, it holds that $\angle ACD = \angle BDC = 45°$ (see Figure 8). Then $\angle DBC = 45°$, so $\angle BCD = 90°$ and it follows that

all four vertex angles are right angles. Hence we have a rectangle with two adjacent equal sides, that is, a square according to Theorem 2.1 *(a)*.

(g) The condition $AP = DP$ together with the symmetry of an isosceles trapezoid imply $AP = DP = CP = BP$ so triangles ADP and CDP are congruent (SSS), see Figure 9. Then $AC \perp BD$ and according to *(f)*, the trapezoid is a square.

(h) The pairs of equal angles imply that AB and DC are parallel, and since $AB = CD$, $ABCD$ is a parallelogram (see Figure 9). But $AB = BC$ then means that the quadrilateral must be a rhombus, and since $\angle A = \angle B$, it's in fact a square. □

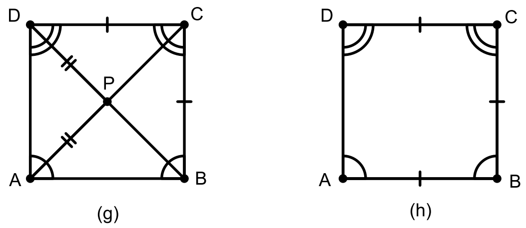

FIGURE 9. Given assumptions in Theorem 5.1 (g) and (h)

6. Kites

Here we study what additional properties a kite must have in order to be a square. There are the following seven criteria. The last three are taken from [8] where no proofs were given.

Theorem 6.1. *A kite $ABCD$ with $AB = BC$ and $CD = DA$ is a square if and only if it satisfies any one of:*

(a) it has equal and bisecting diagonals
(b) it has two opposite equal sides and a right vertex angle
(c) it has two opposite parallel sides and a right vertex angle
(d) $\angle B = \angle D = 90°$
(e) $\angle A = \angle B = \angle D$
(f) $\angle A = \angle B$ and $\angle C = \angle D$
(g) $AP = DP$ and $BP = CP$

Proof. *(a)* With $AC = BD$ and $AP = BP = CP = DP$, where P is the diagonal intersection, triangles ABP, BCP, CDP, DAP are congruent (SAS) since $AC \perp BD$ in a kite (see Figure 10). Then the kite is a rhombus, and since $AC = BD$, it is a square according to Theorem 3.1 *(d)*.

(b) We immediately get that the kite is a rhombus with a right vertex angle, that is, a square by Theorem 3.1 *(a)*, see Figure 10.

(c) Diagonal BD divide the kite into two congruent isosceles triangles, so it is a rhombus, and since one vertex angle is a right angle, it is a square (see Figure 11).

(d) Since right triangles ABC and CDA have hypotenuse AC, from the Pythagorean theorem we get

$$\sqrt{2}AB = \sqrt{2}BC = AC = \sqrt{2}CD = \sqrt{2}DA$$

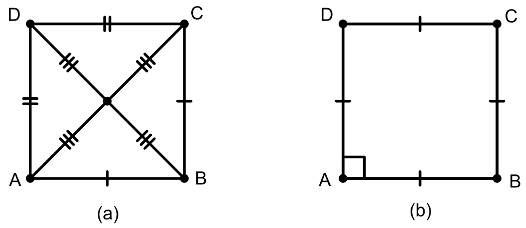

FIGURE 10. Given assumptions in Theorem 6.1 (a) and (b)

so $AB = BC = CD = DA$ (see Figure 11). Hence the kite is a rhombus, and since it has a right vertex angle (in fact two), it's a square.

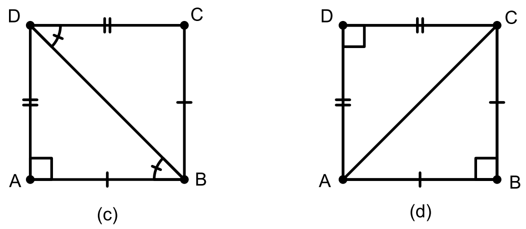

FIGURE 11. Given assumptions in Theorem 6.1 (c) and (d)

(e) We have that triangles ABD and CBD are congruent (SSS), so $\angle A = \angle C$. Thus $\angle A = \angle B = \angle C = \angle D$ (see Figure 12). Hence $ABCD$ is a rectangle, and since two adjacent sides are equal, it's a square.

(f) These conditions mean that *the kite is also an isosceles trapezoid* (see Figure 12). Since triangles ABD and CBD are congruent (SSS), then $\angle A = \angle C$, and in the same way as in *(e)* we have a rectangle ($\angle A = \angle B = \angle C = \angle D$) that is a square (due to $AB = BC$).

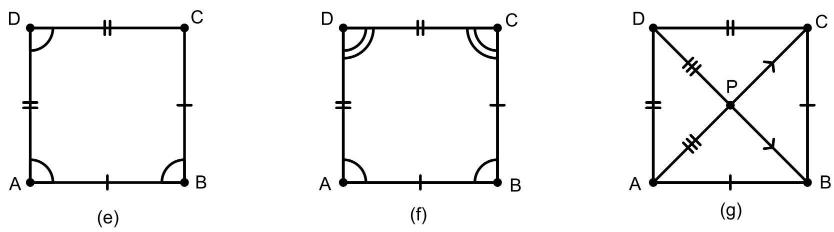

FIGURE 12. Given assumptions in Theorem 6.1 (e), (f) and (g)

(g) In a kite, $AC \perp BD$, so

$$\angle PAD = \angle PDA = \angle PCB = \angle PBC = 45^\circ$$

(see Figure 12). Since triangles BAD and BCD are congruent (SSS), we have $\angle CBP = \angle ABP = 45^\circ$ and $\angle CDP = \angle ADP = 45^\circ$. Then $\angle B = \angle D = 90^\circ$, so the kite is a square according to *(d)*. □

7. Cyclic quadrilaterals

This far we have mostly studied really basic characterizations of squares. Now we will have some more challenging results when we in the next three sections turn the attention to quadrilaterals that can have a circumcircle, an incircle, or both. To prove the first formula in the following theorem was Problem 2 on the second paper in the year 2000 of the Irish Mathematical Olympiad [10].

Theorem 7.1. *A cyclic quadrilateral with sides a, b, c, d, area K and circumradius R is a square if and only if it satisfies any one of:*

(a) $K = \frac{(abcd)^{3/4}}{R\sqrt{2}}$
(b) $K = 2R^2$
(c) it has the largest area of all quadrilaterals inscribed in a given circle
(d) it has the largest area for a given perimeter

Proof. *(a)* Drawing diagonal $p = AC$ and using a well-known triangle formula (see Figure 13), we see that the area of a cyclic quadrilateral is

$$K = \frac{pab}{4R} + \frac{pcd}{4R} = \frac{p(ab+cd)}{4R}$$

and similarly using diagonal $q = BD$, we get

$$K = \frac{q(ad+bc)}{4R}.$$

Multiplying these formulas and applying Ptolemy's theorem $pq = ac + bd$ yields

$$K^2 = \frac{pq(ab+cd)(ad+bc)}{16R^2} = \frac{(ab+cd)(ac+bd)(ad+bc)}{16R^2}. \tag{1}$$

This formula, when solving for R, is called Parameshava's formula. The derivation we used is cited from [3, p. 40].

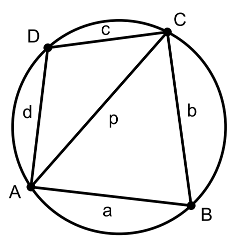

Figure 13. Diagonal in cyclic quadrilateral

Next we apply the AM-GM inequaliy to get

$$K^2 \geq \frac{(2\sqrt{abcd})^3}{16R^2}$$

which upon taking the square root simplifies to

$$K \geq \frac{(abcd)^{3/4}}{R\sqrt{2}}.$$

Equality holds if and only if $ab = cd$, $ac = bd$, and $ad = bd$, which yields $a = b = c = d$ by multiplying the equalities two by two and simplifying. The only cyclic rhombus is the square according to Theorem 3.1 *(e)*.

(b) The area of a convex quadrilateral is given by

$$K = \tfrac{1}{2}pq\sin\theta \tag{2}$$

where θ is the angle between the diagonals p and q (for a proof, see [13]). Applying that in a cyclic quadrilateral $p = 2R\sin B$ and $q = 2R\sin A$ (see Figure 14), we get

$$K = 2R^2\sin A\sin B\sin\theta \leq 2R^2 \tag{3}$$

with equality if and only if $\angle A = \angle B = \angle\theta = 90°$. This implies the cyclic quadrilateral is a rectangle with perpendicular diagonals, that is, a square by Theorem 2.1 *(b)*.

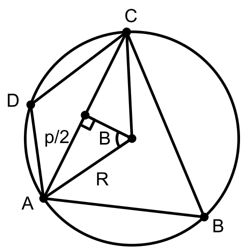

FIGURE 14. $p = 2R\sin B$

(c) When we have a quadrilateral inscribed in a given circle, it means the radius R is constant. This condition is a direct consequence of the inequality $K \leq 2R^2$, where there is equality if and only if the quadrilateral is a square.

(d) If we apply the AM-GM inequality to Brahmagupta's formula, we get

$$\begin{aligned} K &= \sqrt{(s-a)(s-b)(s-c)(s-d)} \\ &\leq \left(\frac{s-a+s-b+s-c+s-d}{4}\right)^2 = \left(\frac{4s-2s}{4}\right)^2 = \left(\frac{L}{4}\right)^2 \end{aligned}$$

where s is the semiperimeter and $L = 2s$ is the given perimeter. Equality holds if and only if

$$s-a = s-b = s-c = s-d$$

which is equivalent to $a = b = c = d$, so we have a cyclic rhombus, that is, a square according to Theorem 3.1 *(e)*. □

8. Tangential quadrilaterals

For a tangential quadrilateral, there is a similar formula as the one in Theorem 7.1 *(b)*. The first two characterizations in the following theorem are closely related via the simple area formula $K = rs$, but instead of just connecting them this way we will present different proofs for the readers benefit. The inequality case of the second formula in this theorem is attributed to T. A. Ivanova in [28, p. 404] (given as a problem in 1975). The proof we give of it is taken from [5].

Theorem 8.1. *A tangential quadrilateral with area K, inradius r and semiperimeter s is a square if and only if it satisfies any one of:*

(a) $K = 4r^2$
(b) $s = 4r$
(c) it has the smallest area of all quadrilaterals circumscribing a given circle
(d) it has the largest area for a given perimeter

Proof. *(a)* If we connect the incenter to the vertices (see Figure 15), it is easy to see that the area of a tangential quadrilateral is given by

$$K = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} + \frac{dr}{2} = \frac{a+b+c+d}{2}\, r \ (= sr).$$

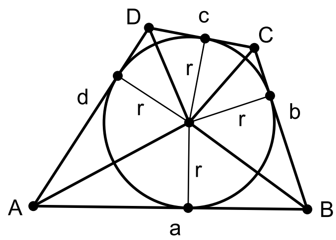

FIGURE 15. The inradius

A double application of the AM-GM inequality yields

$$a + b + c + d \geq 2\sqrt{ab} + 2\sqrt{cd} \geq 4\sqrt[4]{abcd} \tag{4}$$

where equality holds if and only if $a = b = c = d$. Putting these results together, we get

$$K \geq 2r\sqrt[4]{abcd}. \tag{5}$$

Next we use the following formula for the area of a tangential quadrilateral $ABCD$ (a derivation can be found for instance in [3, pp. 56, 18–19])

$$K = \sqrt{abcd}\sin\frac{A+C}{2} \leq \sqrt{abcd} \tag{6}$$

were we have equality if and only if $\angle A + \angle C = 180°$, that is, only when the quadrilateral is also cyclic. Now combining a squared version of (5) with (6), we have

$$\frac{K^2}{4r^2} \geq \sqrt{abcd} \geq K$$

which yields $K \geq 4r^2$, where equality holds if and only if the tangential quadrilateral is a cyclic rhombus, that is, a square by Theorem 3.1 *(e)*.

(b) We connect the incenter to the vertices and to the points where the incircle of a tangential quadrilateral is tangent to the sides and label the eight angles thus created at the incenter in pairs by α, β, γ, δ (see Figure 16). Then $\alpha + \beta + \gamma + \delta = \pi$. We directly get

$$s = r(\tan\alpha + \tan\beta + \tan\gamma + \tan\delta).$$

The tangent function is convex on the interval $\left(0, \frac{\pi}{2}\right)$. Applying Jensen's inequality yields

$$s \geq 4r\tan\frac{\alpha+\beta+\gamma+\delta}{4} = 4r\tan\frac{\pi}{4} = 4r$$

where equality holds if and only if $\alpha = \beta = \gamma = \delta$. In that case all eight triangles where these four different angles are included are congruent (ASA), so the tangential quadrilateral is a rhombus with equal vertex angles: a square by Theorem 3.1 *(b)*.

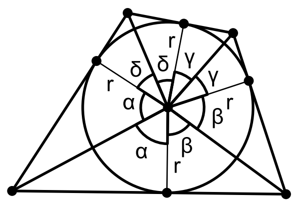

Figure 16. Angles at the incenter

(c) In a similar way as in Theorem 7.1 *(c)*, here we have a circle with a given radius r. This condition is a direct consequence of the inequality $K \geq 4r^2$ with equality if and only if the quadrilateral is a square.

(d) If L is the perimeter, then using the formulas $K = \frac{L}{2}r$ and $K \geq 4r^2$, we get

$$K \geq 4\left(\frac{2K}{L}\right)^2 = \frac{16K^2}{L^2} \qquad \Rightarrow \qquad K \leq \frac{L^2}{16}$$

with equality if and only if the quadrilateral is a square. □

9. Bicentric quadrilaterals

A *bicentric quadrilateral* is both cyclic and tangential. That the inequality $R \geq \sqrt{2}r$ holds in bicentric quadrilaterals was proved by I. Gerasimov and O. A. Kotii in 1964 according to [6, p. 132]. The second characterization in the following theorem was proved by M. Klamkin in 1967 according to [3, pp. 51–52] and the third was proved as Theorem 6.2 in [1]. We have no reference for the fourth formula, it was discovered while preparing to write this paper.

Theorem 9.1. *A bicentric quadrilateral with sides a, b, c, d, circumradius R, inradius r, diagonals p and q, circumcenter O and incenter I is a square if and only if it satisfies any one of:*

(a) $R = \sqrt{2}r$
(b) $8pq = (a+b+c+d)^2$
(c) $O = I$
(d) $\frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b} = 4$
(e) it has the largest area for a given perimeter

Proof. *(a)* Combining inequalities from the proofs of Theorems 7.1 *(b)* and 8.1 *(a)*, we have for the area of a bicentric quadrilateral

$$4r^2 \le K \le 2R^2$$

where equality on either side hold if and only if the quadrilateral is a square. A direct consequence is the inequality

$$2r^2 \le R^2$$

with equality only for a square. Now we just need to take the square root of both sides.

(b) We use Ptolemy's theorem, the AM-GM inequality, and that $a+c = b+d = s$ in tangential quadrilaterals to get

$$8pq = 2(4ac + 4bd) \le 2\left((a+c)^2 + (b+d)^2\right) = 2(2s^2) = (a+b+c+d)^2$$

where equality hold if and only if $a = c$ and $b = d$, so we have a parallelogram, but since the quadrilateral is both cyclic and tangential, it is a square according to Theorem 4.1 *(g)*.

(c) Let the distances from the vertices to the points where the incircle is tangent to the sides be e, f, g, h (the tangent lengths; there are eight of these that are equal in pairs according to the two tangent theorem). Applying the Pythagorean theorem four times in a bicentric quadrilateral where $O = I$ (see Figure 17), we get

$$e^2 = R^2 - r^2 = f^2 = g^2 = h^2$$

so $e = f = g = h$. Then $a = b = c = d$ and the bicentric quadrilateral is a rhombus, which is also a square according to Theorem 3.1 *(e)* or *(f)*.

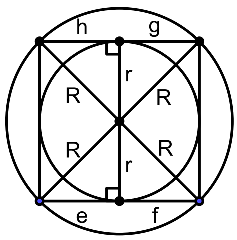

FIGURE 17. A bicentric quadrilateral where $O = I$

(d) The AM-GM inequality yields

$$\left(\frac{a}{c} + \frac{c}{a}\right) + \left(\frac{b}{d} + \frac{d}{b}\right) \ge 2\sqrt{\frac{a}{c} \cdot \frac{c}{a}} + 2\sqrt{\frac{b}{d} \cdot \frac{d}{b}} = 4 \tag{7}$$

with equality if and only if $\frac{a}{c} = \frac{c}{a}$ and $\frac{b}{d} = \frac{d}{b}$, that is, $a = c$ and $b = d$. Again we have that a cyclic and tangential parallelogram is a square.

(e) The area of a bicentric quadrilateral is given by the quite well-known formula

$$K = \sqrt{abcd}.$$

(For a proof, see [14, pp. 155–156].) Combining this with (4), we get

$$K = \left(\sqrt[4]{abcd}\right)^2 \le \left(\frac{a+b+c+d}{4}\right)^2 = \left(\frac{L}{4}\right)^2$$

where equality holds if and only if the quadrilateral is a square. □

We note that the second characterization could be formulated with the semiperimeter s instead, in which case we have that a bicentric quadrilateral is a square if and only if $s^2 = 2pq$.

A circle tangent to one side of a quadrilateral and the extensions of the adjacent two sides will be called an *escribed circle*, see Figure 18. (Note that an excircle to a quadrilateral is tangent to the extensions of all four sides, which can only happen in certain quadrilaterals, see [21], whereas all quadrilaterals always have four escribed circles.)

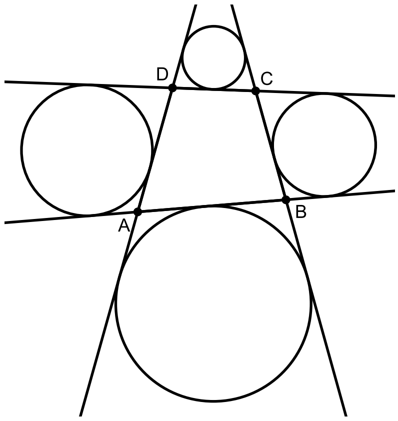

FIGURE 18. The four escribed circles

In the proof of the next two characterizations of squares, for which we have no references, we will need the following formula for the radius of an escribed circle to a bicentric quadrilateral. The proof of the lemma is cited from [29, pp. 100–102, 165].

Lemma 9.1. *The escribed circle tangent to side a of a bicentric quadrilateral with sides a, b, c, d and inradius r has radius*

$$r_a = \frac{a}{c}\,r.$$

Proof. With notations as in Figure 19, we have $\angle FAE = \frac{\pi - \angle A}{2}$. Then

$$\frac{x}{r_a} = \cot\frac{\pi - A}{2} = \tan\frac{A}{2}$$

so side $a = AB$ has length

$$a = x + y = r_a\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right).$$

In a cyclic quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, area K and semiperimeter s, we get

$$\begin{aligned} r_a &= \frac{a}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{aK}{K\tan\frac{A}{2} + K\tan\frac{B}{2}} \\ &= \frac{aK}{(s-d)(s-a) + (s-a)(s-b)} = \frac{aK}{(s-a)(a+c)} \end{aligned}$$

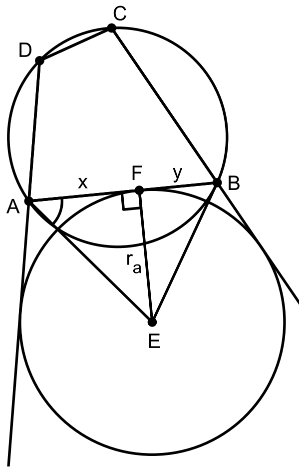

FIGURE 19. The escribed circle radius r_a

where we applied the half-angle formula

$$\tan\frac{A}{2}=\sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}},$$

(its derivation is left as an exercise for the reader, for a hint see [11, p. 25]), a similar one for angle B, and Brahmagupta's formula

$$K=\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

To simplify the escribed radius formula, we use that in a tangential quadrilateral, $s-a=c$ and $K=rs$ which immediately yields the formula in the lemma for a bicentric quadrilateral. □

Now we prove two characterizations about the four escribed radii.

Theorem 9.2. *A bicentric quadrilateral with escribed circles of radii r_a, r_b, r_c, r_d and inradius r is a square if and only if it satisfies any one of:*

(a) $r_a+r_b+r_c+r_d=4r$
(b) $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_d}=\frac{4}{r}$

Proof. *(a)* Applying the lemma (and similar formulas for the other three escribed radii) we get

$$r_a+r_c+r_b+r_d=\left(\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}\right)r\geq 4r$$

where the last step follows from (7), according to which equality holds if and only if the bicentric quadrilateral is a square.

(b) Analogously, we get

$$\frac{1}{r_a}+\frac{1}{r_c}+\frac{1}{r_b}+\frac{1}{r_d}=\left(\frac{c}{a}+\frac{a}{c}+\frac{d}{b}+\frac{b}{d}\right)\frac{1}{r}\geq\frac{4}{r}$$

where equality holds if and only if the bicentric quadrilateral is a square. □

10. Orthodiagonal quadrilaterals

In this section we study what additional properties a quadrilateral with perpendicular diagonals must have in order to be a square. We know of the following eight criteria. The first was stated in [12, p. 50] and also formulated and proved in a different way in [19, p. 138]. Another way to state it is that an orthodiagonal quadrilateral $ABCD$ is a square if and only if $AP = BP = CP = DP$ where P is the diagonal intersection. The other seven characterizations are taken from [9], which was chaired before publication by our friend Mario Dalcín from Uruguay.

Theorem 10.1. *An orthodiagonal quadrilateral $ABCD$ with diagonal intersection P is a square if and only if it satisfies any one of:*

(a) it has equal and bisecting diagonals
(b) $AB = BC = DA$ and $\angle A = \angle B$
(c) $AB = BC = DA$ and $AP = BP$
(d) $AB = DA$ and $AP = BP = CP$
(e) $BC = DA$ and $\angle A = \angle B = \angle C$
(f) $DA = AB$ and $\angle A = \angle B = \angle C$
(g) $\angle A = \angle B$ and $DP = AP = BP$
(h) $AB = CD$ and $AP = BP = CP$

Proof. *(a)* A quadrilateral with perpendicular, equal, and bisecting diagonals is divided by its diagonals into four congruent isosceles triangles (SAS), see Figure 20, so it is a rhombus with four equal vertex angles, which means a square.

(b) Triangles ADB and BAC are congruent (SAS), so (see Figure 20)

$$\angle ADB = \angle ABD = \angle BAC = \angle BCA = 45^\circ$$

since $\angle APB = 90^\circ$. Then $\angle A = \angle B = 90^\circ$. We also have $AC = BD$ by the congruent triangles and $AP = BP$ in isosceles triangle APB, so $DP = CP$. Using that $\angle DPC = 90^\circ$, we get $\angle CDP = \angle DCP = 45^\circ$ and thus $\angle C = \angle D = 90^\circ$. Then $ABCD$ is a rectangle with perpendicular diagonals, that is, a square according to Theorem 2.1 *(b)*.

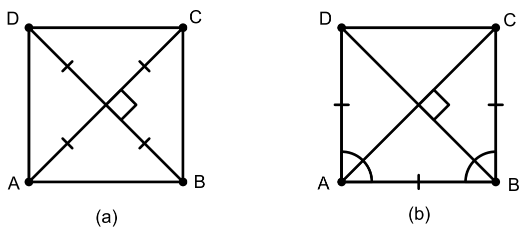

Figure 20. Given assumptions in Theorem 10.1 (a) and (b)

(c) We directly get $\angle BAP = \angle ABP = \angle ACB = \angle BDA = 45^\circ$ from isosceles triangles using that $AC \perp BD$ (see Figure 21). Then $\angle A = \angle B = 90^\circ$ and since triangles ABD and BAC are congruent (SAS), the diagonals are equal and bisect each other. Then triangles APD and CPD are congruent (SAS), so $AD = DC$. Thus $ABCD$ is a rhombus with a right vertex angle, that is, a square by Theorem 3.1 *(a)*.

(d) Here we have $\angle BAP = \angle ABP = \angle PCB = \angle CBP = 45°$, so triangles ABP and BCP are congruent (ASA), see Figure 21. Then $AB = BC$ and by *(c)*, $ABCD$ is a square.

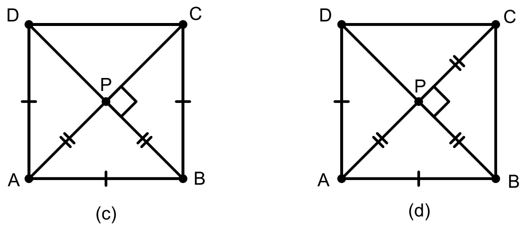

Figure 21. Given assumptions in Theorem 10.1 (c) and (d)

(e) Triangles ABD and ABC are congruent (SAS), so $BD = AC$ and $\angle ABD = \angle BAC$ (see Figure 22). Then $AP = BP$, so $CP = DP$, and thus $\angle ACD = \angle BDC$. We get

$$\angle C = \angle ACB + \angle ACD = \angle ADB + \angle CDB = \angle D,$$

and thus $\angle A = \angle B = \angle C = \angle D$, so we have a rectangle with perpendicular diagonals: a square.

(f) $\angle ADB = \angle ABD$, so triangles ADP and ABP are congruent (AAS), implying $DP = BP$, so triangles DPC and BPC are congruent (SAS), see Figure 22. Then $DC = BC$, so triangles ADC and ABC are congruent (SSS). Hence $\angle A = \angle B = \angle C = \angle D$ and $ABCD$ is a square since $AC \perp BD$.

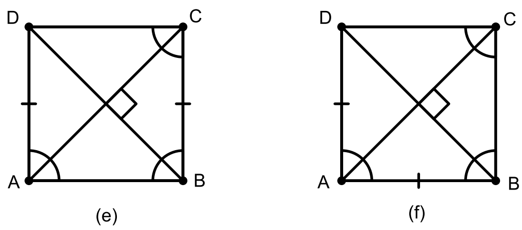

Figure 22. Given assumptions in Theorem 10.1 (e) and (f)

(g) We have $\angle PAB = \angle PBA$ in isosceles triangle ABP, so triangles ABD and BAC are congruent (ASA), see Figure 23. Then $BD = AC$, implying $AP = BP = CP = DP$ and $ABCD$ is a square according to *(a)*.

(h) Triangles ABP and DCP are congruent (RHS) and $\angle PAB = \angle PBA$, so $\angle DCP = \angle CDP = 45°$ (see Figure 23). Thus $CP = DP$, so $AP = BP = CP = DP$ and $ABCD$ is a square. □

11. Convex quadrilaterals

Now we shall study what additional properties a general convex quadrilateral must have in order to be a square. We know of the following seven criteria. The second was discussed in [31, p. 24]. The last five characterizations are taken from the recent paper [8], where no proofs were given.

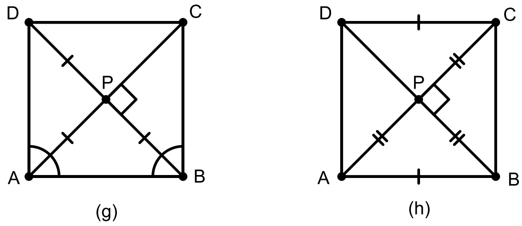

FIGURE 23. Given assumptions in Theorem 10.1 (g) and (h)

Theorem 11.1. *A convex quadrilateral $ABCD$ with diagonal intersection P is a square if and only if it satisfies any one of:*

(a) it has 4 equal sides and 4 right angles
(b) it has rotational symmetry of order 4
(c) $AB = BC = CD$ and $\angle A = \angle C = \angle D$
(d) $AB = BC = CD$ and $AP = BP = CP$
(e) $AB = BC = CD$ and $AP = CP = DP$
(f) $AB = DA$ and $AP = BP = CP = DP$
(g) $BC = CD = DA$, $AP = DP$ and $BP = CP$

Proof. *(a)* This is the definition of a square that we have used.

(b) A rotational symmetry of order 4 means that if we rotate the quadrilateral $\frac{360°}{4} = 90°$ then it is congruent with its original position. Hence all four angles are equal and the four sides have equal length, so it is a square.

(c) From $\angle BAC = \angle BCA$ and $\angle A = \angle C$ we get $\angle DAC = \angle DCA$, so $DA = DC$ and $ABCD$ is a rhombus (see Figure 24). But $\angle C = \angle D$ implies it's a square.

(d) Triangles ABP and CBP are congruent (SSS), so $AC \perp BD$ (see Figure 24). Then triangles APD and CPD are also congruent (SAS), implying that $AD = CD$ and thus that $ABCD$ is a rhombus. But $\angle B = 45° + 45° = 90°$ due to right isosceles triangles ABP and CBP, so $ABCD$ is a square.

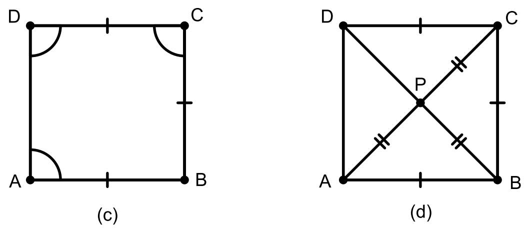

FIGURE 24. Given assumptions in Theorem 11.1 (c) and (d)

(e) Triangles ABP and CBP are congruent (SSS), so $AC \perp BD$ (see Figure 25). Then triangles ABP and DCP are also congruent (RHS), implying that $CP = BP$. Then $ABCD$ is a square by Theorem 10.1 *(a)*.

(f) Triangles DAP and ABP are congruent (SSS), so $AC \perp BD$ (see Figure 25). Then $ABCD$ is a square according to Theorem 10.1 *(a)*.

(g) Vertical angles $\angle APB = \angle CPD$, so triangles APB and DPC are congruent (SAS), see Figure 25. Then all sides are equal, so $ABCD$ is

a rhombus, and since the diagonals have equal length, it is a square by Theorem 3.1 *(d)*. □

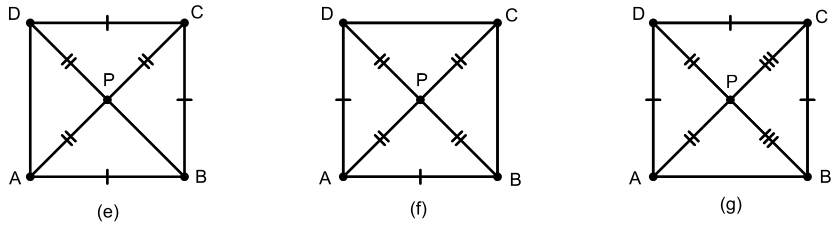

FIGURE 25. Given assumptions in Theorem 11.1 (e), (f) and (g)

12. Area

In this last section we study eight characterizations for when a convex quadrilateral is a square that are different formulas for the area of the quadrilateral expressed in various ways, but mostly in terms of the four sides. The first is attributed to Z. A. Skopec and V. A. Zarov (1962) in [6, p. 132]. The proof we give is cited from [30]. The second and third characterization are proved as Corollary 15 in [19] and Theorem 2.2 in [20], but here we will give a new proof of the second. The inequality case of the fourth was given as an exercise in [2, p. 72]. We found the fifth and sixth in [3, pp. 202–203, 265–266]. The last two were proved in [26, pp. 48–51], which are different formulations of the *isoperimetric theorem for a quadrilateral.* We cite a much shorter proof from [3, p. 16].

Theorem 12.1. *A convex quadrilateral with sides a, b, c, d, area K and distances w, x, y, z from an interior point to the vertices, is a square if and only if it satisfies any one of:*

(a) $K = \frac{1}{4}\left(a^2+b^2+c^2+d^2\right)$
(b) $K = \frac{1}{2}\left(a^2+c^2\right) = \frac{1}{2}\left(b^2+d^2\right)$
(c) $K = \frac{1}{8}\left((a+c)^2+(b+d)^2\right)$
(d) $K = \frac{1}{6}(ab+ac+ad+bc+bd+cd)$
(e) $K = \frac{1}{2}\left(w^2+x^2+y^2+z^2\right)$
(f) $K = \frac{1}{2}\sqrt[3]{(ab+cd)(ac+bd)(ad+bc)}$
(g) $K = \frac{1}{16}(a+b+c+d)^2$
(h) it has the largest area for a given perimeter

Proof. *(a)* The area of a convex quadrilateral $ABCD$ satisfies (see Figure 26)

$$K = \frac{ab\sin B + cd\sin D}{2} \le \frac{ab+cd}{2} \le \frac{\frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}}{2} = \frac{a^2+b^2+c^2+d^2}{4}$$

where equality holds if and only if $\angle B = \angle D = 90^\circ$, $a=b$ and $c=d$. Then the quadrilateral is a square according to Theorem 6.1 *(d)*.

(b) In the same way as in *(a)*, we get

$$K = \frac{ad\sin A + bc\sin C}{2} \le \frac{ad+bc}{2}$$

and adding this and the similar inequality from *(a)* yields

$$2K \le \frac{ab+cd+ad+bc}{2}.$$

This is factorized into

$$K \le \frac{a+c}{2}\cdot\frac{b+d}{2} \le \sqrt{\left(\frac{a^2+c^2}{2}\right)\left(\frac{b^2+d^2}{2}\right)} \tag{8}$$

where we to get the second inequality applied the AM-RMS inequality. Equality holds if and only if $\angle A = \angle B = \angle C = \angle D = 90°$, $a = c$ and $b = d$, that is, only when $ABCD$ is a rectangle. (Five other proofs of the second inequality in (8) were given in [17].) Next we use the well-known characterization that the diagonals in a convex quadrilateral are perpendicular if and only if $a^2 + c^2 = b^2 + d^2$ (for a proof, see [16]). Hence

$$K \le \tfrac{1}{2}\left(a^2+c^2\right) = \tfrac{1}{2}\left(b^2+d^2\right)$$

where we have equality if and only if the rectangle has perpendicular diagonal, which happens only for a square according to Theorem 2.1 *(b)*.

(c) With diagonals p and q and included angle θ, the area is given by

$$K = \frac{pq}{2}\sin\theta \le \frac{pq}{2} \le \frac{ac+bd}{2} \le \frac{1}{2}\left(\left(\frac{a+c}{2}\right)^2 + \left(\frac{b+d}{2}\right)^2\right)$$

according to (2), Ptolemy's inequality and the AM-GM inequality. We have equality if and only if the diagonals are perpendicular, the quadrilateral is cyclic, $a = c$ and $b = d$, that is, if and only if it is a rectangle with perpendicular diagonals, which is a square.

(d) In *(a)*, *(b)* and *(c)* we have proved

$$2K \le ab+cd, \qquad 2K \le ad+bc, \qquad 2K \le ac+bd.$$

Adding these, we get

$$K \le \tfrac{1}{6}(ab+ac+ad+bc+bd+cd)$$

where equality holds if and only if $\angle A = \angle B = \angle C = \angle D = 90°$, the diagonals are perpendicular and the quadrilateral is cyclic. Hence it must be a rectangle with perpendicular diagonals, that is, a square according to Theorem 2.1 *(b)*.

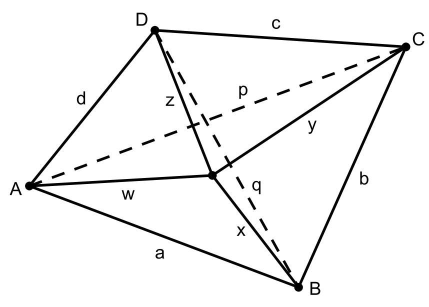

Figure 26. Applying the triangle inequality

(e) The area satisfies (see Figure 26)

$$2K \le pq \le (w+y)(x+z) \le 4\sqrt{\frac{w^2+y^2}{2}}\sqrt{\frac{x^2+z^2}{2}} \le \frac{4}{2}\left(\frac{w^2+y^2}{2}+\frac{x^2+z^2}{2}\right)$$

where we applied the triangle inequality, the AM-RMS inequality and the AM-GM inequality. We have equality if and only if the diagonals are perpendicular, the interior point in question is the diagonal intersection, $w = y$, $x = z$ and $w^2 + y^2 = x^2 + z^2$, where the latter three yields $w = x = y = z$. Altogether this means that we have equality only for a square.

(f) We have

$$8K^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{2R^2} \le \frac{(ab+cd)(ac+bd)(ad+bc)}{K}$$

according to (1) and (3), from which we get the inequality by rearranging and taking the cube root. Equality holds if and only if the quadrilateral is a square. This proof relies on the two quite well-known facts that there is always a cyclic quadrilateral with the same sides as a convex quadrilateral and that the cyclic quadrilateral has the largest area of all convex quadrilaterals with given sides. Proofs can be found in [3, pp. 33–34, 241].

(g) Applying the AM-GM inequality to the first inequality in (8) yields

$$K \le \frac{1}{4}(a+c)(b+d) \le \frac{1}{4}\left(\frac{a+c+b+d}{2}\right)^2 = \frac{(a+b+c+d)^2}{16} \qquad (9)$$

where we have equality if and only if the quadrilateral is a rectangle (so $a = c$ and $b = d$) and $a + c = b + d$, that is, a square.

(h) Denoting the perimeter by L, we have from (9)

$$K \le \frac{(a+b+c+d)^2}{16} = \frac{L^2}{16}$$

with equality if and only if the quadrilateral is a square, so the last criteria is just an interpretation of that inequality. □

Note that the last characterization is a generalization of Theorems 2.1 *(i)*, 4.1 *(h)*, 7.1 *(d)*, 8.1 *(d)* and 9.1 *(e)*.

We conclude by giving a second proof of the inequality in *(a)*, which we have not found any reference for. It is based on three equalities: the area formula (2), the simple algebra rule $(p-q)^2 = p^2 + q^2 - 2pq$, and Euler's quadrilateral formula

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2$$

where v is the distance between the midpoints of diagonals p and q (derived for instance in [3, pp. 9–10]). We get

$$K = \tfrac{1}{2}pq\sin\theta = \tfrac{1}{4}\big(p^2 + q^2 - (p-q)^2\big)\sin\theta$$

and thus the area of a convex quadrilateral is

$$K = \tfrac{1}{4}\big(a^2 + b^2 + c^2 + d^2 - 4v^2 - (p-q)^2\big)\sin\theta \le \tfrac{1}{4}\big(a^2 + b^2 + c^2 + d^2\big)$$

where we have equality if and only if $v = 0$, $p = q$ and $\theta = 90°$, which is a parallelogram with equal and perpendicular diagonals, that is, a square according to Theorem 4.1 *(d)*.

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