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ABOUT A GENERAL INEQUALITY AND SOME APPLICATIONS IN THE GEOMETRY OF THE QUADRILATERAL

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Abstract. In this paper we will demonstrate a general inequality. For particular value of the variables, we will obtain known and new inequalities.

1. Introduction

In this section we will recall some known results, which we will use in the following.

In a bicentric quadrilateral, we denote the lengths of the sides with a, b, c, d, F the area, r, R the radius of the inscribed circle, respectively of the circumscribed circle of the quadrilateral and with $s = \dfrac{a+b+c+d}{2}$ the semiperimeter of the quadrilateral.

Theorem 1 (see [3], or [5]). *In a bicentric quadrilateral the equalities take place*

$$\sum ab = s^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}\,, \tag{1}$$

$$\sum a^2 = 2s^2 - 4r^2 - 4r\sqrt{4R^2 + r^2}\,, \tag{2}$$

$$F = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{abcd} = sr \tag{3}$$

and

$$16F^2 = 2\sum a^2b^2 - \sum a^4 + 8abcd. \tag{4}$$

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Theorem 2 (Finsler-Hadwiger's type inequality, see [3], pages 58–61)**.** *In every bicentric quadrilateral, the following inequality*

$$a^2+b^2+c^2+d^2 \geq \tag{5}$$
$$\geq 4F+\frac{1}{2}\left((a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2\right)$$

is true.
We denote $s_1=\sqrt{8r\left(\sqrt{4R^2+r^2}\,-r\right)}$ *and* $s_2=\sqrt{4R^2+r^2}+r$.

Theorem 3 (Blundon–Eddy's Inequality, see [2], or [3] page 58, or [5] pages 168–171)**.** *A bicentric quadrilateral is given.*
The inequality

$$s_1 \leq s \tag{6}$$

holds. If $R=r\sqrt{2}$ *then the quadrilateral is square, both circles are concentric and the equality holds. If* $R\neq r\sqrt{2}$ *, then the equality holds if and only if the quadrilateral is an isosceles trapezoid.*
The inequality

$$s \leq s_2 \tag{7}$$

holds, with equality if and only if the quadrilateral is orthodiagonal.
The inequalities

$$s_1 \leq s \leq s_2 \tag{8}$$

holds. If $R=r\sqrt{2}$*, then both inequalities becomes equalities and in this case the quadrilateral is square. If* $R\neq r\sqrt{2}$*, then at least one of the inequalities is strict.*

Theorem 4 (L. Fejes Tóth's Inequality, see [3] page 147, or [5] page 166)**.** *In any bicentric quadrilateral, the inequality*

$$R \geq r\sqrt{2} \quad \textit{holds.} \tag{9}$$

2. Main results and applications

In the following, we denote
$16T(a,b,c,d)=2(ab+ac+ad+bc+bd+cd-\,-a^2-b^2-c^2-d^2)$,
with $a,b,c,d\in\mathbb{R}$.

Remark 1. If $x,y,z,t\geq 0$ and $(x+y+z)(y+z+t)(z+t+x)(t+x+y)\neq 0$, then at most two numbers between x,y,z,t can be equal with 0 and $x+y+z+t>0$.

Theorem 5. *Let* $a_1,a_2,\dots,a_n>0$, $x_1,x_2,\dots,x_n\geq 0$, $s=\sum\limits_{i=1}^{n}x_i$ *such that* $\prod\limits_{i=1}^{n}(s-x_i)\neq 0$. *Then*

$$\frac{x_1}{s-x_1}a_1^2+\frac{x_2}{s-x_2}a_2^2+\cdots+\frac{x_n}{s-x_n}a_n^2 \geq \tag{10}$$
$$\geq \frac{1}{n-1}\Bigg(2\sum_{1\leq i<j\leq n}a_ia_j-(n-2)\sum_{i=1}^{n}a_i^2\Bigg),$$

with equality if and only if

$$\frac{a_1}{s-x_1} = \frac{a_2}{s-x_2} = \cdots = \frac{a_n}{s-x_n}\,. \tag{11}$$

Proof. We denote $s - x_1 = u_1$, $s - x_2 = u_2, \ldots, s - x_n = u_n$.
So $x_i = \dfrac{u_2 + u_3 + \cdots + u_n - (n-2)u_1}{n-1}$ and analogously.

We have

$$\sum_{i=1}^{n} \frac{x_i}{s-x_i}\,a_i^2 = \frac{1}{n-1}\Bigg(\frac{u_2+u_3+\cdots+u_n}{u_1}\,a_1^2 - (n-2)a_1^2 +$$

$$+\frac{u_1+u_3+\ldots+u_n}{u_2}\,a_2^2 - (n-2)a_2^2 + \cdots + \frac{u_1+u_2+\cdots+u_{n-1}}{u_n}\,a_n^2 - (n-2)a_n^2\Bigg) =$$

$$= \frac{1}{n-1}\Bigg(\left(\frac{u_2}{u_1}\,a_1^2 + \frac{u_1}{u_2}\,a_2^2\right) + \left(\frac{u_3}{u_1}\,a_1^2 + \frac{u_1}{u_3}\,a_3^2\right) + \cdots + \left(\frac{u_1}{u_n}\,a_n^2 + \frac{u_n}{u_1}\,a_1^2\right) +$$

$$+ \cdots + \left(\frac{u_{n-1}}{u_n}\,a_n^2 + \frac{u_n}{u_{n-1}}\,a_{n-1}^2\right) - (n-2)\sum_{i=1}^{n} a_i^2\Bigg).$$

Applying the inequality of means, we get $\dfrac{u_2}{u_1}a_1^2 + \dfrac{u_1}{u_2}a_2^2 \geq 2a_1a_2$ and analogs and then, the inequality from statement follows. The equality holds if and only if $\dfrac{u_2}{u_1}a_1^2 = \dfrac{u_1}{u_2}a_2^2$ and analogs, equivalent with $\dfrac{a_1}{u_1} = \dfrac{a_2}{u_2}$ and analogs, from where (11) result.

Theorem 6. *Let $a, b, c, d > 0$, $x, y, z, t \geq 0$ such that $(x+y+z)(y+z+t)(z+t+x)(t+x+y) \neq 0$. Then the inequality*

$$\frac{x}{y+z+t}\,a^2 + \frac{y}{z+t+x}\,b^2 + \frac{z}{t+x+y}\,c^2 + \frac{t^2}{x+y+z}\,d^2 \geq \frac{16T(a,b,c,d)}{3} \tag{12}$$

holds, with equality if and only if $k > 0$ exists such that

$$\begin{cases} -2a+b+c+d \geq 0 \\ a-2b+c+d \geq 0 \\ a+b-2c+d \geq 0 \\ a+b+c-2d \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} x = (-2a+b+c+d)k \\ y = (a-2b+c+d)k \\ z = (a+b-2c+d)k \\ t = (a+b+c-2d)k. \end{cases} \tag{13}$$

Proof. If we replace in (10) $n = 4$, $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$, $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$, then the inequality from (12) follows.
From (11) results that the equality holds if and only if

$$\frac{a}{y+z+t} = \frac{b}{z+t+x} = \frac{c}{t+x+y} = \frac{d}{x+y+z} = \frac{a+b+c+d}{3(x+y+z+t)}\,.$$

We note that $\dfrac{a+b+c+d}{3(x+y+z+t)} = \dfrac{1}{3k}$, where $k > 0$ since $a, b, c, d > 0$ and $x, y, z, t \geq 0$, $x+y+z+t > 0$ from Remark 1.
Then $x+y+z+t = k(a+b+c+d)$ and from $\dfrac{a}{y+z+t} = \dfrac{1}{3k}$ it results that $y+z+t = 3ka$. From the last equality it results that $x = (-2a+b+c+d)k$ and analogs and then (13) holds.

Remark 2. According to (13), if $-2a+b+c+d<0$ or $a-2b+c+d<0$, or $a+b-2c+d<0$, or $a+b+c-2d<0$, then the inequality from (12) is strictly.

Theorem 7. *If $a,b,c,d>0$, $x,y,z,t\geq 0$ and $(x+y+z)(y+z+t)(z+t+x)(t+x+y)\neq 0$, then*

$$\text{(14)}\quad \frac{x}{dcy+dbz+cbt}+\frac{y}{adz+act+dcx}+\frac{z}{bct+bdx+ady}+\frac{t}{cbx+cay+baz}\geq \geq \frac{16T(a,b,c,d)}{3abcd},$$

with equality if and only if $k>0$ exists such that

$$\text{(15)}\quad \begin{cases}-2a+b+c+d\geq 0\\ a-2b+c+d\geq 0\\ a+b-2c+d\geq 0\\ a+b+c-2d\geq 0\end{cases} \quad \text{and} \quad \begin{cases}x=a(-2a+b+c+d)k\\ y=b(a-2b+c+d)k\\ z=c(a+b-2c+d)k\\ t=d(a+b+c-2d)k.\end{cases}$$

Proof. If we replace in (12) x,y,z,t with $\frac{x}{a},\frac{y}{b},\frac{z}{c}$, respectively $\frac{t}{d}$, we obtain (14).

In the following, we will particularize the inequality from (12).

Theorem 8. *If $a,b,c>0$, $x,y,z\geq 0$ such that $(x+y)(y+z)(z+x)\neq 0$, then the inequality*

$$\text{(16)}\quad \frac{x}{cy+bz}+\frac{y}{az+cx}+\frac{z}{bx+ay}\geq \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{2abc}$$

holds. The equality occurs if and only if

$$\text{(17)}\quad \begin{cases}-a+b+c\geq 0\\ a-b+c\geq 0\\ a+b-c\geq 0\end{cases} \quad \text{and} \quad \begin{cases}x=a(-a+b+c)k\\ y=b(a-b+c)k\\ z=c(a+b-c)k,\end{cases}$$

where $k>0$.

Proof. In inequality (14) we consider $t=0$ and we have

$$\frac{x}{d(cy+bz)}+\frac{y}{d(az+cx)}+\frac{z}{d(bx+ay)}\geq \frac{2\left(\sum ab-\sum a^2\right)}{3abcd},$$

equivalent to

$$\text{(18)}\quad \frac{x}{cy+bz}+\frac{y}{az+cx}+\frac{z}{bx+ay}\geq \geq \frac{2\big(ab+ac+ad+bc+bd+cd-(a^2+b^2+c^2+d^2)\big)}{4abc}.$$

Taking $t=0$ and (15) into account, a necessary condition for (18) not to be a strict inequality is that $a+b+c-2d=0$. Substituting in the right member of the inequality (18) and by $d=\frac{a+b+c}{2}$, we have

$$\frac{2\big(ab+ac+bc+d(a+b+c)-(a^2+b^2+c^2+d^2)\big)}{3abc}=$$

$$= \frac{2\Big(ab+ac+bc+\dfrac{(a+b+c)^2}{2}-\Big(a^2+b^2+c^2+\Big(\dfrac{a+b+c}{2}\Big)^2\Big)\Big)}{3abc} =$$

$$= \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{2abc},$$

so from (18) the inequality (16) follows. Replacing d with $\dfrac{a+b+c}{2}$ in relations from (15), we obtain the conditions from (17).

Remark 3. The inequality from Theorem 8 is demonstrated in [6].

Corollary 1. *If $a, b, c, d > 0$ then the following inequality*

(19) $$\frac{3}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}+\frac{3}{\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}}+\frac{3}{\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{a^2}}+\frac{3}{\frac{1}{d^2}+\frac{1}{a^2}+\frac{1}{b^2}} \geq$$
$$\geq 16T(a,b,c,d)$$

holds.

Proof. In inequality (14) we replace x, y, z, t with bcd, cda, dab and abc from which it follows the statement.

In the following we consider a bicentric quadrilateral with a, b, c, d the lengths of side and F the area.

The inequality from (5) is equivalent with

$2(a^2+b^2+c^2+d^2) \geq 8F+3(a^2+b^2+c^2+d^2)-2(ab+ac+ad+bc+bd+cd)$,

equivalent with

$2\big(ab+ac+ad+bc+bd+cd-(a^2+b^2+c^2+d^2)\big) \geq 8F-(a^2+b^2+c^2+d^2)$,

from where

(20) $$16T(a,b,c,d) \geq 8F-(a^2+b^2+c^2+d^2).$$

Corollary 2 (Tsintsifas' type Inequality)**.** *If $x, y, z, t > 0$ such that $(x+y+z)(y+z+t)(z+t+x)(t+x+y) \neq 0$, then the inequality*

(21) $$\left(\frac{x}{y+z+t}+\frac{1}{3}\right)a^2+\left(\frac{y}{z+t+x}+\frac{1}{3}\right)b^2+\left(\frac{z}{t+x+y}+\frac{1}{3}\right)c^2+$$
$$+\left(\frac{t}{x+y+z}+\frac{1}{3}\right)d^2 \geq \frac{8}{3}F$$

holds.

Proof. The inequality from (21) follows from the inequalities (12) and (20).

Corollary 3 (Ionescu–Weitzenböck's type Inequality)**.** *The following inequality*

(22) $$a^2+b^2+c^2+d^2 \geq 4F$$

is true.

Proof. The inequality (22) is obtained from (21) for $x = y = z = t$.

Lemma 1. *In a bicentric quadrilateral we have*

(23) $$8F^2 = 16T(a^2,b^2,c^2,d^2)+a^4+b^4+c^4+d^4$$

and

(24) $$F^2 \geq 4T(a^2,b^2,c^2,d^2).$$

Proof. Taking (3), (4) and the definition of $T(a,b,c,d)$ into account, we get the identity (23).
But from the means inequality we have that $a^4+b^4+c^4+d^4 \geq 4abcd = 4F^2$, so from (23) we obtain the inequality (24).

Corollary 4 (Goldner's type Inequality)**.** *In every bicentric quadrilateral the inequality*

$$a^4+b^4+c^4+d^4 \geq 4F^2. \tag{25}$$

holds.

Proof. In inequality from (14) we put a^2, b^2, c^2, d^2 instead of x, y, z, respectively t and a^2, b^2, c^2, d^2 instead a, b, c, respectively d. Then we have that

$$\frac{a^2}{3b^2c^2d^2}+\frac{b^2}{3a^2c^2d^2}+\frac{c^2}{3a^2b^2d^2}+\frac{d^2}{3a^2b^2c^2} \geq \frac{16T(a^2,b^2,c^2,d^2)}{3a^2b^2c^2d^2},$$

from where

$$a^4+b^4+c^4+d^4 \geq 16T(a^2,b^2,c^2,d^2). \tag{26}$$

Taking (23) and (26) into account, the inequality from (25) follows.

From (1), (2) and the expression of $T(a,b,c,d) = \frac{1}{8}\left(\sum ab - \sum a^2\right)$, we have

$$T(a,b,c,d) = \frac{1}{8}\left(6r\sqrt{4R^2+r^2}+6r^2-s^2\right). \tag{27}$$

Theorem 9. *In every bicentric quadrilateral, the following inequalities*

$$\frac{1}{2}\left(r\sqrt{4R^2+r^2}-R^2+r^2\right) \leq T(a,b,c,d) \leq \frac{1}{4}\left(7r^2-r\sqrt{4R^2+r^2}\right) \tag{28}$$

hold.

Proof. From (8) we have $s_1 \leq s \leq s_2$ and let $f : [s_1, s_2] \to \mathbb{R}$ be a function defined by $f(s) = T(a,b,c,d)$, where $T(a,b,c,d)$ is found in (27).
Since f is a decreasing function on $[s_1, s_2]$, it results that $f(s_1) \geq f(s) \geq f(s_2)$. But $s_1 = \sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$ and $s_2 = \sqrt{4R^2+r^2}+r$ and by calculus, from the inequalities above (28) is obtained.

Corollary 5. *In every bicentric quadrilateral, the following inequalities are true*

$$T(a,b,c,d) \geq 0 \quad \textit{if} \quad \frac{R}{r} \in \left[\sqrt{2}, \sqrt{6}\right], \tag{29}$$

$$T(a,b,c,d) \leq 0 \quad \textit{if} \quad \frac{R}{r} \geq 2\sqrt{3} \tag{30}$$

and

$$T(a,b,c,d) \leq r^2. \tag{31}$$

Proof. From (28), if we denote $\frac{R}{r} = x$, we obtain

$$\frac{1}{2}\left(\sqrt{4x^2+1}-x^2+1\right) \leq \frac{T(a,b,c,d)}{r^2} \leq \frac{1}{4}\left(7-\sqrt{4x^2+1}\right). \tag{32}$$

Taking (9) into account, let $g, h : [\sqrt{2}, \infty) \to \mathbb{R}$ be the functions defined by $g(x) = \frac{1}{2}\left(\sqrt{4x^2+1} - x^2 + 1\right)$ and $h(x) = \frac{1}{4}\left(7 - \sqrt{4x^2+1}\right)$, $x \in [\sqrt{2}, \infty)$. We have immediately by calculation that $g'(x) = \dfrac{x\left(2 - \sqrt{4x^2+1}\right)}{\sqrt{4x^2+1}} < 0$, $h'(x) = -\dfrac{x}{\sqrt{4x^2+1}} < 0$ for $x \in [\sqrt{2}, \infty)$, so the functions g and h are decreasing. Since $h\left(\sqrt{2}\right) = 1$, from (32) the inequality (31) results. From $g(x) = 0$ and $h(x) = 0$ it results that $x = \sqrt{6}$, respectively $x = 2\sqrt{3}$ and since the functions g and h are decreasing, the inequalities from (29) and (30) it results.

Corollary 6. *In every bicentric quadrilateral the inequality*

$$4T(a,b,c,d) \le F \tag{33}$$

holds.

Proof. Inequality from statement is equivalent with
$\frac{1}{8}\left(6r\sqrt{4R^2+r^2} + 6r^2 - s^2\right) \le \frac{1}{4}rs$, equivalent with
$s^2 + 2rs \ge 6r\sqrt{4R^2+r^2} + 6r^2$. Since $s \ge s_1$, it will be sufficient to show that $s_1^2 + 2rs_1 \ge 6r\sqrt{4R^2+r^2} + 6r^2$. Substituting s_1, the last inequality is equivalent to $2r\sqrt{4R^2+r^2} + 2r\sqrt{8r\left(\sqrt{4R^2+r^2} - r\right)} \ge 14r^2$. This inequality is true since $R \ge r\sqrt{2}$.

Corollary 7. *In every bicentric quadrilateral, the inequalities*

$$\sum \frac{x}{y+z+t}\, a^2 \ge \frac{2}{3}\left(6r\sqrt{4R^2+r^2} + 6r^2 - s^2\right) \ge \frac{8}{3}\left(r\sqrt{4R^2+r^2} - R^2 + r^2\right) \tag{34}$$

and

$$\sum \frac{x}{dcy+dbz+cbt} \ge \frac{2}{3} \cdot \frac{6r\sqrt{4R^2+r^2} + 6r^2 - s^2}{s^2 r^2} \ge \frac{8}{3} \cdot \frac{r\sqrt{4R^2+r^2} + r^2 - R^2}{r^2\left(\sqrt{4R^2+r^2} + r^2\right)^2} \tag{35}$$

hold.

Proof. It follows from (12), (27) and (8), respectively (14), (27) and (8).
We consider the expression

$$16K(a,b,c,d) = 2\sum ab - \sum a^2 + 8\sqrt{abcd}\,, \tag{36}$$

and taking (1), (2) and (3) into account, we obtain

$$K = \frac{1}{2}\, r\left(\sqrt{4R^2+r^2} + r + s\right). \tag{37}$$

Corollary 8. *In every bicentric quadrilateral, the following inequalities*

$$4T(a,b,c,d) \le F \le K(a,b,c,d) \tag{38}$$

hold.

Proof. The inequality $F \leq K(a, b, c, d)$ is equivalent taking (37) into account with $s \leq \sqrt{4R^2 + r^2} + r$ which is a true inequality from (8). The inequality $4T(a, b, c, d) \leq F$ was demonstrated in Corollary 6.

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