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A DYNAMICAL SYSTEM IN THE PLANE GEOMETRY

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Abstract. We explore the convergence of the sequence of some concurrency points defined by the incenter and we prove that the limit is the Hofstadter zero-point $X(360)$ of a triangle.

1. Introduction

Numerous dynamic geometries in the triangle geometry have been inspired by simple geometric iterations, some of which are reviewed in the articles [3] or [4]: iterations generated by the median, orthic, bisector, orthic or pedal triangles, or by the incircle and the circumcircle.

In the present note we study the convergence of a sequence of concurency points in the triangle geometry. Let us consider an arbitrary triangle ABC with the incenter I and let us define the sequence of points $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$, $(Z_n)_{n\geq 1}$, where:

$$\begin{cases} X_1 = I \\ Y_1 = I \\ Z_1 = I \end{cases}$$

and for every $n \geq 1$:

$$\begin{cases} X_{n+1} = \text{the incenter of } \triangle X_n BC \\ Y_{n+1} = \text{the incenter of } \triangle Y_n CA \\ Z_{n+1} = \text{the incenter of } \triangle Z_n AB. \end{cases}$$

In what follows α, β, γ are the measures in radians of the angles A, B, C, respectively, and

$$\alpha_n = \frac{\alpha}{2^n},\ \beta_n = \frac{\beta}{2^n},\ \gamma_n = \frac{\gamma}{2^n},\ \forall\ n \in \mathbb{N}^*.$$

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2. The main result

Proposition 2.1. *For every $n \in \mathbb{N}$, $n \geq 1$, the lines AX_n, BY_n, CZ_n are concurrent.*

Proof. From the definition of the sequence $(X_n)_{n\geq 1}$ we have :

$$m(\sphericalangle X_nBC) = \beta_n,\ m(\sphericalangle X_nCB) = \gamma_n,\ \forall\ n \geq 1.$$

From the concurrency in X_n of the lines AX_n, BX_n, CX_n and the trigonometric form of Ceva Theorem it follows

$$\frac{\sin(\sphericalangle BAX_n)}{\sin(\sphericalangle CAX_n)} \cdot \frac{\sin\beta_n}{\sin(\beta-\beta_n)} \cdot \frac{\sin(\gamma-\gamma_n)}{\sin\gamma_n} = 1,$$

and similarly

$$\frac{\sin(\sphericalangle CBY_n)}{\sin(\sphericalangle ABY_n)} \cdot \frac{\sin\gamma_n}{\sin(\gamma-\gamma_n)} \cdot \frac{\sin(\alpha-\alpha_n)}{\sin\alpha_n} = 1,$$

$$\frac{\sin(\sphericalangle ACZ_n)}{\sin(\sphericalangle BCZ_n)} \cdot \frac{\sin\alpha_n}{\sin(\alpha-\alpha_n)} \cdot \frac{\sin(\beta-\beta_n)}{\sin\beta_n} = 1.$$

Multiplying the above relations we obtain

$$\frac{\sin(\sphericalangle BAX_n)}{\sin(\sphericalangle CAX_n)} \cdot \frac{\sin(\sphericalangle CBY_n)}{\sin(\sphericalangle ABY_n)} \cdot \frac{\sin(\sphericalangle ACZ_n)}{\sin(\sphericalangle BCZ_n)} = 1$$

and the concurrency of the lines AX_n, BY_n, CZ_n follows by the converse of Ceva Theorem in trigonometric form. $\square$

The above result shows that the sequence of points $(I_n)_{n\geq 1}$, defined by

$$\begin{cases} I_1 = I \\ \{I_n\} = AX_n \cap BY_n \cap CZ_n,\ \forall\ n \in \mathbb{N}^* \end{cases}$$

is well defined, and consider

$$\{A_n\} = AI_n \cap BC,\ \{B_n\} = BI_n \cap CA,\ \{C_n\} = CI_n \cap AB,\ \forall\ n \in \mathbb{N}^*.$$

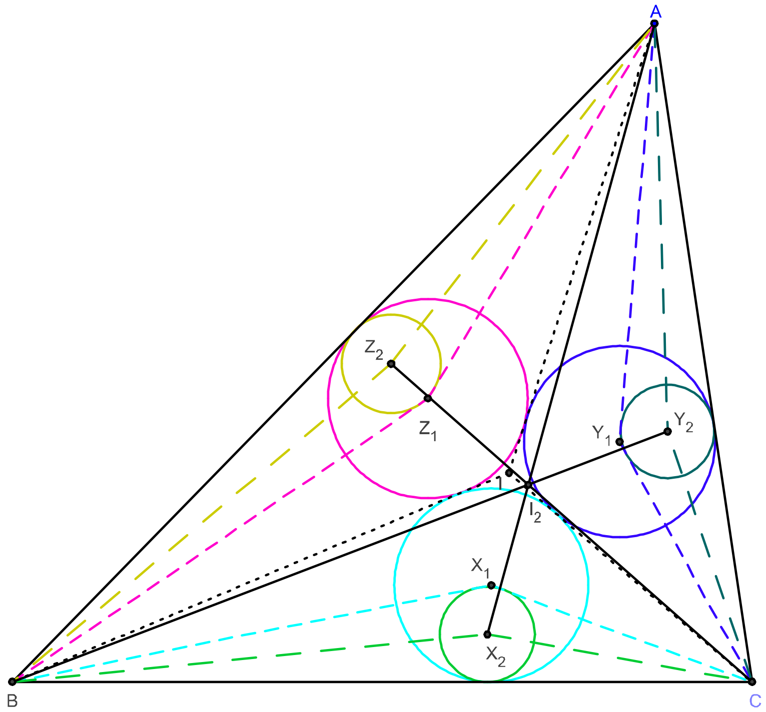

FIGURE 1

Concerning the sequences of points $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$, $(C_n)_{n\geq 1}$ the following result holds.

Proposition 2.2. *The sequences*

$$\left(\frac{BA_n}{CA_n}\right)_{n\geq 1}, \quad \left(\frac{CB_n}{AB_n}\right)_{n\geq 1}, \quad \left(\frac{AC_n}{BC_n}\right)_{n\geq 1},$$

are convergent and we have

$$\lim_{n\to\infty}\frac{BA_n}{CA_n}=\frac{\gamma}{\beta},\ \lim_{n\to\infty}\frac{CB_n}{AB_n}=\frac{\alpha}{\gamma},\ \lim_{n\to\infty}\frac{AC_n}{BC_n}=\frac{\beta}{\alpha}.$$

Proof. Using the areas or the Law of Sines and the first relation in the proof of Proposition 1, we have :

$$\frac{BA_n}{CA_n}=\frac{c\cdot\sin(\sphericalangle BAX_n)}{b\cdot\sin(\sphericalangle CAX_n)}=\frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin(\beta-\beta_n)}{\sin\beta_n}\cdot\frac{\sin\gamma_n}{\sin(\gamma-\gamma_n)}. \tag{1}$$

Because $\beta_n, \gamma_n \to 0$ and $\dfrac{\sin\gamma_n}{\gamma_n}, \dfrac{\sin\beta_n}{\beta_n} \to 1$, from (1) we get

$$\lim_{n\to\infty}\frac{BA_n}{CA_n}=\frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin\beta}{\sin\gamma}\cdot\frac{\gamma}{\beta}=\frac{\gamma}{\beta}$$

and the conclusion is obtained. □

In what follows, we recall that the sequence of points in the Euclidean plane $(P_n)_{n\geq 1}$ is convergent to the point P if $\lim\limits_{n\to\infty} P_nP = 0$, where P_nP is the length of the segment $[P_nP]$. For a point X let us denote by x it complex coordinate. Clearly we have the the sequence $(P_n)_{n\in\mathbb{N}^*}$ is convergent to the point P if and only if $\lim\limits_{n\to\infty} |p_n - p| = 0$.

The main result of this note is contained in the following theorem.

Theorem 2.1. *With the notations introduced above, we have :*

(i) The sequences $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$, $(C_n)_{n\geq 1}$ *are convergent to the points* A', B', C', *where*

$$\begin{cases} a' = \dfrac{\gamma}{\beta+\gamma}\cdot c + \dfrac{\beta}{\beta+\gamma}\cdot b, \\ b' = \dfrac{\alpha}{\gamma+\alpha}\cdot a + \dfrac{\gamma}{\gamma+\alpha}\cdot c \\ c' = \dfrac{\beta}{\alpha+\beta}\cdot b + \dfrac{\alpha}{\alpha+\beta}\cdot a. \end{cases}$$

(ii) The sequence $(I_n)_{n\geq 1}$ *is convergent to* I', *the intersection point of the lines* AA', BB', CC'.

Proof. Considering

$$\frac{BA_n}{CA_n} = \frac{t_n}{v_n},\ \frac{CB_n}{AB_n} = \frac{u_n}{t_n},\ \frac{AC_n}{BC_n} = \frac{v_n}{u_n},$$

we have

$$\frac{t_n}{v_n} \to \frac{\gamma}{\beta},\ \frac{u_n}{t_n} \to \frac{\alpha}{\gamma},\ \frac{v_n}{u_n} \to \frac{\beta}{\alpha},\ \text{for } n\to\infty.$$

But

$$\frac{BA_n}{a} = \frac{t_n}{t_n+v_n},$$

hence

$$a_n = \frac{t_n}{t_n+v_n}\cdot c + \frac{v_n}{t_n+v_n}\cdot b,$$

and it follows

$$\begin{aligned} |a_n - a'| &= \left|\left(\frac{t_n}{t_n+v_n} - \frac{\gamma}{\beta+\gamma}\right)c + \left(\frac{v_n}{t_n+v_n} - \frac{\gamma}{\beta+\gamma}\right)b\right| \\ &\leq |c|\cdot\left|\frac{t_n}{t_n+v_n} - \frac{\gamma}{\beta+\gamma}\right| + |b|\cdot\left|\frac{v_n}{t_n+v_n} - \frac{\beta}{\beta+\gamma}\right|. \end{aligned} \tag{1}$$

Taking in to account that $\dfrac{t_n}{v_n} \to \dfrac{\gamma}{\beta}$, we have

$$\lim_{n\to\infty}\frac{t_n}{t_n+v_n} = \lim_{n\to\infty}\frac{1}{1+\dfrac{v_n}{t_n}} = \frac{1}{1+\dfrac{\beta}{\gamma}} = \frac{\gamma}{\beta+\gamma}$$

and

$$\lim_{n\to\infty}\frac{v_n}{t_n+v_n} = \frac{\beta}{\beta+\gamma}.$$

These relations and (1) we obtain $|a_n - a'| \to 0$, therefore $A_n \to A'$.

We proceed in a similar way to prove the other two convergences.

(ii) From (i) and Proposition 2 it follows

$$\frac{BA'}{CA'} = \frac{\gamma}{\beta},\ \frac{CB'}{AB'} = \frac{\alpha}{\gamma},\ \frac{AC'}{BC'} = \frac{\beta}{\alpha},$$

hence the lines AA', BB', CC' are concurrent at the point I'. The barycentric coordinates of I' with respect to the triangle ABC are

$$\left(\frac{\alpha}{\alpha+\beta+\gamma}, \frac{\beta}{\alpha+\beta+\gamma}, \frac{\gamma}{\alpha+\beta+\gamma}\right).$$

With the notations above, the point I_n has the barycentric coordinates with respect to the triangle ABC given by

$$\left(\frac{u_n}{u_n+v_n+t_n}, \frac{v_n}{u_n+v_n+t_n}, \frac{t_n}{u_n+v_n+t_n}\right),$$

It follows

$$\begin{aligned}
|i_n - i'| &= \left|\frac{u_n}{u_n+v_n+t_n}\cdot a + \frac{v_n}{u_n+v_n+t_n}\cdot b + \frac{t_n}{u_n+v_n+t_n}\cdot c\right.\\
&\quad \left. -\frac{\alpha}{\alpha+\beta+\gamma}\cdot a - \frac{\beta}{\alpha+\beta+\gamma}\cdot b - \frac{\gamma}{\alpha+\beta+\gamma}\cdot c\right|\\
&= \left|\left(\frac{u_n}{u_n+v_n+t_n} - \frac{\alpha}{\alpha+\beta+\gamma}\right)a + \left(\frac{v_n}{u_n+v_n+t_n} - \frac{\beta}{\alpha+\beta+\gamma}\right)b\right.\\
&\quad + \left.\left(\frac{t_n}{u_n+v_n+t_n} - \frac{\gamma}{\alpha+\beta+\gamma}\right)c\right|\\
&\le |a|\cdot\left|\frac{u_n}{u_n+v_n+t_n} - \frac{\alpha}{\alpha+\beta+\gamma}\right| + |b|\cdot\left|\frac{v_n}{u_n+v_n+t_n} - \frac{\beta}{\alpha+\beta+\gamma}\right|\\
&\quad + |c|\cdot\left|\frac{t_n}{u_n+v_n+t_n} - \frac{\gamma}{\alpha+\beta+\gamma}\right| \qquad (2)
\end{aligned}$$

Because we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{u_n}{u_n+v_n+t_n} &= \lim_{n\to\infty}\frac{1}{1+\dfrac{v_n}{u_n}+\dfrac{t_n}{u_n}}\\
&= \frac{1}{1+\dfrac{\beta}{\alpha}+\dfrac{\gamma}{\alpha}} = \frac{\alpha}{\alpha+\beta+\gamma}
\end{aligned}$$

and the other two similar relations, using (2) we obtain

$$\lim_{n\to\infty}|i_n - i'| = 0,$$

therefore $I_n \to I'$, $n\to\infty$, and we are done. □

Remark 2.1. *1) All the presented results remain valid if we consider* $\alpha_n = r^n\alpha, \beta_n = r^n\beta, \gamma_n = r^n\gamma$, *where* $r \in (0,1)$.

2) Also, an approach involving the exterior angles of the triangle ABC gives an other concurrency property and the convergence of the related sequences of points.

3) Let $(X_n^i)_{n\ge1}$ *be the sequences of points situated in the interior of* $\triangle ABC$, *and let* $(X_n^e)_{n\ge1}$ *be the sequence of points situated in the exterior of* $\triangle ABC$. *For every point* $Y_n \in [X_n^i X_n^e]$, $n\in\mathbb{N}$, *the sequence* (Y_n) *is convergent to the limit of* $(X_n^i)_{n\ge1}$.

4) We have seen that the sequence $(I_n)_{n\geq 1}$ *is convergent to the point* I' *having the barycentric coordinates*

$$I'\left(\frac{\alpha}{\alpha+\beta+\gamma},\frac{\beta}{\alpha+\beta+\gamma},\frac{\gamma}{\alpha+\beta+\gamma}\right).$$

This is a triangle center called the Hofstadter zero-point and designated as X(360) in the C.Kimberling Encyclopedia of Triangle Centers [6]*. It was discovered by Douglas Hofstadter in 1992.*

References

[1] Andrica, D., Bagdasar, O., and Marinescu, D., *Dynamic geometry of Kasner triangles with a fixed weight*, International Journal of Geometry **11 (2) (2022)**, 101-110.

[2] Andrica, D., and Marinescu, D., *Dynamic Geometry Generated by the Circumcircle Midarc Triangle*, Chapter 5 in "Analysis, Geometry, Nonlinear Optimization and Applications", Rassias, Th. M. and Pardalos, P. M. (eds.), World Scientific Publishing Company Ltd, Singapore, 2023.

[3] Dragomir, L., and Braica, P., *Problema 4*, Concurs Viitori Olimpici, ediţia a XIII-a, clasa a IX-a.

[4] Preda, T., *O metodă de a demonstra concurenţa unor drepte*, GMB 5/2018 (articol metodic).

[5] Braica, P., *În legătură cu convergenţa unui şir de puncte de concurenţă în geometria triunghiului*, prezentare Conferinţa Naţională a SSMR, Bucureşti, 29.10.2022.

[6] Kimberling, C., *Hofstadter points*, Nieuw Archief voor Wiskunde, 12(1994), 109-114.

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