INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 12 (2023), No. 2, 89 - 116

N.A. COURT'S CENTROID LOCUS PROBLEM

YU CHEN AND R.J. FISHER

Abstract. Let $\mathcal{C}$ be a circle with center O and radius r. Let $K \neq O$ be a point lying inside $\mathcal{C}$. The paper studies the locus of triangles inscribed in $\mathcal{C}$ having K as symmedian point. This locus is referred to as the $(\mathcal{C}, K)$-locus. In his book *College Geometry* [2], N. A. Court includes the problem/conjecture that the locus of centroids of the triangles in the $(\mathcal{C}, K)$-locus is a circle. The paper proves this fact in three steps using the canonical vector space $\mathcal{V}$ of geometric vectors associated to the plane $\mathcal{E}$.

Step one proves the following: Given $A \in \mathcal{C}$, assume that the dot product $(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} \neq 0$. Let D be the point defined by the vector equation

$$\overrightarrow{AD} = \frac{3\overrightarrow{AO} \cdot \overrightarrow{AK}}{(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK}} \overrightarrow{AK}.$$

Let $\overline{BC}$ be the polar of D. Then $\triangle ABC$ is the unique triangle $(\mathcal{C}, K)$-locus that has A as a vertex. See Theorems 2.4 and 2.7. Next, Lemma 2.2 addresses the case $(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} = 0$ proving that the triangle in the $(\mathcal{C}, K)$-locus having A as a vertex is a right triangle where $\angle A = 90$ and K is the midpoint of the altitude at vertex A.

Step two is the following theorem: Let O_g be the point defined by the vector equation

$$\overrightarrow{OO_g} = \frac{2r^2}{4r^2 - k^2} \overrightarrow{OK}$$

where $k = OK$, the distance from O to K. Then the centroid G of a triangle in the $(\mathcal{C}, K)$-locus lies on the circle $\mathcal{C}_g$ with center O_g and radius $\frac{r^2 k}{4r^2 - k^2}$. See Theorem 3.1.

Step three solves the inverse problem in section §4. Specifically, given a point $G \in \mathcal{C}_g$, the inverse problem is a method for finding the triangle in the $(\mathcal{C}, K)$-locus having G as its centroid. The method depends on the roots of a cubic polynomial determined by G. Let $g = OG$. Then this polynomial has the form $X^3 + pX + q$ where

$$p = -\frac{3k^2}{r^2}, \qquad q = \frac{2[(4r^2 - 3k^2)g^2 - r^2 k^2]}{r^2(r^2 - g^2)}.$$

Keywords and phrases: Circumcircle, Centroid, Locus, Symmedian point, Harmonic division, Pole and Polar

(2020)Mathematics Subject Classification: 51M04

Received: 07.08.2022. In revised form: 12.01.2023. Accepted: 06.12.2022

The polynomial is shown to have three real roots, with multiplicity counted, that explicitly determine the triangle in the $(\mathcal{C}, K)$-locus having centroid G.

1. Introduction

In his celebrated textbook *College Geometry* [2], N. A. Court includes the following problem:

Exercise 1.1. A variable triangle has a fixed circumcircle and a fixed symmedian point. Show that the locus of the centroid is a circle.

See problem 6 on page 292 of [2]. The problem though simple to state is not so simple to solve. Court's book includes many problems of this sort that in Court's words *may appeal primarily to those who have an enduring interest, either professional or avocational, in the subject of modern geometry.* Based upon our proposed solution to Exercise 1.1, it is our opinion that Court regarded the exercise as a conjecture based upon his intuition.

To formulate Exercise 1.1 more precisely, fix a circle $\mathcal{C}$ with center O, radius r, and a point $K \neq O$ lying inside $\mathcal{C}$. We refer to the locus of triangles inscribed in $\mathcal{C}$ having K as symmedian point as the $(\mathcal{C}, K)$-locus. In an involved proof using the transversality of a harmonic pencil, Court proves that for each point $A \in \mathcal{C}$, there is a unique triangle $(\mathcal{C}, K)$-locus having the point A as a vertex. We call this result Court's Uniqueness Theorem. See Theorem 2.7 in section §2.3 and also item 623, page 266 of [2].

Given the variable nature of the $(\mathcal{C}, K)$-locus, we use the canonical vector space $\mathcal{V}$ of geometric vectors associated to the plane $\mathcal{E}$ to first describe the locus and then prove Exercise 1.1 in sections §3 and §4.

For organizational purposes, the vector calculations, which are used multiple times in the paper to study $(\mathcal{C}, K)$-locus, are outlined in section §2.

Section §2 begins with a review of some known facts about the symmedians of a triangle. Theorem 2.4 in section §2.2 relates the concepts of vector projection to harmonic division; see Chapter 7 of [2]. Together Theorems 2.4 and 2.6 provide an alternate proof of the Court Uniqueness Theorem; see Theorem 2.7 in §2.3.

In section §3 the forward direction of Exercise 1.1 is given. Specifically Theorem 3.1 shows that there is a circle $\mathcal{C}_g$, called the centroid circle, having radius $\frac{rk^2}{4r^2-k^2}$ where $k = OK$, with the property that the centroid of a triangle in the $(\mathcal{C}, K)$-locus lies on the circle $\mathcal{C}_g$.

In section §4, the backward direction of Exercise 1.1, referred to as the inverse construction, is proved. Specifically, given a point G from the centroid circle, the inverse construction is a method for finding a triangle in the $(\mathcal{C}, K)$-locus having G as its centroid. The method depends on the roots of a cubic polynomial determined by G. This polynomial (4.7) is shown to have three real roots with multiplicity counted; see Lemmas 4.1 and 4.3.

2. A Vector Method for Constructing a Triangle with a Fixed Symmedian Point

Canonically associated to the plane $\mathcal{E}$ is its vector space $\mathcal{V}$ of geometric vectors. By definition the elements of $\mathcal{V}$ are equivalence classes of directed line segments, which roughly are defined as follows: given four points $A, B, C, D \in \mathcal{E}$ no three of which are collinear, the directed line segments $[A, B] \sim [C, D]$ iff the quadrilateral $\square ABDC$ is a parallelogram. Special care has to be given to the case when A, B, C, and D are collinear. We ignore this. To establish notation, the vector $\overrightarrow{AB}$ is taken to mean the equivalence class represented by the directed line segment $[A, B]$.

It is well known that $\mathcal{V}$ has a canonical vector addition, a canonical scalar multiplication, a dot product, and has dimension two. Note that $\overrightarrow{AB} \cdot \overrightarrow{CD}$ will denote the dot product of the two vectors. Following [5], let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be vectors in $\mathcal{V}$. Then

$$\begin{aligned} BC^2 &= \overrightarrow{BC} \cdot \overrightarrow{BC} \\ &= (\overrightarrow{AC} - \overrightarrow{AB}) \cdot (\overrightarrow{AC} - \overrightarrow{AB}) \\ &= \overrightarrow{AC} \cdot \overrightarrow{AC} - 2\overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{AB} \cdot \overrightarrow{AB} \\ &= AC^2 - 2\overrightarrow{AB} \cdot \overrightarrow{AC} + AB^2. \end{aligned}$$

Hence

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \frac{AB^2 + AC^2 - BC^2}{2}. \tag{2.1}$$

This form of the dot product is a generalization of the Law of Cosines and will be used freely through out the paper. Also used throughout is the well known concept of vector projection. Specifically, the projection of the vector $\overrightarrow{AC}$ parallel to $\overrightarrow{AB}$ is

$$\mathrm{Proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{AB^2} \overrightarrow{AB}.$$

Finally, for us the "vector method" is the use of the vector space $\mathcal{V}$ to study the problems to follow. In particular, the use of a variable basis, i.e. moving frame, for $\mathcal{V}$ is both natural and efficient in discussing the problems of the paper.

2.1. Harmonic Division Characterization of the Symmedian Point.

This section presents a known description of the symmedian point of a triangle in terms of harmonic division (equivalently inversion in a circle). The description, uses the basic midpoint property of the symmedians of a triangle together with the midpoint characterization of a harmonic pencil; see Theorem 351 and 352, page 169 of [2].

Theorem 2.1. ***(Midpoint Property of a Harmonic Range)*** *Let A, S, K, and D be four collinear points situated as in the figure. Let B be a point not lying on the line $(ASKD)$. Let l_K be the line through K that is parallel to $\overline{BD}$. Let U and V be the points where l_K intersects l_{BA} and l_{BS}, resp. Then K is the midpoint of U and V iff K and D harmonically divide $\overline{AS}$ in the direction from A to S, that is, K and D are inverse in the circle $\mathcal{C}(AS)$.*

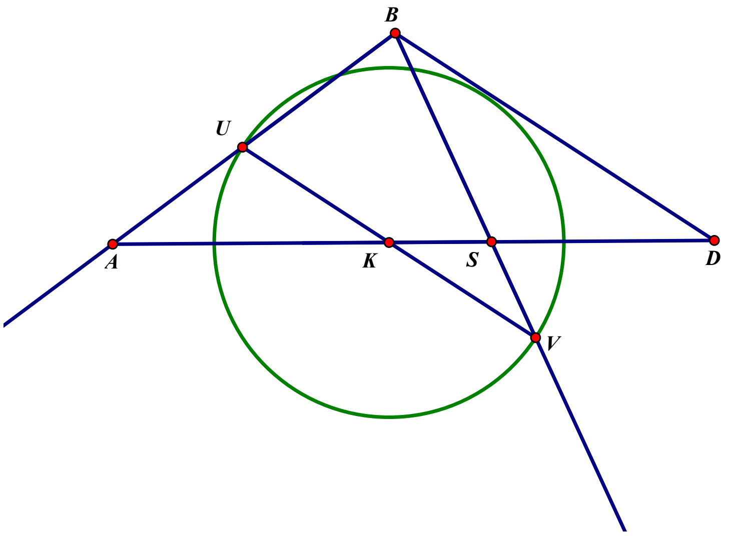

FIGURE 1

Proof. Since $\triangle AUK \sim \triangle ABD$ and $\triangle SKV \sim \triangle SDB$, we get

$$\frac{AK}{AD} = \frac{UK}{BD} \qquad \text{and} \qquad \frac{SK}{SD} = \frac{VK}{BD}.$$

Then

$$\begin{aligned}(ASKD) \text{ is a harmonic range} &\iff \frac{AK}{SK} = \frac{AD}{SD} \\ &\iff \frac{AK}{AD} = \frac{SK}{SD} \\ &\iff \frac{UK}{BD} = \frac{VK}{BD} \\ &\iff UK = VK \\ &\iff K \text{ is the midpoint of } \overline{UV}.\end{aligned}$$

Hence K and D harmonically divide $\overline{AS}$ in the A-direction. □

Remark 2.1. Theorem 2.1 can be stated in a more general way. For example, take the line l through A that is parallel to l_{BD}. Let U and V be the two points where l meets the lines l_{BK} and l_{BS}. Then the same argument used in the proof of Theorem 2.1 shows that $(ASKD)$ is a harmonic range iff U is the midpoint of A and V. The narrower statement of Theorem 2.1 is for convenience.

Remark 2.2. (**Basic Midpoint Property of the Symmedians of a Triangle**) To keep the discussion self-contained, the symmetric median (i.e. symmedian) of a triangle $\triangle ABC$ at a given vertex, say A, is the reflection of the median at A across the angle bisector at A. The three symmedians intersect. The common point K is called the symmedian point of $\triangle ABC$.

A chord of a triangle $\triangle ABC$ at a vertex, say B, is a line segment obtained by intersecting a line not passing through B with the lines l_{BA} and l_{BC}. In terms of chords, the median m_B at vertex B is characterized as the point B along with those points P such that P is the midpoint of the unique chord through P that is parallel to l_{AC}. Next, let l_B be the unique line through B that is parallel to l_{AC}. The reflection of l_B across the angle bisector at vertex B is the tangent line l_{BB} to the circumcircle $\mathcal{C}$ of $\triangle ABC$ at B. On the other hand, the symmedian at B is the reflection sm_B of the median m_B across the angle bisector of $\angle B$. Hence, since a reflection preserves midpoints and parallel lines, the symmedian sm_B consists of the point B and all points P' where P' is the midpoint of the unique chord that is parallel to the tangent line l_{BB}. This "basic midpoint" characterization of sm_B along with several other characterizations are nicely discussed in [3].

The following well known fact is an immediate consequence of the basic midpoint characterization. There are other proofs of this lemma. See page 58 of [6] and Theorem 348 on page 216 of [7].

Lemma 2.1. *Let $\triangle ABC$ be a right triangle with $\angle A = 90$. Let A_h be the foot of the altitude at vertex A. Then the symmedian point of $\triangle ABC$ is the midpoint of A and A_h.*

Proof. Since $\overline{BC}$ is a diameter of the circumcircle $\mathcal{C}$ of $\triangle ABC$, the altitude $\overline{AA_h}$ is a chord at vertices B and C that is parallel to the tangent lines l_{BB} and l_{CC}. Hence the midpoint of $\overline{AA_h}$ lies on the symmedians at vertices B and C by the basic midpoint characterization of a symmedian. $\square$

Theorem 2.1 together with the basic-midpoint characterization of a symmedian leads to an interesting characterization of the symmedian point of a triangle. We use this theorem in an essential way in the proof Theorem 2.4 just ahead.

Theorem 2.2. *Let $\triangle ABC$ be a triangle with circumcircle $\mathcal{C}$ and symmedian point K. Suppose that $\angle A \neq 90$. Let D be the point where the tangent lines to $\mathcal{C}$ at B and C intersect. Let S be the point where the line l_{AD} (i.e. the symmedian at vertex A) intersects $\overline{BC}$. Then K is the inverse of D in the circle $\mathcal{C}(AS)$.*

Proof. Suppose that L is the inverse of D in the circle $\mathcal{C}(AS)$, that is, $(ASLD)$ is a harmonic range. As in Theorem 2.1, let l_L be the line through L that is parallel to $\overline{BD}$. Let U and V be the two points where l_L intersects l_{BA} and $l_{BS} = l_{BC}$. Then L is the midpoint of the chord $\overline{UV}$ of $\triangle ABC$ at vertex B. So, since l_{BD} is the tangent line to $\mathcal{C}$ at B, the chord $\overline{UV}$ is parallel to the tangent line at vertex B by construction. Hence the point L lies on the symmedian at vertex B by the basic midpoint characterization of a symmedian. On the other hand, L also lies on the symmedian at vertex A by construction. Therefore, $L = K$. $\square$

Theorem 2.2 and Theorem 2.1(in its general form; see Remark 2.1) lead to a second midpoint characterization of the symmedian point in terms of the altitudes of a triangle. See Theorem 586, page 256 of [2] and Theorem 2.3 just below. This theorem is used in an essential way in the proof of Theorem 2.6 in §2.3 . A different proof of Theorem 2.3 than the one given below can be found on page 66 of [6].

Theorem 2.3. *Given a triangle* $\triangle ABC$ *, let* A_h *be the foot of the altitude at* A *; let* M_h *be the midpoint of* A *and* A_h *. Next, let* M *be the midpoint of* $\overline{BC}$ *. Assume* $\angle A \neq 90$ *. Then the intersection of the symmedian at vertex* A *and the segment* $\overline{MM_h}$ *is the symmedian point* K *of the triangle.*

Proof. In the notation of Theorem 2.2, consider the harmonic range $(ASKD)$. Consider the line $l_{_{MD}}$. The line through A that is parallel to $l_{_{MD}}$ is the altitude at vertex A. Next by Theorem 2.1, the midpoint M_h of A and A_h is the intersection of $l_{_{MK}}$ and the altitude at A. Hence K is the point where the segment $\overline{MM_h}$ intersects the symmedian point K . □

2.2. **Harmonic Division and the Vector Method.**

Theorem 2.4. *Let* $\mathcal{C}$ *be a circle with center* O *and radius* r *. Fix a point* $K \neq O$ *lying inside* $\mathcal{C}$ *. Let* O' *denote the rotation of* O *about* K *by* 180 *. Let* $\mathcal{C}(OK)$ [1] *and* $\mathcal{C}(O'K)$ *be the circles with diameters* $\overline{OK}$ *and* $\overline{O'K}$ *, resp.*

For each $A \in \mathcal{C}, A \notin \mathcal{C}(O'K)$ *, let* $R \in \mathcal{C}(OK)$ *and* $Q \in \mathcal{C}(O'K)$ *be the second points where the line* $l_{_{AK}}$ *intersects the circles* $\mathcal{C}(OK)$ *and* $\mathcal{C}(O'K)$ *, resp.*

Let $\overline{AA'}$ *be the unique chord of* $\mathcal{C}$ *through* K *having* A *as an endpoint. Let* R' *be the rotation of* R *about* A' *by* 180 *.*

Choose any line l *through* A *. Choose any line* m *through* Q *that is not parallel to* l *. Let* m' *be the parallel to* m *through* R' *. Let* Q' *and* R'' *be the points where the lines* m *and* m' *meet* l *, resp. Let* n *be the parallel to* $l_{_{KQ'}}$ *through* R'' *. Let* D *be the point where* n *meets* $l_{_{AK}}$ *. Then* A *is either the external or internal point that harmonically divides* $\overline{KD}$ *in the* K*-direction in the "signed" ratio*

$$\delta = \frac{3\overrightarrow{AO}\cdot\overrightarrow{AK}}{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot\overrightarrow{AK}}. \tag{2.2}$$

Geometrically, up to sign,

$$\delta = \frac{DA}{AK} = 3\frac{AR}{AQ}. \tag{2.3}$$

Also the point D *lies outside the circle* $\mathcal{C}$ *and is given by the vector equation*

$$\overrightarrow{AD} = \frac{3\overrightarrow{AO}\cdot\overrightarrow{AK}}{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot\overrightarrow{AK}}\overrightarrow{AK}. \tag{2.4}$$

Let $\overline{BC}$ *be the polar of* D *, i.e. the chord through the inverse* M *of* D *that is perpendicular to the line* $l_{_{OD}}$ *. Then the harmonic conjugate of* A *in* $\overline{KD}$ *is the point* S *where the line* $l_{_{AD}}$ *intersects* $\overline{BC}$ *. Consequently, the point* K *is the symmedian point of* $\triangle ABC$ *.*

[1]The circle $\mathcal{C}(OK)$ is commonly referred to as the First LeMoine circle. See [2], page 258.

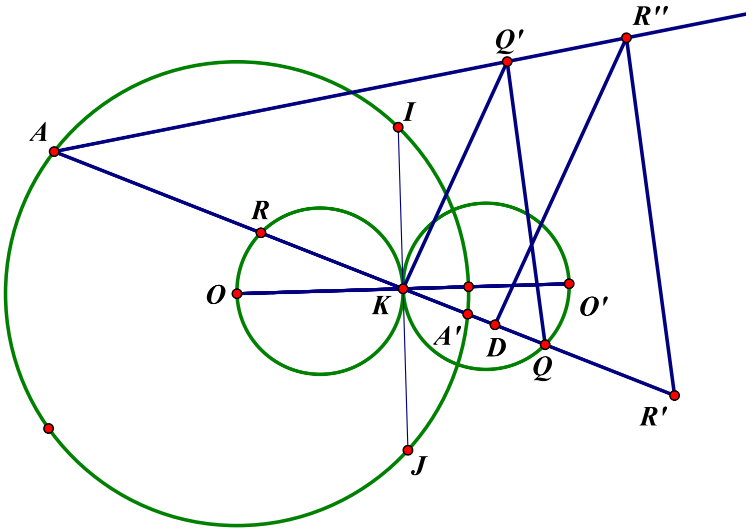

FIGURE 2

Proof. Suppose first that $AK > A'K$, i.e. (2.2) is positive. When $AK < A'K$, (2.2) < 0, the proof is analogous. As the figures illustrates, we have the triangle similarities $\triangle AQQ' \sim \triangle AR'R''$ and $\triangle AKQ' \sim \triangle ADR''$. Hence $\frac{AR'}{AQ} = \frac{AR''}{AQ'} = \frac{AD}{AK}$. Let δ denote the common ratio. Since $\overline{OR} \perp \overline{AA'}$, R is the midpoint of $\overline{AA'}$ so that $AR' = 3AR$. Hence, $\delta = 3\frac{AR}{AQ}$.

Regardless of whether $AK > A'K$ or $AK < A'K$, the point Q lies between A and R'. Indeed, in terms of signed distances $|KR| < AR$ so that

$$AQ = AR + 2KR < 3AR = AR'.$$

Hence $\delta > 1$. Consequently, A is on the side of K opposite to D. Hence, A is the external point that harmonically divides $\overline{KD}$ in the K-direction in the ratio δ.

To argue that D lies outside the circle $\mathcal{C}$, it suffices to show that $AD > AA'$. First note that K is the midpoint of Q and R. Next, assuming $AK > A'K$,

$$\begin{aligned} AD - AA' &= \delta AK - 2AR \\ &= \frac{3(AR)(AK) - 2(AR)(AQ)}{AQ} \\ &= \frac{3(AR)(AK) - 2(AR)(AK + KR)}{AQ} \\ &= \frac{(AR)(AK - 2KR)}{AQ} \\ &= \frac{(AR)(AR - KR)}{AQ} \\ &> 0. \end{aligned}$$

If $AK < A'K$, a similar argument shows that D lies outside $\mathcal{C}$.

To prove (2.4), observe that $\overrightarrow{OO'} = 2\overrightarrow{OK}$,

$$\begin{aligned} \overrightarrow{AR} &= \mathrm{Proj}_{\overrightarrow{AK}}(\overrightarrow{AO}) = \tfrac{\overrightarrow{AO}\cdot\overrightarrow{AK}}{AK^2}\overrightarrow{AK}, \\ \overrightarrow{AQ} &= \mathrm{Proj}_{\overrightarrow{AK}}(\overrightarrow{AO'}) = \tfrac{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot\overrightarrow{AK}}{AK^2}\overrightarrow{AK}. \end{aligned} \tag{2.5}$$

Then (2.4) follows immediately.

To prove $S \in \overline{BC}$, first observe that $\overrightarrow{OM} = \frac{r^2}{OD^2}\overrightarrow{OD}$ since M and D are inverse in the circle $\mathcal{C}$. Then

$$\begin{aligned} S \in l_{BC} \qquad &\Longleftrightarrow \qquad \mathrm{Proj}_{\overrightarrow{OD}}(\overrightarrow{OS}) = \overrightarrow{OM} \\ &\Longleftrightarrow \qquad \overrightarrow{OS}\cdot\overrightarrow{OD} = r^2. \end{aligned}$$

Next, we express the vectors $\overrightarrow{OD}$ and $\overrightarrow{OS}$ in terms of $\overrightarrow{OK}$ and $\overrightarrow{OA}$. In fact, we will show that for all $A \notin \mathcal{C}(KO')$,

$$\begin{aligned} \overrightarrow{OD} &= \delta\overrightarrow{OK} + (1-\delta)\overrightarrow{OA} \\ \overrightarrow{OS} &= \tfrac{2\delta}{\delta+1}\overrightarrow{OK} + (1-\delta)\overrightarrow{OA}. \end{aligned} \tag{2.6}$$

The equation $\overrightarrow{AD} = \delta\overrightarrow{AK}$ implies

$$\overrightarrow{OD} = \delta\overrightarrow{OK} + (1-\delta)\overrightarrow{OA}.$$

Let $k = OK$ and $\mu = \overrightarrow{OA}\cdot\overrightarrow{OK}$. From the formulas (2.5) for $\overrightarrow{AR}$ and $\overrightarrow{AQ}$ just above,

$$\delta = \frac{3(r^2-\mu)}{2k^2-3\mu+r^2}, \qquad 1-\delta = \frac{2(k^2-r^2)}{2k^2-3\mu+r^2}.$$

Since A and S harmonically divide $\overline{KD}$ in the ratio δ, K and D harmonically divide A and S in the ratio

$$\epsilon = \frac{\delta+1}{\delta-1} = \frac{k^2-3\mu+2r^2}{r^2-k^2}.$$

In particular, $\overrightarrow{AK} = \epsilon\overrightarrow{KS}$ so that

$$\begin{aligned} \overrightarrow{OS} &= \tfrac{1+\epsilon}{\epsilon}\overrightarrow{OK} - \tfrac{1}{\epsilon}\overrightarrow{OA} \\ &= \tfrac{2\delta}{\delta+1}\overrightarrow{OK} + (1-\delta)\overrightarrow{OA}. \end{aligned}$$

To show that dot product $\overrightarrow{OS}\cdot\overrightarrow{OD} = r^2$, observe as follows:

$$\begin{aligned} \overrightarrow{OS}\cdot\overrightarrow{OD} &= \tfrac{1}{\delta+1}\Big(2\delta\overrightarrow{OK} + (1-\delta)\overrightarrow{OA}\Big)\cdot\Big(\delta\overrightarrow{OK} + (1-\delta)\overrightarrow{OA}\Big) \\ &= \tfrac{1}{\delta+1}\Big[2\delta^2k^2 + 3\delta(1-\delta)\mu + (\delta-1)^2r^2\Big]. \end{aligned}$$

Next,

$$\begin{aligned} 2\delta^2k^2 &+ 3\delta(1-\delta)\mu + (\delta-1)^2r^2 \\ &= \delta^2(2k^2-3\mu+r^2) + \delta(3\mu-2r^2) + r^2 \\ &= \frac{9(r^2-\mu)^2 + 3(r^2-\mu)(3\mu-2r^2) + r^2(2k^2-3\mu+r^2)}{2k^2-3\mu+r^2}. \end{aligned}$$

Hence

$$\begin{aligned} (2k^2-3\mu+r^2)(\delta+1)&\overrightarrow{OS}\cdot\overrightarrow{OD} \\ &= 9(r^2-\mu)^2 + 3(r^2-\mu)(3\mu-2r^2) \\ &\qquad + r^2(2k^2-3\mu+r^2) \\ &= 3(r^2-\mu)r^2 + r^2(2k^2-3\mu+r^2) \\ &= r^2(2k^2-6\mu+4r^2). \end{aligned}$$

On the other hand,

$$\delta+1 = \frac{2k^2-6\mu+4r^2}{2k^2-3\mu+r^2}$$

so that

$$(2k^2-3\mu+r^2)(\delta+1) = 2k^2-6\mu+4r^2.$$

Hence $\overrightarrow{OS}\cdot\overrightarrow{OD} = r^2$.

By construction the point K lies on the symmedian of $\triangle ABC$ at vertex A. So to argue that K is the symmedian point of $\triangle ABC$, we need to argue that K also lies on either the symmedian at vertex B or C. However, this is an immediate consequence of Theorem 2.2 since $B(ASKD)$ is a harmonic pencil. □

Remark 2.3. Let $\overline{IJ}$ be the unique chord of $\mathcal{C}$ where K is the midpoint. When $A = I$ or J, Theorem 2.4 holds. Indeed, suppose $A = I$. Then $K = Q = R$. The point D is defined by the vector equation $\overrightarrow{ID} = 3\overrightarrow{IK}$. In turn, the harmonic cojungate of I in $\overline{KD}$ is the midpoint S of K and J. Next, S and D are inverse in the circle $\mathcal{C}(IJ)$ so that the point S lies on the polar $\overline{BC}$ of D in the circle $\mathcal{C}$. Hence the hypothesis of Theorem 2.2 holds so that K is the symmedian point of $\triangle IBC$. See the figure.

Remark 2.4.

(1) For each $A \in \mathcal{C}$, the angle $\angle OAK$ is acute so that the dot product $\overrightarrow{AO}\cdot\overrightarrow{AK} > 0$.

(2) Let $k = OK$. If $r/2 < k$, then the point O' lies outside $\mathcal{C}$. Also the two circles $\mathcal{C}$ and $\mathcal{C}(O'K)$ intersect at two points, say A_1 and A_2. Label the intersection points so that the arc $\overset{\frown}{A_1A_2} \subset \mathcal{C}$ lies inside $\mathcal{C}(O'K)$. Then $\angle KAO'$ is obtuse for all $A \in \overset{\frown}{A_1A_2}$, $A \neq A_1, A_2$. Hence the dot product $(2\overrightarrow{OK} - \overrightarrow{OA})\cdot\overrightarrow{AK} < 0$.

(3) When $A = A_1, A_2$, $\angle KAO' = 90$ so that $(2\overrightarrow{OK} - \overrightarrow{OA})\cdot\overrightarrow{AK} = 0$. The triangle $\triangle ABC$ can be constructed in this case though δ is not given by (2.2). See Lemma 2.2 just ahead.

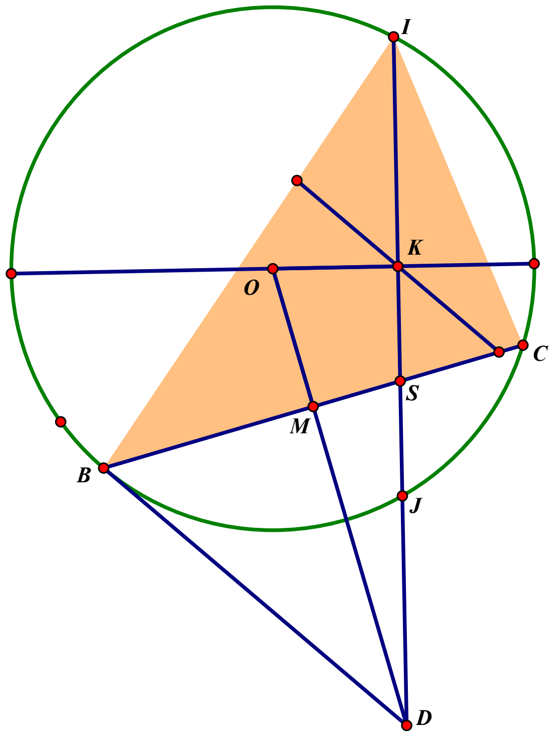

FIGURE 3

Lemma 2.2. *Let $\mathcal{C}$ be a circle with center O and radius r. Let K be a point inside $\mathcal{C}$. Let O' be the rotation of O about K by 180; let $\mathcal{C}(O'K)$ denote the circle with diameter $\overline{O'K}$. Let $k = OK$. Assume $k > r/2$. Then $\mathcal{C}(O'K)$ and $\mathcal{C}$ intersect at two points.*

Let $\triangle ABC$ be a triangle inscribed in $\mathcal{C}$ with symmedian point K. Then $\triangle ABC$ is a right triangle iff one of the vertices of $\triangle ABC$ lies in the intersection $\mathcal{C}(O'K) \cap \mathcal{C}$.

Proof. Since $2k < 2r$, the two circles $\mathcal{C}(O'K)$ and $\mathcal{C}$ intersect at two points.

Up to relabelling, we can assume that $\angle A = 90$.

Let ρ denote rotation by 180 about K. Then $O' = \rho(O)$. Consequently, for each $A \in \mathcal{C}$, the line $\rho(l_{AO'})$ is an extended diameter of $\mathcal{C}$. Let $\overline{BC}$ be the diameter so determined. Then $\triangle ABC$ is a right triangle inscribed in $\mathcal{C}$ with $\angle A = 90$. Next, let $A_h = \rho(A)$. Since ρ preserves angles,

$$\angle KA_hO = \angle KAO'.$$

Suppose now that $A \in \mathcal{C}(KO')$. Then $90 = \angle KAO' = \angle KA_hO$ so that since $l_{BC} = l_{OA_h}$, $\overline{AA_h}$ is the altitude at vertex A of $\triangle ABC$ with K as its midpoint. By Lemma 2.1, K is the symmedian point of $\triangle ABC$.

Conversely, if the right triangle $\triangle ABC$ has K as its symmedian point, then

$$90 = \angle KA_hO = \angle KAO'$$

so that $A \in \mathcal{C}(KO')$. □

Remark 2.5. When $k = r/2$, Lemma 2.2 is also true. In this case, $\mathcal{C}(O'K) \cap \mathcal{C} = \{O'\}$. If $A = O'$, the line $l_{AO'}$ is replaced by the tangent line l_{AA} to $\mathcal{C}$ at O'. Then $\rho(l_{AA})$ is the extended diameter l_{BC} parallel to l_{AA}. The altitude at vertex A is $\overline{AO}$ with midpoint K.

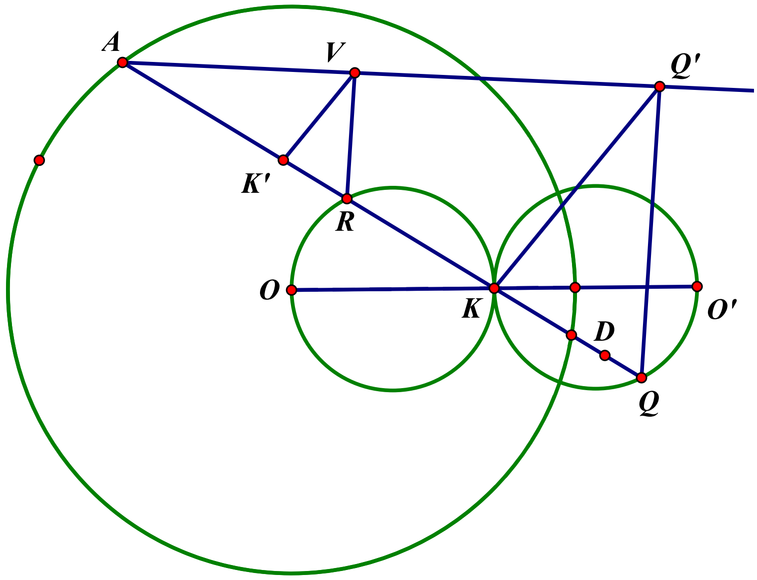

FIGURE 4

Corollary 2.1. *In the notation of Theorem 2.4, the point D can also be constructed as follows:*

- *Choose any line l through A and any line m through Q that is not parallel to l. Let Q' be the point where l and m meet.*
- *Let the point V be defined by $l_{RV} \parallel m$ with $V \in l$.*
- *Let K' be defined by $l_{KQ'} \parallel l_{K'V}$ with $K' \in l_{AK}$.*

Then D is the dilation of K' through A by a factor of 3.

Proof. As illustrated in the figure, $\triangle AQQ' \sim \triangle ARV$ and $\triangle AKQ' \sim \triangle AK'V$. Hence

$$\frac{AK'}{AK} = \frac{AV}{AQ'} = \frac{AR}{AQ}.$$

Moreoever, the point K' lies on the ray $\overrightarrow{AK}$. Hence D is the dilation of K' through A by a factor of 3 by Theorem 2.4. □

Remark 2.6. Let $\overline{A_1A_2}$ be the diameter of $\mathcal{C}$ that contains the point K. Then it can be argued routinely that the triangles $\triangle A_iB_iC_i$ in the $(\mathcal{C},K)$-locus are isosceles. Indeed, the triangles are symmetric across the line l_{OK}. Hence the two centroids G_i also lie on l_{OK}. Assuming that $A_1K > A_2K$, the two centroids lie between O and K such that if $k = OK$, then

$$OG_1 = \frac{rk}{2r+k} < OG_2 = \frac{rk}{2r-k}.$$

2.3. **The Symmedian Point in Vector Form/Court Uniqueness Theorem.** In this section, we prove some known results about the symmedian point of a triangle using the vector method.

Let $\mathcal{C}$ be a circle and K a point lying inside $\mathcal{C}$. In [2], Court proves that for each point $A \in \mathcal{C}$, there is a unique triangle $\triangle ABC$ inscribed in $\mathcal{C}$ having K as its symmedian point. See 623, page 266 of [2]. As indicated in the introduction, we refer to this locus of triangles as the $(\mathcal{C},K)$-locus. Continuing, Court's proof of the existence and uniqueness uses the duality of poles and polars, the transversality property of a harmonic pencil, and the Apollonian circles of a triangle. The three concepts are discussed in detail in [2]. Using Theorem 2.4 and Theorem 2.6, the Court Uniqueness Theorem is an immediate consequence; see Theorem 2.7.

The following theorem gives a natural vector description of the symmedian point.

Theorem 2.5. *Let O be the circumcenter of a triangle $\triangle ABC$. Let $a = BC, b = CA$, and $c = AB$. The symmedian point K of $\triangle ABC$ is given by the vector equation*

$$\overrightarrow{OK} = \frac{a^2}{a^2+b^2+c^2}\overrightarrow{OA} + \frac{b^2}{a^2+b^2+c^2}\overrightarrow{OB} + \frac{c^2}{a^2+b^2+c^2}\overrightarrow{OC}. \tag{2.7}$$

Proof. Theorem 2.5 can be proved in the general context of isogonal points of a triangle. Our proof is for K only. Additionally, Theorem 2.5 is similar in spirit to the trilinear coordinates of the symmedian point of a triangle; see [8]. In particular, if $\Delta = \text{Area}(\triangle ABC)$, then the trilinear coordinates of K are

$$k_a = \frac{2a\Delta}{a^2+b^2+c^2}, \qquad k_b = \frac{2b\Delta}{a^2+b^2+c^2}, \qquad k_c = \frac{2c\Delta}{a^2+b^2+c^2}.$$

Let S_a be the point of intersection of l_{AK} and l_{BC}. Let K_a be the foot of the perpendicular through K to the line l_{BC}. Let H_a be the foot of the altitude at vertex A. Since $\frac{CS_a}{BS_a} = \frac{b^2}{c^2}$ (see [2], Theorem 561, page 248),

$$\begin{aligned}\overrightarrow{BC} &= \left(1 + \frac{CS_a}{BS_a}\right)\overrightarrow{BS_a}\\ &= \frac{b^2+c^2}{c^2}\overrightarrow{BS_a}.\end{aligned}$$

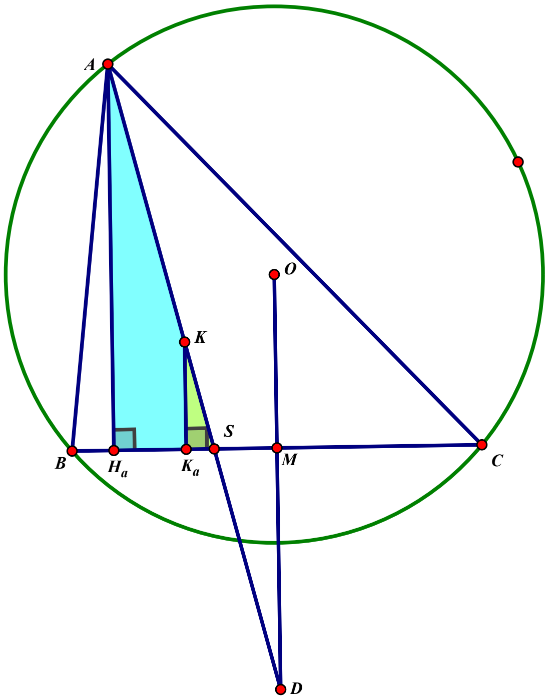

FIGURE 5

Hence,

$$\begin{aligned}\overrightarrow{AS_a} &= \overrightarrow{AB} + \overrightarrow{BS_a} \\ &= \overrightarrow{AB} + \frac{c^2}{b^2+c^2}\overrightarrow{BC} \\ &= \frac{b^2}{b^2+c^2}\overrightarrow{AB} + \frac{c^2}{b^2+c^2}\overrightarrow{AC}.\end{aligned}$$

Next,

$$KK_a = \frac{2a\Delta}{a^2+b^2+c^2};$$

see Corollary 343, page 214 of [7]. Of course, $AH_a = 2\Delta/a$. Hence, since $\triangle KK_aS_a \sim \triangle AH_aS_a$,

$$\frac{KS_a}{AS_a} = \frac{KK_a}{AH_a} = \frac{a^2}{a^2+b^2+c^2}.$$

In turn, $AK = AS_a - KS_a$ so that

$$\frac{AK}{AS_a} = \frac{b^2+c^2}{a^2+b^2+c^2}.$$

Finally,

$$\begin{aligned}\overrightarrow{AK} &= \tfrac{b^2+c^2}{a^2+b^2+c^2}\overrightarrow{AS_a} \\ &= \tfrac{b^2}{a^2+b^2+c^2}\overrightarrow{AB} + \tfrac{c^2}{a^2+b^2+c^2}\overrightarrow{AC} \\ &= \tfrac{b^2}{a^2+b^2+c^2}(\overrightarrow{OB} - \overrightarrow{OA}) + \tfrac{c^2}{a^2+b^2+c^2}(\overrightarrow{OC} - \overrightarrow{OA}).\end{aligned}$$

Using $\overrightarrow{AK} = \overrightarrow{OK} - \overrightarrow{OA}$, the above equation is rewritten in the intended form (2.7). □

Theorem 2.6. *Let $\triangle ABC$ be a triangle inscribed in a circle $\mathcal{C}$ of radius r centered at O. Let K be the symmedian point of $\triangle ABC$.*

(1) *If $\angle A = 90$, then $(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} = 0$.*
(2) *If $\angle A \neq 90$, let M be the midpoint of $\overline{BC}$ and D be the inverse point of M with respect to the circle $\mathcal{C}$. Then*

$$\overrightarrow{AD} = \frac{3(\overrightarrow{AO} \cdot \overrightarrow{AK})}{(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK}}\overrightarrow{AK};$$

in particular, we have $(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} \neq 0$.

Proof. For notational convenience in the proof to follow, let $a = BC$, $b = CA$, and $c = AB$. Next, let $k = OK$ and $d = OD$. Finally, since the line l_{AD} is the symmedian of $\triangle ABC$ at vertex A, see Theorem 560, page 248 of [2], $\overrightarrow{AK} = \delta\overrightarrow{AD}$ for some $\delta \in \mathbb{R}$.

First suppose $\triangle ABC$ is a right triangle with $\angle A = 90$. Let $\overline{AA_h}$ be the altitude of $\triangle ABC$ at the vertex A, where A_h lies on l_{BC}. As observed earlier, K is the midpoint of $\overline{AA_h}$; see Lemma 2.1. On the other hand, since $\overline{BC}$ is a diameter of the circle $\mathcal{C}$ and K lies on the altitude at A, we have $\overrightarrow{OB} \cdot \overrightarrow{AK} = 0$ and $\overrightarrow{OC} \cdot \overrightarrow{AK} = 0$. Next, $a^2 = b^2 + c^2$ so that by Theorem 2.5,

$$\begin{aligned}\overrightarrow{OK} &= \frac{a^2}{a^2+b^2+c^2}\overrightarrow{OA} + \frac{b^2}{a^2+b^2+c^2}\overrightarrow{OB} + \frac{c^2}{a^2+b^2+c^2}\overrightarrow{OC} \\ &= \frac{1}{2}\overrightarrow{OA} + \frac{b^2}{2a^2}\overrightarrow{OB} + \frac{c^2}{2a^2}\overrightarrow{OC}.\end{aligned}$$

Hence,

$$2\overrightarrow{OK} - \overrightarrow{OA} = \frac{b^2}{a^2}\overrightarrow{OB} + \frac{c^2}{a^2}\overrightarrow{OC}$$

so that $(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} = 0$.

The proof of statement (2) is broken into two cases.

Case 1: Assume the points A, O, and K are noncollinear. Let $\mu = \overrightarrow{OA} \cdot \overrightarrow{OK}$. Let $\overline{AA_h}$ be the altitude of $\triangle ABC$ at the vertex A, where A_h lies on the line l_{BC}.

Let M_h be the midpoint of $\overline{AA_h}$. The point K is the intersection point of the lines l_{MM_h} and l_{AD}. See Theorem 2.3. Next,

$$\overrightarrow{OD} = \overrightarrow{OA} + \delta\overrightarrow{AK} = \overrightarrow{OA} + \delta(\overrightarrow{OK} - \overrightarrow{OA}) = (1-\delta)\overrightarrow{OA} + \delta\overrightarrow{OK}$$

$$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA} = \frac{r^2}{d^2}\overrightarrow{OD} - \overrightarrow{OA}$$

so that

$$\begin{aligned}\overrightarrow{AM} \cdot \overrightarrow{OD} &= (\overrightarrow{OM} - \overrightarrow{OA}) \cdot \overrightarrow{OD} \\ &= \overrightarrow{OM} \cdot \overrightarrow{OD} - \overrightarrow{OA} \cdot [(1-\delta)\overrightarrow{OA} + \delta\overrightarrow{OK}] \\ &= r^2 - [(1-\delta)r^2 + \delta\mu] \\ &= \delta(r^2 - \mu).\end{aligned}$$

Additionally,

$$\overrightarrow{AM_h} = \tfrac{1}{2}\mathrm{Proj}_{\overrightarrow{OD}}(\overrightarrow{AM}) = \frac{\overrightarrow{AM} \cdot \overrightarrow{OD}}{2d^2}\overrightarrow{OD} = \frac{\delta(r^2-\mu)}{2d^2}\overrightarrow{OD}.$$

Since K is the intersection point of l_{MM_h} and l_{AD}, there exists $t \in \mathbb{R}$ such that

$$\begin{aligned}\overrightarrow{AK} &= (1-t)\overrightarrow{AM} + t\overrightarrow{AM_h} \\ &= (1-t)\left(\frac{r^2}{d^2}\overrightarrow{OD} - \overrightarrow{OA}\right) + \frac{t\delta(r^2-\mu)}{2d^2}\overrightarrow{OD} \\ &= \frac{2r^2 + t[\delta(r^2-\mu) - 2r^2]}{2d^2}\overrightarrow{OD} - (1-t)\overrightarrow{OA}.\end{aligned}$$

Hence,

$$\begin{aligned}\overrightarrow{OD} - \overrightarrow{OA} &= \overrightarrow{AD} \\ &= \delta\overrightarrow{AK} \\ &= \tfrac{2\delta r^2 + t[\delta^2(r^2-\mu) - 2\delta r^2]}{2d^2}\overrightarrow{OD} - (1-t)\delta\overrightarrow{OA}.\end{aligned}$$

Since $\overrightarrow{OD}$ and $\overrightarrow{OA}$ are linearly independent vectors, we have

$$\begin{aligned}2\delta r^2 + t[\delta^2(r^2-\mu) - 2\delta r^2] &= 2d^2, \\ (1-t)\delta &= 1.\end{aligned}$$

Then $t = (\delta-1)/\delta$ and hence

$$\begin{aligned}2d^2 &= 2\delta r^2 + \frac{\delta-1}{\delta}[\delta^2(r^2-\mu) - 2\delta r^2] \\ &= \delta(\delta-1)(r^2-\mu) + 2r^2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}d^2 &= [(1-\delta)\overrightarrow{OA} + \delta\overrightarrow{OK}] \cdot [(1-\delta)\overrightarrow{OA} + \delta\overrightarrow{OK}] \\ &= (1-\delta)^2 r^2 + 2\delta(1-\delta)\mu + \delta^2 k^2 \\ &= \delta^2(r^2 - 2\mu + k^2) - 2\delta(r^2-\mu) + r^2.\end{aligned}$$

Then

$$2[\delta^2(r^2-2\mu+k^2)-2\delta(r^2-\mu)+r^2]=\delta(\delta-1)(r^2-\mu)+2r^2$$

and $\delta^2(r^2-3\mu+2k^2)=3\delta(r^2-\mu)$. Note that $\delta\neq 0$. We get

$$\begin{aligned}\delta&=\frac{3(r^2-\mu)}{r^2-3\mu+2k^2}\\&=-\frac{3(\overrightarrow{OA}\cdot\overrightarrow{OK}-\overrightarrow{OA}\cdot\overrightarrow{OA})}{\overrightarrow{OA}\cdot\overrightarrow{OA}-3\overrightarrow{OA}\cdot\overrightarrow{OK}+2\overrightarrow{OK}\cdot\overrightarrow{OK}}\\&=\frac{-3[\overrightarrow{OA}\cdot(\overrightarrow{OK}-\overrightarrow{OA})]}{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot(\overrightarrow{OK}-\overrightarrow{OA})}\\&=\frac{-3(\overrightarrow{OA}\cdot\overrightarrow{AK})}{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot\overrightarrow{AK}}.\end{aligned}$$

Hence

$$\overrightarrow{AD}=\frac{3(\overrightarrow{AO}\cdot\overrightarrow{AK})}{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot\overrightarrow{AK}}\overrightarrow{AK}.$$

Case 2: Suppose the points A, O, and K are collinear. Then $\triangle ABC$ is an isosceles triangle with $AB=AC$. Moreover, $\overrightarrow{AD}=\delta'\overrightarrow{OA}$ for some $\delta'\in\mathbb{R}$ so that

$$\overrightarrow{OD}=\overrightarrow{OA}+\delta'\overrightarrow{OA}=(\delta'+1)\overrightarrow{OA}. \tag{2.8}$$

Note that $\delta'+1\neq 0$. We will first prove that

$$\overrightarrow{AD}=-(2\delta'+3)\overrightarrow{AK} \tag{2.9}$$

and then argue that

$$\frac{-3(\overrightarrow{OA}\cdot\overrightarrow{AK})}{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot\overrightarrow{AK}}=-(2\delta'+3). \tag{2.10}$$

First of all,

$$\begin{aligned}\overrightarrow{OK}&=\frac{a^2}{a^2+b^2+c^2}\overrightarrow{OA}+\frac{b^2}{a^2+b^2+c^2}\overrightarrow{OB}+\frac{c^2}{a^2+b^2+c^2}\overrightarrow{OC}\\&=\frac{a^2}{a^2+2b^2}\overrightarrow{OA}+\frac{b^2}{a^2+2b^2}\overrightarrow{OB}+\frac{b^2}{a^2+2b^2}\overrightarrow{OC}\\&=\frac{a^2}{a^2+2b^2}\overrightarrow{OA}+\frac{b^2}{a^2+2b^2}(\overrightarrow{OB}+\overrightarrow{OC})\\&=\frac{a^2}{a^2+2b^2}\overrightarrow{OA}+\frac{2b^2}{a^2+2b^2}\overrightarrow{OM}.\end{aligned} \tag{2.11}$$

Next,

$$\begin{aligned}\overrightarrow{OM}=\frac{r^2}{d^2}\overrightarrow{OD}&=\frac{r^2}{[(\delta'+1)r]^2}\overrightarrow{OD}\qquad\text{(by (2.8))}\\&=\frac{1}{(\delta'+1)^2}\cdot(\delta'+1)\overrightarrow{OA}\\&=\frac{1}{\delta'+1}\overrightarrow{OA}.\end{aligned}$$

Hence

$$OM^2 = \frac{r^2}{(\delta'+1)^2}.$$

Additionally, $\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA} = \frac{-\delta'}{\delta'+1}\overrightarrow{OA}$ so that

$$AM^2 = \left(\frac{\delta' r}{\delta'+1}\right)^2.$$

Both $\triangle OMB$ and $\triangle AMB$ are right triangles. So using the Pythagorean Theorem twice leads to

$$a^2 = \frac{4\delta'(\delta'+2)r^2}{(\delta'+1)^2}, \qquad b^2 = \frac{2\delta' r^2}{\delta'+1},$$
$$a^2 + 2b^2 = \frac{4\delta'(2\delta'+3)r^2}{(\delta'+1)^2}.$$

Hence (2.11) becomes

$$\overrightarrow{OK} = \frac{\delta'+3}{2\delta'+3}\,\overrightarrow{OA}.$$

In turn,

$$\begin{aligned}\overrightarrow{AK} &= \overrightarrow{OK} - \overrightarrow{OA} \\ &= \frac{-1}{2\delta'+3}\overrightarrow{AD}\end{aligned}$$

so that (2.9) holds. Note that $2\delta'+3 \neq 0$.

On the other hand, we have

$$2\overrightarrow{OK} - \overrightarrow{OA} = \frac{2(\delta'+3)}{2\delta'+3}\overrightarrow{OA} - \overrightarrow{OA} = \frac{3}{2\delta'+3}\overrightarrow{OA}.$$

Hence,

$$\begin{aligned}\frac{-3(\overrightarrow{OA}\cdot\overrightarrow{AK})}{(2\overrightarrow{OK}-\overrightarrow{OA})\cdot\overrightarrow{AK}} &= \frac{-3(\overrightarrow{OA}\cdot\overrightarrow{OA})}{\{[3/(2\delta'+3)]\overrightarrow{OA}\}\cdot\overrightarrow{OA}} \\ &= \frac{-3r^2}{3r^2/(2\delta'+3)} \\ &= -(2\delta'+3).\end{aligned}$$

Hence (2.10) holds. □

Theorem 2.7. ***(Court Uniqueness Theorem)*** *Let $\mathcal{C}$ be a circle with center O and radius r. Let $K \neq O$ be a point lying inside $\mathcal{C}$. For each point $A \in \mathcal{C}$, there is a unique triangle $\triangle ABC$ inscribed in $\mathcal{C}$ having K as its symmedian point. Additionally, if $(\mathcal{C}, K)$ denotes the locus of all such triangles, i.e. the $(\mathcal{C}, K)$-locus, then the map from $\mathcal{C}$ to $(\mathcal{C}, K)$ is 3-1.*

Proof. Let $A \in \mathcal{C}$. By Theorem 2.4 there is a triangle $\triangle ABC$ inscribed in $\mathcal{C}$ having K as its symmedian point. However, this triangle is unique by Theorem 2.6. Consequently, there is a well defined map

$$\mathcal{C} \longrightarrow (\mathcal{C}, K).$$

It is clear that the map is 3-1. □

3. The Centroid Locus Theorem

Let $\mathcal{C}$ be a circle of radius r centered at O and let $K \neq O$ be a fixed point inside $\mathcal{C}$. For each point $A \in \mathcal{C}$, let $\triangle ABC$ be the unique triangle in the $(\mathcal{C}, K)$-locus. Let G be the centroid of $\triangle ABC$. The objective of this section is to prove that the variable centroid G is an arc on a circle; see Theorem 3.1 just ahead. In preparation for the proof, we first express G in terms of the moving frame $\{\overrightarrow{OA}, \overrightarrow{OK}\}$.

Assuming that $A \notin \mathcal{C}(KO')$, let D and M be defined as in the statement of Theorem 2.4. Let $d = OD$, $k = OK$, and $\mu = \overrightarrow{OA} \cdot \overrightarrow{OK}$. Let

$$\lambda = (2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} = 2k^2 - 3\mu + r^2, \tag{3.1}$$

$$x = \frac{\lambda}{\overrightarrow{AO} \cdot \overrightarrow{AK}}. \tag{3.2}$$

Note that, up to $\pm$, x is the ratio $\frac{AQ}{AR}$. Continuing, the harmonic ratio from Theorem 2.4 is given by

$$\begin{aligned} \delta &= \frac{3}{x} \\ &= \frac{3\overrightarrow{AO} \cdot \overrightarrow{AK}}{(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK}} \\ &= \frac{3(r^2 - \mu)}{\lambda}. \end{aligned} \tag{3.3}$$

Hence $x = \dfrac{\lambda}{r^2 - \mu}$. As the discussion to follow will show, the number x (3.2) is an efficient parameter that is geometrically natural.

Lemma 3.1. *The scalar* $\mu = \overrightarrow{OA} \cdot \overrightarrow{OK}$ *is the rational function of* x *given by*

$$\mu = \frac{r^2(x-1) - 2k^2}{x - 3}. \tag{3.4}$$

Proof.

$$\begin{aligned} x = \frac{\lambda}{r^2 - \mu} \quad &\iff \quad x(r^2 - \mu) = \lambda \\ &\iff \quad x\mu = r^2 x - (2k^2 - 3\mu + r^2) \\ &\iff \quad (x-3)\mu = r^2 x - (r^2 + 2k^2) \\ &\iff \quad \mu = \frac{r^2 x - (r^2 + 2k^2)}{x - 3}. \end{aligned}$$

□

Lemma 3.2. *The centroid is given by the vector equation*

$$\overrightarrow{OG} = \alpha \overrightarrow{OA} + \beta \overrightarrow{OK} \tag{3.5}$$

where

$$\alpha = \tfrac{1}{3} + \tfrac{2r^2}{3d^2}(1 - \delta), \tag{3.6}$$

$$\beta = \tfrac{2r^2\delta}{3d^2}. \tag{3.7}$$

Proof. First of all,

$$\begin{aligned} 3\overrightarrow{OG} &= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \\ &= \overrightarrow{OA} + 2\overrightarrow{OM}. \end{aligned}$$

Next,

$$\begin{aligned} 2\overrightarrow{OM} &= \tfrac{2r^2}{d^2}\overrightarrow{OD} \\ &= \tfrac{2r^2}{d^2}(\overrightarrow{OA} + \overrightarrow{AD}) \\ &= \tfrac{2r^2}{d^2}(\overrightarrow{OA} + \delta\overrightarrow{AK}) \\ &= \tfrac{2r^2}{d^2}\Big[\overrightarrow{OA} + \delta(\overrightarrow{OK} - \overrightarrow{OA})\Big] \\ &= \tfrac{2r^2}{d^2}\Big[(1-\delta)\overrightarrow{OA} + \delta\overrightarrow{OK}\Big]. \end{aligned}$$

Hence

$$3\overrightarrow{OG} = \Big(1 + \tfrac{2r^2}{d^2}(1-\delta)\Big)\overrightarrow{OA} + \tfrac{2r^2\delta}{d^2}\overrightarrow{OK}.$$

□

The point D in Theorem 2.4 can also be shown to lie outside the circle $\mathcal{C}$ using the dot product. Though the calculation is somewhat tedious, equation (3.8) in Lemma 3.3 will be used Lemma 3.4, Theorem 3.1, and again in Lemma 4.3 in section §4 .

Lemma 3.3. *Let $\mathcal{C}$ be a circle of radius r centered at O. Let K be a point with $OK < r$ and A be a point on $\mathcal{C}$ satisfying $(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} \neq 0$. The point D determined by*

$$\overrightarrow{AD} = \frac{3(\overrightarrow{AO} \cdot \overrightarrow{AK})}{(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK}} \overrightarrow{AK}$$

lies outside of the circle $\mathcal{C}$. In particular, set $k = OK$, $d = OD$, $\mu = \overrightarrow{OA} \cdot \overrightarrow{OK}$, and $\lambda = 2k^2 - 3\mu + r^2$. Then

$$d^2 = r^2 + \frac{3(r^2 - k^2)(r^2 - \mu)^2}{\lambda^2}. \tag{3.8}$$

Proof. First of all,

$$\overrightarrow{OA} \cdot \overrightarrow{AK} = \overrightarrow{OA} \cdot (\overrightarrow{OK} - \overrightarrow{OA}) = \mu - r^2$$

$$(2\overrightarrow{OK} - \overrightarrow{OA}) \cdot \overrightarrow{AK} = (2\overrightarrow{OK} - \overrightarrow{OA}) \cdot (\overrightarrow{OK} - \overrightarrow{OA}) = 2k^2 - 3\mu + r^2.$$

Then

$$\overrightarrow{AD} = \frac{3(r^2 - \mu)}{\lambda}(\overrightarrow{OK} - \overrightarrow{OA}).$$

On the other hand, $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$. Hence

$$\lambda\overrightarrow{OD} = 3(r^2 - \mu)\overrightarrow{OK} - 2(r^2 - k^2)\overrightarrow{OA}. \tag{3.9}$$

Letting $d = OD$,

$$\begin{aligned}\lambda^2 d^2 = (12r^2 - 3k^2)\mu^2 &- 6r^2(2r^2 + k^2)\mu \\ &+ r^2(4r^4 + k^2r^2 + 4k^4).\end{aligned}$$

Next,

$$\begin{aligned}\lambda^2(d^2 - r^2) &= (12r^2 - 3k^2)\mu^2 - 6r^2(2r^2 + k^2)\mu \\ &\qquad + r^2(4r^4 + r^2k^2 + 4k^4) - r^2(2k^2 - 3\mu + r^2)^2 \\ &= (12r^2 - 3k^2)\mu^2 - 6r^2(2r^2 + k^2)\mu \\ &\qquad + r^2(6r^2\mu + 12k^2\mu - 9\mu^2) + 3r^4(r^2 - k^2) \\ &= 3(r^2 - k^2)\mu^2 - 6r^2(r^2 - k^2)\mu + 3r^4(r^2 - k^2) \\ &= 3(r^2 - k^2)(\mu - r^2)^2.\end{aligned}$$

Since $0 < k < r$, $-r^2 < -kr \leq \mu \leq kr < r^2$ by the Cauchy-Schwarz inequality. Hence $\mu - r \neq 0$ so that $d^2 - r^2 > 0$ by the above equation, that is, D lies outside $\mathcal{C}$. In addition, equation (3.8) holds. □

Lemma 3.4. *The coefficients α (3.6) and β (3.7) are the rational functions of x given by*

$$\alpha = \frac{r^2(x - 1)^2 - k^2}{r^2(x^2 + 3) - 3k^2}$$

$$\beta = \frac{2r^2 x}{r^2(x^2 + 3) - 3k^2}.$$

Proof. First of all, the squared distance d^2 is a rational function of x. Indeed, by (3.8) from Lemma 3.3,

$$\begin{aligned} d^2 &= r^2 + \frac{3(r^2-k^2)(r^2-\mu)^2}{\lambda^2} \\ &= r^2 + \frac{(r^2-k^2)\delta^2}{3} \qquad \left(\because\ \delta = \frac{3(r^2-\mu)}{\lambda}\right) \\ &= r^2 + \frac{3(r^2-k^2)}{x^2} \qquad \left(\because\ \delta = \frac{3}{x}\right) \\ &= \frac{r^2x^2+3(r^2-k^2)}{x^2}. \end{aligned}$$

Next,

$$\begin{aligned} 3\alpha &= \frac{d^2+2r^2(1-\delta)}{d^2} \\ &= \frac{r^2x^2+3(r^2-k^2)+2r^2x(x-3)}{r^2x^2+3(r^2-k^2)} \\ &= \frac{3r^2(x-1)^2-3k^2}{r^2(x^2+3)-3k^2}. \end{aligned}$$

Finally,

$$\begin{aligned} 3\beta &= \frac{2r^2\delta}{d^2} \\ &= \frac{6r^2x}{r^2(x^2+3)-3k^2}. \end{aligned}$$

□

Theorem 3.1. ***(Centroid Locus)*** *Let $\mathcal{C}$ be a circle of radius r centered at O and $K \neq O$* [2] *be a fixed point inside $\mathcal{C}$. Let $k = OK$. For any point $A \in \mathcal{C}$, let $\triangle ABC$ be the unique triangle in the $(\mathcal{C}, K)$-locus having A as a vertex and K as its symmedian point; see Theorem 2.7. As A varies along $\mathcal{C}$, the locus of centroids G lie on the circle $\mathcal{C}_g$ of radius $rk^2/(4r^2-k^2)$ with center O_g defined by the vector equation*

$$\overrightarrow{OO_g} = \frac{2r^2}{4r^2-k^2}\overrightarrow{OK}. \tag{3.10}$$

Proof. Let $\overline{A_1A_2}$ be the diameter of $\mathcal{C}$ containing the point K. The centroids G_i of the two triangles $\triangle A_iB_iC_i$ in the $(\mathcal{C}, K)$-locus lie on the diameter $\overline{A_1A_2}$; see Remark 2.6. The proof to follow shows that the midpoint of G_1 and G_2 is the point O_g defined by equation (3.10).

[2] If $K = O$, then the triangles in the $(\mathcal{C}, K)$-locus are equilateral. Hence, the variable centroid G degenerates to O.

Using Lemma 3.2 and the point O_g defined by (3.10),

$$\begin{aligned}\overrightarrow{O_gG} &= \overrightarrow{OG} - \overrightarrow{OO_g} \\ &= \alpha\overrightarrow{OA} + \tilde{\beta}\overrightarrow{OK}\end{aligned}$$

where

$$\begin{aligned}\tilde{\beta} &= \left(\beta - \tfrac{2r^2}{4r^2-k^2}\right) \\ &= \frac{-2r^2(x-3)(r^2(x-1)+k^2)}{(4r^2-k^2)(r^2x^2+3(r^2-k^2))}.\end{aligned}$$

Next using the algebraic properties of the dot product

$$O_gG^2 = r^2\alpha^2 + 2\alpha\tilde{\beta}\mu + k^2\tilde{\beta}^2. \tag{3.11}$$

By Lemmas 3.1 and 3.4,

$$\begin{aligned}r^2\alpha^2 &= \frac{r^2(r^2(x-1)^2-k^2)^2}{(r^2x^2+3(r^2-k^2))^2}, \\ 2\alpha\tilde{\beta}\mu &= \frac{-4r^2(r^2(x-1)^2-k^2)(r^2(x-1)+k^2)(r^2(x-1)-2k^2)}{(4r^2-k^2)(r^2x^2+3(r^2-k^2))^2}, \\ k^2\tilde{\beta}^2 &= \frac{4r^4k^2(x-3)^2(r^2(x-1)+k^2)^2}{(4r^2-k^2)^2(r^2x^2+3(r^2-k^2))^2}.\end{aligned}$$

Hence equation (3.11) is equivalent to

$$\begin{aligned}&(4r^2-k^2)^2(r^2x^2+3(r^2-k^2))^2O_gG^2 = \\ &\qquad r^2(4r^2-k^2)^2x_1 - 4r^2(4r^2-k^2)x_2 + 4r^4k^2x_3\end{aligned}$$

where

$$\begin{aligned}x_1 &= (r^2(x-1)^2-k^2)^2, \\ x_2 &= (r^2(x-1)^2-k^2)(r^2(x-1)+k^2)(r^2(x-1)-2k^2), \\ x_3 &= (x-3)^2(r^2(x-1)+k^2)^2.\end{aligned}$$

Expanding the polynomials (patiently), we get

$$x_1 = r^4x^4 - 4r^4x^3 + 2r^2(3r^2-k^2)x^2 - 4r^2(r^2-k^2)x + (r^2-k^2)^2,$$

$$\begin{aligned}x_2 = r^6x^4 &- r^4(4r^2+k^2)x^3 + 2r^2(3r^4+r^2k^2-k^4)x^2 \\ &- r^2(r^2-k^2)(4r^2+5k^2)x + (r^2-k^2)^2(r^2+2k^2),\end{aligned}$$

$$\begin{aligned}x_3 = r^4x^4 &- 2r^2(4r^2-k^2)x^3 + (22r^4-14r^2k^2+k^4)x^2 \\ &- 6(r^2-k^2)(4r^2-k^2)x + 9(r^2-k^2)^2.\end{aligned}$$

Consequently,

$$(4r^2-k^2)^2(r^2x^2+3(r^2-k^2))^2O_gG^2 =$$
$$r^2\left[r^4k^4x^4+6r^2k^4(r^2-k^2)x^2+9k^4(r^2-k^2)^2\right]$$
$$= r^2k^4\left[r^2x^2+3(r^2-k^2)\right]^2,$$

that is,

$$O_gG^2 = \frac{r^2k^4}{(4r^2-k^2)^2}.$$

□

4. Inverse Construction

In this section, the inverse problem is addressed, namely, given a point G from the centroid circle $\mathcal{C}_g$, we provide a method for finding the triangle in the $(\mathcal{C},K)$-locus having the point G as its centroid. The method depends on the roots of a cubic polynomial determined by G. This polynomial is shown to have three real roots with multiplicity counted; see (4.7) ahead.

From section §3, the centroid G of a triangle $\triangle ABC$ in the $(\mathcal{C},K)$-locus is given by the vector equation

$$\overrightarrow{OG} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OK};$$

see Lemma 3.4. Let θ_A (resp, θ_G) denote the angle between $\overrightarrow{OA}$ and $\overrightarrow{OK}$ (resp, $\overrightarrow{OG}$ and $\overrightarrow{OK}$). Geometrically, the equation

$$\overrightarrow{OG}\cdot\overrightarrow{OK} = \alpha\overrightarrow{OA}\cdot\overrightarrow{OK} + \beta k^2 \tag{4.1}$$

explains how the angles θ_A and θ_G are related.

Conversely, let $G \in \mathcal{C}_g$. Let $g = OG$. By (3.10)

$$\overrightarrow{O_gG} = \overrightarrow{OG} - \overrightarrow{OO_g} = \overrightarrow{OG} - \tfrac{2r^2}{4r^2-k^2}\overrightarrow{OK}$$

so that

$$O_gG^2 = g^2 - \tfrac{4r^2}{4r^2-k^2}\overrightarrow{OG}\cdot\overrightarrow{OK} + \tfrac{4r^4k^2}{(4r^2-k^2)^2}$$

and hence

$$\overrightarrow{OG}\cdot\overrightarrow{OK} = \frac{(4r^2-k^2)g^2+r^2k^2}{4r^2}. \tag{4.2}$$

Rewrite equation (4.1) as

$$\overrightarrow{OA}\cdot\overrightarrow{OK} = \frac{1}{\alpha}\overrightarrow{OG}\cdot\overrightarrow{OK} - \frac{\beta}{\alpha}k^2. \tag{4.3}$$

The inverse problem reduces to finding α and β so that the point $A \in \mathcal{C}$ and also the corresponding triangle in the $(\mathcal{C}, K)$-locus has centroid G. Using Lemma 3.4, rewrite (4.3) as

$$\overrightarrow{OA} \cdot \overrightarrow{OK} = \frac{r^2(x^2+3) - 3k^2}{r^2(x-1)^2 - k^2} \overrightarrow{OG} \cdot \overrightarrow{OK} - \frac{2r^2k^2x}{r^2(x-1)^2 - k^2}.$$

Using $\mu = \overrightarrow{OA} \cdot \overrightarrow{OK}$, (4.3) is equivalent to

$$(r^2(x^2+3) - 3k^2)(\overrightarrow{OG} \cdot \overrightarrow{OK}) - (r^2(x-1)^2 - k^2)\mu - 2r^2k^2x = 0. \tag{4.4}$$

By Lemmas 3.1 and 3.4, as well as (4.2), equation (4.4) is

$$\begin{aligned} \left[\frac{r^2(x^2+3) - 3k^2}{r^2(x-1)^2 - k^2}\right]\left[\frac{(4r^2 - k^2)g^2 + r^2k^2}{4r^2}\right] &- \frac{r^2(x-1) - 2k^2}{x-3} \\ &- \frac{2r^2k^2x}{r^2(x-1)^2 - k^2} = 0. \end{aligned} \tag{4.5}$$

Since $x \neq 3$, multiply the above equation by $4r^2(r^2(x-1)^2 - k^2)(x-3)$ to get the equivalent cubic equation

$$\begin{aligned} 0 = [(4r^2 - k^2)g^2 + r^2k^2]&[r^2(x^2+3) - 3k^2](x-3) \\ &- 4r^2[r^2(x-1)^2 - k^2][r^2(x-1) - 2k^2] \\ &- 8r^4k^2x(x-3). \end{aligned} \tag{4.6}$$

Next,

$$[r^2(x^2+3) - 3k^2](x-3) = r^2(x-1)^3 - 3k^2(x-1) - 2(r^2 - 3k^2),$$

$$\begin{aligned} [r^2(x-1)^2 - k^2][r^2(x-1) - 2k^2] = r^4(x-1)^3 &- 2r^2k^2(x-1)^2 \\ &- r^2k^2(x-1) + 2k^4, \end{aligned}$$

$$-8r^4k^2x(x-3) = -8r^4k^2(x-1)^2 + 8r^4k^2(x-1) + 16r^4k^2.$$

Using this information, rewrite (4.6) as

$$\begin{aligned} -r^2(4r^2 - k^2)(r^2 - g^2)(x-1)^3 &+ 3k^2(4r^2 - k^2)(r^2 - g^2)(x-1) \\ &- 2(4r^2 - k^2)[(4r^2 - 3k^2)g^2 - r^2k^2] = 0. \end{aligned}$$

Letting $X = x - 1$ and dividing by $-r^2(4r^2 - k^2)(r^2 - g^2)$, we get the cubic equation

$$X^3 + pX + q = 0 \tag{4.7}$$

where

$$p = -\frac{3k^2}{r^2}, \qquad q = \frac{2[(4r^2 - 3k^2)g^2 - r^2k^2]}{r^2(r^2 - g^2)}. \tag{4.8}$$

Lemma 4.1. *Let* $\overline{G_1G_2}$ *be the diameter of the centroid circle* $\mathcal{C}_g$ *lying between* O *and* K *. Given* $G \in \mathcal{C}_g$ *such that* $G \neq G_i, i = 1, 2$ *, the corresponding cubic equation (4.7) has three distinct real roots.*

Proof. First of all, the diameter $\overline{G_1G_2}$ of the centroid circle is determined by the vector equations

$$\overrightarrow{OG_1} = \tfrac{r}{2r+k}\overrightarrow{OK}, \qquad \overrightarrow{OG_2} = \tfrac{r}{2r-k}\overrightarrow{OK};$$

see Remark 2.6 in section §2.2 . Next, given any $G \in \mathcal{C}_g, G \neq G_i$

$$OG_1 = \tfrac{rk}{2r+k} < g = OG < \tfrac{rk}{2r-k} = OG_2.$$

Hence

$$g(2r-k) < rk < g(2r+k). \tag{4.9}$$

Next, by routine algebra

$$\frac{p^3}{27} + \frac{q^2}{4} = \frac{-(r^2-k^2)[(2r+k)^2g^2 - r^2k^2][r^2k^2 - (2r-k)^2g^2]}{r^6(r^2-g^2)^2}.$$

By (4.9), both $(2r+k)^2g^2 - r^2k^2$ and $r^2k^2 - (2r-k)^2g^2$ are positive. Hence $\frac{p^3}{27} + \frac{q^2}{4} < 0$ so that the discriminant of the cubic, namely, $\Delta = -(4p^3 + 27q^2) > 0$. This means that (4.7) has three distinct real roots. See Chapter 14, §2 of [1]. □

Lemma 4.2. *Let* X *be a real solution to (4.7). Set* $x = X + 1$ *. Since* $x - 1$ *is a solution of (4.7),*

$$\begin{aligned} x^3 &= 3x^2 - (p+3)x + (p-q+1) \\ &= 3x^2 - \tfrac{3(r^2-k^2)}{r^2}x + \tfrac{(r^2-k^2)(r^2-9g^2)}{r^2(r^2-g^2)} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} x^4 &= (6-p)x^2 - (2p+q+8)x + 3(p-q+1) \\ &= \tfrac{3(2r^2+k^2)}{r^2}x^2 - \tfrac{8(r^2-k^2)}{r^2-g^2}x + \tfrac{3(r^2-k^2)(r^2-9g^2)}{r^2(r^2-g^2)}. \end{aligned} \tag{4.11}$$

Using (4.10) and (4.11),

$$[r^2(x^2+3) - 3k^2](x-3) = -\tfrac{8r^2(r^2-k^2)}{r^2-g^2} \tag{4.12}$$

and

$$\begin{aligned} [r^2(x-1)^2 - k^2]^2 = r^2k^2x^2 &- \tfrac{8r^2g^2(r^2-k^2)}{r^2-g^2}x \\ &+ \tfrac{(r^2-k^2)[(8r^2+k^2)g^2 - r^2k^2]}{r^2-g^2}. \end{aligned} \tag{4.13}$$

Proof. Equations (4.10) and (4.11) follows immediately from the assumption that $x - 1$ is a solution of the cubic equation (4.7). In turn, equations (4.12) and (4.13) follow routinely. However, they are used in an essential way in the proof of Lemma 4.3 just ahead. □

Lemma 4.3. *Let X be a solution to (4.7). Set $x = X + 1$. Define α and β by Lemma 3.4. In turn, define A by the vector equation $\overrightarrow{OA} = \frac{1}{\alpha}\overrightarrow{OG} - \frac{\beta}{\alpha}\overrightarrow{OK}$; see (4.3). Then $A \in \mathcal{C}$ and*

$$\overrightarrow{OA} \cdot \overrightarrow{OK} = \frac{r^2(x-1) - 2k^2}{x-3} = \mu,$$

i.e. equation (3.4) in Lemma 3.1 holds. Consequently, the given point G is the centroid of the unique triangle in the $(\mathcal{C}, K)$-locus having the point A as a vertex.

Proof. First of all, from $\overrightarrow{OA} = \frac{1}{\alpha}\overrightarrow{OG} - \frac{\beta}{\alpha}\overrightarrow{OK}$,

$$OA^2 = \frac{g^2}{\alpha^2} - 2\frac{\beta}{\alpha^2}\overrightarrow{OG} \cdot \overrightarrow{OK} + \frac{\beta^2 k^2}{\alpha^2}.$$

Using Lemma 3.4, this equation is equivalent to

$$\begin{aligned}[r^2(x-1)^2 - k^2]^2 OA^2 = [r^2(x^2+3) - 3k^2]^2 g^2 \\ - 4r^2 x[r^2(x^2+3) - 3k^2]\overrightarrow{OG} \cdot \overrightarrow{OK} \\ + 4r^4k^2x^2.\end{aligned} \tag{4.14}$$

Using (4.2), the right side of equation (4.14) is the quartic polynomial

$$\begin{aligned}Q(x) &= r^4g^2x^4 \\ &\quad - r^2[(4r^2 - k^2)g^2 + r^2k^2]x^3 + [6r^2g^2(r^2 - k^2) + 4r^4k^2]x^2 \\ &\quad - 3(r^2 - k^2)[(4r^2 - k^2)g^2 + r^2k^2]x \\ &\quad + 9g^2(r^2 - k^2)^2.\end{aligned}$$

Next use Lemma 4.2 to rewrite the above quartic as

$$\begin{aligned}Q(x) &= r^4g^2x^4 - r^2[(4r^2 - k^2)g^2 + r^2k^2]x^3 + [6r^2g^2(r^2 - k^2) + 4r^4k^2]x^2 \\ &\quad - 3(r^2 - k^2)[(4r^2 - k^2)g^2 + r^2k^2]x + 9g^2(r^2 - k^2)^2 \\ &= r^2\left[r^2k^2x^2 - \frac{8r^2g^2(r^2 - k^2)}{r^2 - g^2}x + \frac{(r^2 - g^2)[(8r^2 + k^2)g^2 - r^2k^2]}{r^2 - g^2}\right] \\ &= r^2[r^2(x-1)^2 - k^2] \qquad \text{by (4.13).}\end{aligned}$$

Hence (4.14) becomes

$$[r^2(x-1)^2 - k^2]OA^2 = [r^2(x-1)^2 - k^2]r^2,$$

that is, $OA^2 = r^2$ so that $A \in \mathcal{C}$.

To argue the equation

$$\overrightarrow{OA} \cdot \overrightarrow{OK} = \frac{r^2x - r^2 - 2k^2}{x-3}, \tag{4.15}$$

proceed as follows:

$$\begin{aligned}\overrightarrow{OA} \cdot \overrightarrow{OK} &= \tfrac{1}{\alpha}\overrightarrow{OG} \cdot \overrightarrow{OK} - \tfrac{\beta}{\alpha}k^2 \\ &= \frac{r^2(x^2+3) - 3k^2}{r^2(x-1)^2 - k^2}\overrightarrow{OG} \cdot \overrightarrow{OK} - \frac{2r^2k^2x}{r^2(x-1)^2 - k^2}\end{aligned}$$

Hence (4.15) holds iff

$$\begin{aligned}[r^2(x-1)^2-k^2][r^2(x-1)-2k^2] &= \overrightarrow{OG}\cdot\overrightarrow{OK}[r^2(x^2+3)-3k^2](x-3)\\ &\quad -2r^2k^2x(x-3)\\ &= \left[\tfrac{(4r^2-k^2)g^2+r^2k^2}{4r^2}\right][r^2(x^2+3)-3k^2](x-3)\\ &\quad -2r^2k^2x(x-3).\end{aligned}$$

Clearing the denominator, (4.15) holds iff

$$\begin{aligned}&4r^2[r^2(x-1)^2-k^2][r^2(x-1)-2k^2]=\\ &\qquad [(4r^2-k^2)g^2+r^2k^2][r^2(x^2+3)-3k^2](x-3)\\ &\qquad -8r^4k^2x(x-3).\end{aligned} \tag{4.16}$$

Using equation (4.13) from Lemma 4.2,

$$\begin{aligned}4r^2[r^2(x-1)^2-k^2][r^2(x-1)-2k^2] = 4r^2\Big[&-2r^2k^2x^2+6r^2k^2x\\ &-\tfrac{2(r^2-k^2)[(4r^2-k^2)g^2+r^2k^2]}{r^2-g^2}\Big].\end{aligned}$$

Using equation (4.12) from Lemma 4.2,

$$[(4r^2-k^2)g^2+r^2k^2][r^2(x^2+3)-3k^2](x-3)=\frac{-8r^2(r^2-k^2)[(4r^2-k^2)g^2+r^2k^2]}{r^2-g^2}.$$

Hence the right side of (4.16) is

$$4r^2\left[-2r^2k^2x^2+6r^2k^2x-\tfrac{2(r^2-k^2)[(4r^2-k^2)g^2+r^2k^2]}{r^2-g^2}\right].$$

Therefore (4.15) holds.

Finally, having shown that $\mu=\overrightarrow{OA}\cdot\overrightarrow{OK}=\frac{r^2(x-1)-2k^2}{x-3}$,

$$\delta=\frac{3(r^2-\mu)}{\lambda}=\frac{3}{x}\qquad\text{and}\qquad d^2=\frac{r^2(x^2+3)-3k^2}{x^2}.$$

Hence

$$\frac{d^2+2r^2(1-\delta)}{d^2}=\frac{3(r^2(x-1)^2-k^2)}{r^2(x^2+3)-3k^2}=3\alpha.$$

Likewise,

$$\frac{2r^2\delta}{d^2}=\frac{6r^2x}{r^2(x^2+3)-3k^2}=3\beta.$$

Consequently, by Lemma 3.2 the given point G is the centroid of the unique triangle in the $(\mathcal{C},K)$-locus having the point A as a vertex. □

Remark 4.1. Finding the roots of (4.7) and in turn illustrating Lemma 4.3 are done using basic complex variables and the fact that $4p^3 + 27q^2 < 0$. Let

$$\begin{aligned} z &= -\frac{q}{2} + i\sqrt{-\left(\frac{p^3}{27} + \frac{q^2}{4}\right)} \\ &= x + iy \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned}$$

where $|z| = \frac{-p}{3}\sqrt{-\frac{p}{3}}$ since $p < 0$. Let θ be the principal polar angle of z. Then

$$\theta = \cos^{-1}(x/|z|) = \cos^{-1}\left(\tfrac{3q}{2p}\sqrt{-\tfrac{3}{p}}\right).$$

On the other hand, the number

$$\frac{3q}{2p}\sqrt{-\frac{3}{p}} = \frac{r[r^2k^2 - g^2(4r^2 - 3k^2)]}{k^3(r^2 - g^2)}.$$

Finally, in symbolic form the roots of (4.7) are

$$\begin{aligned} x - 1 &= \sqrt[3]{z} + \overline{\sqrt[3]{z}} = 2|z|^{1/3}\cos(\theta/3) \\ &= \frac{2k}{r}\cos(\theta/3) \end{aligned}$$

where $\sqrt[3]{z}$ is understood to mean an arbitrary cube root of z. Consequently, using $x = 1 + \frac{2k}{r}\cos(\theta/3)$, α, β, and

$$\overrightarrow{OA} = \tfrac{1}{\alpha}\overrightarrow{OG} - \tfrac{\beta}{\alpha}\overrightarrow{OK}$$

are determined.

References

[1] Artin, M., *Algebra*, Prentice Hall, Inc., 1991.

[2] Altshiller-Court, N., *College Geometry*, Dover Publications, Inc., 2007.

[3] Bataille, M., *Characterizing A Symmedian*, Crux Mathematicorum, Vol. 43(4), April 2017, 145-150.

[4] Chen, Y. and Fisher, R.J., *Symmedian Locus Configuration*, Preprint, August, 2021.

[5] Dray, T. and Manogue, C.A., *The Geometry of the Dot and Cross Products*, Journal of Online Mathematics and Its Applications, Vol 6, June 2006, Article ID 1156.

[6] Honsberger, R., *Episodes in Nineteenth and Twentieth Century Euclidean Geometry,* Washington: Mathematical Association of America, 1995.

[7] Johnson, R.A., *Advanced Euclidean Geometry: An Elementary Treatise on the Geometry of the Triangle and Circle*, Dover Publications, Inc., New York, 1960.

[8] Kimberling, C., *Central Points and Central Lines in the Plane of a Triangle*, Math. Magazine 67 (June 1994), 163–187.

Department of Mathematics and Statistics

Idaho State University, Pocatello, ID 83209-8085, U.S.A.

E-mail address: chenyu@isu.edu

E-mail address: robertfisher@isu.edu