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ON CYCLOTOMIC ARRANGEMENTS OF LINES IN THE PLANE

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Abstract. In this paper, we consider the seemingly simple problem of describing and enumerating the set of connected components delimited by the cyclotomic arrangement of straight lines in the plane generated from the edges of a regular n-gon. Knowing the exact position of each intersection point of the arrangement, an exhaustive study of the different areas of the components is also provided.

1. Introduction and notation

We address mainly in this work the problem of providing a detailed list of the various connected components generated and bounded by a particular cyclotomic arrangement of straight lines in the plane which coincide with the edges of a regular n-gon.

The very first result related to this subject, valid for any arrangement, is the Jacobus Steiner's bound for the number of connected components in the plane, that is to say $\frac{1}{2}(n^2+n+2)$ where n denotes the cardinality of the arrangement and which is easy to prove by induction (see for example [5, 9]). However, this result does not give insight in some features of interest including listing the number of triangles, quadrilaterals and so on, generated by the arrangement, nor prove the existence of some relationships between those amounts, or even provide the numbers of compact or non-compact components. In [8], Wetzel gave an historical account of the Steiner's problem and focus on generalization of Broussaud's formula for the defect number, that is the difference between the Steiner's upperbound $\frac{1}{2}(n^2+n+2)$ and the actual value of the number of connected components. Wetzel highlighted the multiplicities of the intersection points as well as the existence of parallel straight lines but did not use symmetries of the arrangements. For their part, Poonen and Rubinstein [7] got interested in the number of intersection points formed inside a regular n-gon by its diagonals while making clear the distinct multiplicities. While Poonen and Rubinstein focused mostly on formulas describing the various situations, some other authors worked on

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efficient deterministic algorithms to report and to count geometric intersections convex polygons, inducing naturally time and storage constraints (see for example [1], [4]).

We emphasize on the fact that none of the authors mentioned previously nor those who are cited in their respective bibliographies have studied the distribution of connected components according to their shape, characterized by the numbers of vertices or the property of being compact or not compact. It should be however specified that the question of counting special shapes among those connected components has been discussed from an algorithmic point of view in [4] and literature therein. Nevertheless, the computation of the areas of these shapes is not considered contrary to what is being done in the present paper. Indeed, providing an exhaustive description of the straight lines and the intersection points generated by the arrangement, we are able to specify the exact area of each polygon as well as an asymptotic expansion of it.

To state our results and the convenient formulation, let us give notation used hereafter. First, we identify $\mathbb{R}^2$ and $\mathbb{C}$ and we denote by I the complex number $I = \sqrt{-1}$. As usual, if $x \in \mathbb{R}$ then $[x]$ denotes the greatest integer function. Given an integer n, let us set

$$c_\alpha = \cos\left(\frac{\alpha\pi}{n}\right) \text{ and } s_\alpha = \sin\left(\frac{\alpha\pi}{n}\right), \ \forall \alpha \in \mathbb{R}, \tag{1}$$

provided the value of n is obvious from the context.
Let $\mathcal{A} = (\mathcal{L}_0, \ldots, \mathcal{L}_{n-1})$ be an arrangement of n lines in the euclidean plane, with $\mathcal{L}_i : a_i x + b_i y + c_i = 0$. If $\mathcal{L}_k \cap \mathcal{L}_\ell \neq \emptyset$ for $k, \ell \in \{0, \ldots, n-1\}$, $k \neq \ell$, we denote by $z_{k,\ell}$ this intersection point. We suppose throughout the paper that $n \geq 3$ and that at least two lines are not parallel. The space $\mathcal{Z} = \mathbb{R}^2 - \bigcup_{i=0}^{n-1} \mathcal{L}_i$ that we may denote $\mathcal{Z} = \mathbb{R}^2 - \mathcal{A}$, is locally compact and locally arcwise connected. The set $\pi_0(\mathcal{Z})$ of arcwise connected components is finite with cardinality $\sharp\pi_0(\mathcal{Z})$. By Steiner's bound, we have $\sharp\pi_0(\mathcal{Z}) \leq \frac{1}{2}(n^2 + n + 2)$. At last, we will denote by $\pi_C(\mathcal{Z})$ and $\pi_{NC}(\mathcal{Z})$ the respective sets of compact and non-compact components of $\mathcal{Z}$.
In the remainder of the paper we call *chamber* the closure of any connected component of a space $\mathbb{R}^2 - \mathcal{A}$, this terminology is justified as in the theory of Lie groups (see [3][Lie, chap 5, §3]) and in differential topology. Chambers correspond to intersections of a finite number of closed half-planes, they are all convex and may be compact or non-compact.

The rest of this paper is organized as follows. In section 2, we present some preliminaries on arrangements of lines in the plane. We remind Robert's formula and we present the defects of the arrangements. We define the *boundification* $\mathcal{A}^b$ of the complement $\mathbb{R}^2 - \mathcal{A}$ of an arrangement of lines $\mathcal{A} = (\mathcal{L}_0, \ldots, \mathcal{L}_{n-1})$ and we provide some results on the numbers of connected components of $\mathcal{A}^b$ whether or not they are compact. In Section 3, we introduce for any integer $n \geq 3$ the cyclotomic minimal arrangement $\mathcal{R}(n)$ containing the regular n-gon $\mathcal{P}_n$ (see for example Figure 1) and we get interested in the exhaustive description of $\mathbb{R}^2 - \mathcal{R}(n)$. In Section 4, taking into account the rich symmetry of the space $\mathcal{R}(n)^b$, we provide some nice

results for the areas of each kind of chambers of $\mathcal{R}(n)^b$ and especially the value of the sums

$$\sum_{m=2}^{\nu-1} \sin\left(\frac{m\pi}{n}\right) \sec\left((m-1)\frac{\pi}{n}\right) \sec\left(m\frac{\pi}{n}\right) \sec\left((m+1)\frac{\pi}{n}\right).$$

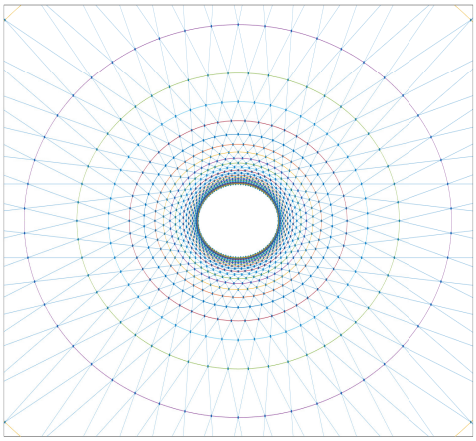

FIGURE 1. A partial geometric view of the cyclotomic arrangement of lines constructed from the regular 50-gon together with its circular orbits

2. Preliminaries on arrangements of lines in the plane.

The final result of Samuel Robert in 1889, that we quote from Wetzel [8] is stated as follows

Proposition 2.1 (Robert's formula)**.** *Let $\mathcal{A} = (\mathcal{L}_0, \ldots, \mathcal{L}_{n-1})$ be an arrangement of n lines in the euclidean plane. Let p be the number of points of multiplicity ≥ 3 and, for each $1 \leq i \leq p$, let $\lambda_i \geq 3$ be the multiplicity of any multiple intersection point. Let q be the number of parallel straight lines in $\mathcal{A}$ and, for each $1 \leq j \leq q$, let $\mu_j \geq 2$ be the number of parallel lines in some direction. Then the number of connected components of $\mathbb{R}^2 - \mathcal{A}$ is equal to*

$$\sharp(\pi_0(\mathbb{R}^2 - \mathcal{A})) = 1 + n + \binom{n}{2} - \sum_{i=1}^{p} \binom{\lambda_i - 1}{2} - \sum_{j=1}^{q} \binom{\mu_j}{2}. \tag{2}$$

It is interesting to observe that the proof given in [8] proceeds by subtracting the various degeneracies in connected components being lost due to multiplicities of intersection points or parallelism of lines, while, on the contrary, we will obtain our main results of the next section by adding the various numbers of connected components of several shapes. We may use the phrasing of Wetzel to speak of the sum $\delta_M(\mathcal{A}) = \sum_{i=1}^{p} \binom{\lambda_i - 1}{2}$ as the number of regions lost because of the multiple points and to speak of the

sum $\delta_P(\mathcal{A}) = \sum_{j=1}^{q} \binom{\mu_j}{2}$ as the number of regions lost because of parallellism. We will call hereafter those two sums $\delta_M(\mathcal{A}), \delta_P(\mathcal{A})$ defects of the arrangements of type M and P respectively and call defect of the arrangement $\mathcal{A}$ the quantity $\delta(\mathcal{A}) = \delta_M(\mathcal{A}) + \delta_P(\mathcal{A})$.

Let $\mathcal{A}$ be an arrangement of lines. There exists a real number R such that the compact disk $\mathcal{D}_R$ of radius R, centered at the origin, contains all the intersection points from $\mathcal{A}$ in its interior. Let $\kappa : \mathbb{C} \to \mathbb{C}$ defined as follows: if $|z| \geq R$, then $\kappa(z) = Rz/|z|$, and $\kappa(z) = z$ otherwise. The mapping κ is a retraction of $\mathbb{R}^2$ on the disk $\mathcal{D}_R$. Moreover, κ preserves the nature and the number of connected components of the complement of the arrangement $\mathcal{A}$ in $\mathbb{R}^2$. However, the mapping κ does not preserve convexity. For instance for any $\varepsilon > 0$, the triangle T with vertices $\{\gamma(1+I), a+bI, b+aI\}$, $a, b > 1$, $a \neq b$, $\gamma \in]0, \frac{\sqrt{2}}{2}[$, has a non convex range $\kappa(T)$. We may define the boundification of $\mathcal{A}$ as the relatively compact subset $\mathcal{A}^b = \kappa(\mathbb{R}^2 - \mathcal{A})$, and the topology of this space does not depend on R.

If one really wants to obtain a compact space, it will be preferable to consider an *open tubular neighborhood* $\mathcal{B}$ of $\mathcal{A}$, of width $\varepsilon > 0$ small enough, and then the space $(\mathbb{R}^2 - \mathcal{B}) \cap \mathcal{D}(0, R)$. In that way, it is possible to compactify the complement set of an arrangement of lines. A naive illustration of this is depicted in [9]. We warn the reader not to confuse this compactification with the Alexandrov usual compactification; this last one indeed maps the complement of the whole "finite" space to one point at infinity, while the retraction κ maps on a given large circle all points at infinity on rays emanating from the origin. For illustrative purpose, let us mention that we may substitute to the circle $|z| = R$ "at infinity" any convex polygon, say a square or a lozenge, which contains all the intersection points in its interior. Doing so, some constructions given hereafter do not use smooth polygons inducing circular segments but ordinary polygons instead.

We easily prove that the connected components of the complement $\mathbb{R}^2 - \mathcal{A}$ of a general arrangement $\mathcal{A}$ are convex. We may distinguish among them compact and non-compact components, and in the first category bounded polygonal components of which the boundary has $k = 3, 4, \ldots$ vertices. If we consider the images of the various components by the retraction κ, we see that the images of bounded polygonal components remain compact while the images of non-compact components become polygonal components implying only one curvilinear arc along their boundary.

Lemma 2.1. *If the arrangement $\mathcal{A} = (\mathcal{L}_0, \ldots, \mathcal{L}_{n-1})$ contains at least one intersection point, we have*

$$\sharp(\pi_{NC}(\mathcal{A}^b)) = 2n \text{ and } \sharp(\pi_C(\mathcal{A}^b)) \leq \frac{1}{2}(n^2 - 3n + 2).$$

Proof. Let us consider R large enough to ensure that the disc $x^2 + y^2 < R^2$ contains all the intersection points $\mathcal{L}_i \cap \mathcal{L}_j$. Due to this hypothesis, each straight line $\mathcal{L}_i$ has some points belonging to the interior of the disc and cuts the circle in two distinct points because $\mathcal{L}_i$ cannot be tangent to the

circle. Let us define $(R\cos\eta_{i,1}, R\sin\eta_{i,1})$ and $(R\cos\eta_{i,2}, R\sin\eta_{i,2})$ as the two intersection points between $\mathcal{L}_i$ and $x^2+y^2=R^2$, with $0 \le \eta_{i,1} < \eta_{i,2} \le 2\pi$. Let us consider the sequence obtained by rearranging in ascending order the roots $\eta_{i,1}, \eta_{i,2}$, $0 \le i \le n-1$, i.e.

$$0 \le \eta_1 < \eta_2 < \ldots < \eta_{2n} < 2\pi. \tag{3}$$

Since the arrangement contains at least two secant straight lines, we do not have $\eta_i = \eta_{j,1}$ and $\eta_{i+1} = \eta_{j,2}$ for some index j. Therefore, if $\widehat{uv}$ denotes the arc between points u and v along the circle $\mathcal{D}_R$, each non-compact component is partially defined through a circular arc $\widehat{R_iR_{i+1}}$, where $R_i = (R\cos\eta_i, R\sin\eta_i)$, together with the two straight lines $\mathcal{L}_j$ and $\mathcal{L}_k$ the two points R_i and R_{i+1} belong to respectively. We have in this way $2n$ non-compact components. Next, Steiner's bound becomes

$$\sharp(\pi_C(\mathcal{A}^b)) \le \frac{1}{2}(n^2+n+2) - \sharp(\pi_{NC}(\mathcal{A}^b)) = \frac{1}{2}(n^2-3n+2),$$

which ends the proof. Let us mention that when the arrangement consists exclusively of parallel straight lines, $\sharp(\pi_{NC}(\mathcal{A}^b)) = n+1$.

To conclude this section, let us discuss the compacity of the chambers. Each chamber, say $\mathcal{Y}$, of $\mathbb{R}^2 - \mathcal{A}$ becomes a chamber $\kappa(\mathcal{Y})$ of $\mathcal{A}^b$. Moreover, the boundary of $\mathcal{Y}$ is sent to the boundary of the chamber $\kappa(\mathcal{Y})$. Lastly, the number of points of intersection lying on the boundary of a chamber $\mathcal{Y}$ of $\mathbb{R}^2 - \mathcal{A}$ is preserved under the retraction κ since R is large enough such that $\mathcal{D}_R$ contains all the intersection points of $\mathcal{A}$. Hence, the space $\mathcal{A}^b$ has a set of connected components $\pi_0(\mathcal{A}^b)$ in 1-to-1 correspondence with the set $\pi_0(\mathbb{R}^2 - \mathcal{A})$. The number of vertices and edges of each chamber of $\mathbb{R}^2 - \mathcal{A}$ remains conserved through the retraction κ. By the way, we warn the reader that some chambers are not anymore polygonal, but have boundary consisting of zero or one arc of circle $\mathcal{D}_R$ and of several segments. Although all connected components of $\mathcal{A}^b$ are compact, we speak anyway of *non-compact* components when considering components of $\mathcal{A}^b$ intersecting the boundary $\partial\mathcal{D}_R$.

As a consequence of the Krein-Milman theorem (see [2]), the chambers are the convex hulls of the sets of their extremal points, i.e. $\mathcal{Y} = \mathrm{Conv}(\mathrm{Ext}(\mathcal{Y}))$ where Conv and Ext denote respectively the convex hull of a set and the set of extremal points. For non-compact chambers $\mathcal{Y}$, $\mathcal{Y}$ and $\mathrm{Ext}(\mathcal{Y})$ contain circular arcs of $\mathcal{D}_R$.

3. The cyclotomic arrangement $\mathcal{R}(n)$.

In this and following sections, we deal with the regular arrangement of $n \ge 3$ lines $\mathcal{R}(n)$ constructed from the regular polygon with n vertices. Let us remark that in the particular case where $n = 2$, the arrangement $\mathcal{R}(2)$ consists only in a single straight line passing through the center of the unit circle which yields that $\mathcal{R}(2)^b$ has two connected components all being (non-compact) half-planes.

Given an integer $n \ge 3$, let us construct the regular polygon $\mathcal{P}_n$ with n vertices $z_k = \exp\left(\frac{2Ik\pi}{n}\right)$, $0 \le k < n$, uniformly distributed on the unit circle. As a geometric graph, the vertices of $\mathcal{P}_n$ are the points z_k and each vertex z_k of $\mathcal{P}_n$ is connected to the next one z_{k+1} where $k+1$ is computed modulo

n. For any index $0 \leq i < n$, let $\mathcal{L}_i$ be the affine straight line passing through z_i and z_{i+1} (indices being computed modulo n). Then $\mathcal{R}(n)$ is the union of all lines $\mathcal{L}_i$.

Before introducing the main result of this paper, let us recall that if $\mathcal{S}_n$ denotes the symmetric group on the set $\{1, 2, \ldots, n\}$, then, to each permutation $\sigma \in \mathcal{S}_n$, we may associate in a 1-to-1 way an invertible boolean matrix $\mu(\sigma) \in \mathcal{M}_n(\mathbb{R})$. For all $\sigma \in \mathcal{S}_n$, the transpose of $\mu(\sigma)$ is obviously the matrix $\mu(\sigma)^T = \mu(\sigma^{-1})$. For instance, if we consider the n-cycle $\tau_n = (1\ 2\ 3\ \ldots\ n) \in \mathcal{S}_n$, then $\tau_n' = (n\ \ldots\ 3\ 2\ 1) \in \mathcal{S}_n$ is nothing but the reverse n-cycle of τ_n.

Theorem 3.1. *Let $n \geq 5$ and $\mathcal{R}(n)$ be the regular arrangement of n lines constructed from the regular polygon $\mathcal{P}_n$ with n vertices. We set $\nu = \left[\frac{n-1}{2}\right]$ and we denote $P = \mu(\tau_n) \in \mathcal{M}_n(\mathbb{R})$. The number of vertices of $\mathcal{R}(n)^b$ is equal to $N = n(\nu + 2)$ while the distribution of cardinalities of chambers of $\mathcal{R}(n)^b$ is given as follows*

Vertices	3	3	4	5	n
Type	C	NC	C	NC	C
$\sharp$	n	n	$n(\nu-2)$	n	1

TABLE 1. Types and cardinalities of the chambers of $\mathcal{R}(n)^b$

Moreover, there is a convenient numbering of the N vertices of $\mathcal{R}(n)^b$ so that the adjacency matrix A of the unoriented geometric graph $\mathcal{R}(n)^b$ may be written as the following boolean symmetric square block matrix of size N

$$A = \left(\begin{array}{c|cccccc}
P+P^T & P+I_n & 0_n & \ldots & 0_n & 0_n & 0_n \\
\hline
P^T+I_n & 0_n & P+I_n & \ddots & \vdots & \multicolumn{2}{c}{\multirow{4}{*}{$\mathbf{0}_{n(\nu-2)\times 2n}$}} \\
0_n & P^T+I_n & 0_n & \ddots & 0_n & & \\
\vdots & \ddots & \ddots & \ddots & P+I_n & & \\
0_n & \ldots & 0_n & P^T+I_n & 0_n & I_n & I_n \\
\hline
0_n & \multicolumn{4}{c}{\multirow{2}{*}{$\mathbf{0}_{2n\times n(\nu-2)}$}} & I_n & 0_n \quad P^T+I_n \\
0_n & & & & I_n & P+I_n & 0_n
\end{array}\right) \tag{4}$$

where 0_n and I_n stand respectively for the usual zero and identity matrices of size n and $\mathbf{0}_{\mathcal{J}}$ denotes the zero matrix of order $\mathcal{J}$.

Proof. Clearly, the matrix A consists of $\nu + 2$ rows of boolean blocks of size $n \times n$. Since the matrix $P = \mu(\tau_n) \in \mathcal{M}_n(\mathbb{R})$ is the adjacency matrix of the path consisting of the n points $z_k = \exp(\frac{2Ik\pi}{n})$ for k from 0 to $n-1$ with edges (z_k, z_{k+1}), the$(1,1)$-block $P+P^T$ stands for the adjacency matrix of the circular chain consisting of the n points z_k, $k \in \{0, \ldots, n-1\}$ with unoriented edges $\{z_k, z_{k+1}\}$.

Using notation (1), a bit of computations gives for all $k \in \{0, \ldots, n-1\}$ the equation

$$\mathcal{L}_k : y = \frac{1}{s_{2k+1}} \left(-c_{2k+1}x + c_1\right). \tag{5}$$

Because of the nonzero y-intercept, none of those n straight lines emanate from the origin. In order to determine the intersection point between $\mathcal{L}_k$ and $\mathcal{L}_\ell$ where $k, \ell \in \{0, \ldots, n-1\}$ and $k \neq \ell$, we have to solve the following system

$$\left\{ \begin{array}{rcl} c_{2k+1}x + s_{2k+1}y & = & c_1 \\ c_{2\ell+1}x + s_{2\ell+1}y & = & c_1 \end{array} \right. . \tag{6}$$

Since the determinant of (6) is equal to $s_{2(k-\ell)}$, $\mathcal{L}_k$ and $\mathcal{L}_\ell$ are parallel if and only if $|k-\ell| = \frac{n}{2}$, which occurs only when n is even. Since $\delta_P(\mathcal{R}(n))$ is zero when n is odd and is equal to $\frac{n}{2}$ when n is even, it rewrites as $\delta_P(\mathcal{R}(n)) = n\left(\frac{n-1}{2} - \nu\right)$ which provides, for any integer $n \geq 5$, the number of directions admitting parallel lines in the arrangement $\mathcal{R}(n)$. When the determinant of (6) is nonzero, the coordinates $z_{k,\ell} = z_{\ell,k} = (x_{k,\ell}, y_{k,\ell})$ of $\mathcal{L}_k \cap \mathcal{L}_\ell$ are equal to

$$\left\{ \begin{array}{rcl} x_{k,\ell} & = & \frac{c_1}{c_{k-\ell}} c_{k+\ell+1} \\ y_{k,\ell} & = & \frac{c_1}{c_{k-\ell}} s_{k+\ell+1} \end{array} \right. . \tag{7}$$

Throughout this paper, it is further assumed that $z_{k,\ell}$ is mentioned if and only if $|k-\ell| \neq \frac{n}{2}$. Moreover, the indices of z have always to be considered modulo n. Clearly, $\forall k \in \{0, \ldots, n-1\}$, $z_{k,k-1} = z_k$. The number of distinct couples (k, ℓ) satisfying the conditions $k, \ell \in \{0, \ldots, n-1\}$, $k \neq \ell$ and $|k-\ell| \neq \frac{n}{2}$, that is to say the number of points of intersection $z_{k,\ell}$, is obviously $N' = n\nu$. Since there exist $2n$ points at infinity as mentioned in Lemma 2.1, we conclude that the number of vertices of $\mathcal{R}(n)^b$ is equal to $N = N' + 2n = n(\nu + 2)$.

Let us consider the sequence of real numbers $(r_m) = \left(\frac{c_1}{c_m}\right)$, with $1 \leq m \leq n-1$ and $m \neq \frac{n}{2}$, which occurs in (7). We may easily prove that the sequence (r_m) contains actually ν distinct values since $r_{n-m} = r_m$. Due to the behaviour of the cosinus mapping over $[0, \frac{\pi}{2}]$, the sequence $(r_m)_{1 \leq m \leq \nu}$ is positive and strictly increasing.

In polar coordinates, the point $z_{k,\ell} = x_{k,\ell} + I y_{k,\ell}$ has a modulus and an argument given respectively by

$$|z_{k,\ell}| = r_{|k-\ell|}, \qquad \theta_{k+\ell} = (k+\ell+1)\frac{\pi}{n}. \tag{8}$$

Let $\mathcal{C}_m$ be the circle of radius r_m centered at the origin, $\mathcal{C}_1$ defining naturally the unit circle. Let ρ be the rotation with center the origin and angle of rotation $\frac{2\pi}{n}$. Since ρ preserves $\mathcal{R}(n)$, each point $z_{k,\ell} \in \mathcal{C}_m$ gives rise to homologous points $\rho^p(z_{k,\ell}) = z_{k+p,\ell+p}$ for all integers p modulo n, and where the indices $k+p$ and $\ell+p$ have also to be considered modulo n. Thus, all these points are, at the same time, intersection points of the arrangement as well as points distributed along a same circle $\mathcal{C}_m$ for some m. In other words, the $n\nu$ points $z_{k,\ell}$ are regularly distributed on the ν circles $\mathcal{C}_m$. Now each circle $\mathcal{C}_m$ contains at least one point $z_{k,\ell}$ and thus at least n such points because of rotation invariance. The amount of points being $n\nu$, we conclude that each circle $\mathcal{C}_m$ contains exactly n points $z_{k,\ell}$. We call those circles the *orbits* of the arrangement $\mathcal{R}(n)$.

Now, let us prove that the points $\mathcal{L}_k \cap \mathcal{L}_\ell = z_{k,\ell}$ are simple, that is to say each multiplicity is equal to 2. In other words, if (x, y) is given as in (7), let

us prove that the integers k and ℓ are unique up to permutation. Indeed, we deduce easily from (8) a system to recover the indices k and ℓ from the polar coordinates (r, θ) of any intersection point $z_{k,\ell}$ and which provides two couples (k, ℓ) and (ℓ, k) uniquely defined modulo n. Since no multiple point occurs, $\delta_M(\mathcal{R}(n)) = 0$ holds for all n.

Let S be the segment in $\mathbb{N}$ defined as $S = [\![1, n(\nu+2)]\!]$. We may reindex the intersection points $z_{k,\ell}$ using lexicographical ordering with respect first to radius and next to polar angles. The points $z_k = z_{k,k-1}$, $0 \le k \le n-1$, which are located on the circle $\mathcal{C}_1$, keep naturally their original labels shifted by one unit. For all the other orbits characterized by $2 \le m \le \nu$, we proceed as follows. Let $m \in \{2, \dots, \nu\}$, then the points $z_{k,\ell}$ belonging to $\mathcal{C}_m$ are indexed by $g(k, \ell) = (m-1)n + \ell + 1$, $\ell \in \{0, \dots, n-1\}$, $m = |k - \ell|$. What preceeds defines without ambiguity a 1-to-1 mapping from the set of vertices not lying at infinity to the segment $[\![1, n\nu]\!]$.

From now on, let us assume that $R > r_\nu = \frac{c_1}{c_\nu}$ in compliance with Section 2. In such a way, the points at infinity $R_i = (R\cos\eta_i, R\sin\eta_i)$ located on the circle $|z| = R$ may also be renumbered. Using formulas in the proof of Lemma 2.1, we show that the polar angles of the two points $\mathcal{L}_k \cap \mathcal{D}_R$ are equal to

$$\eta'_k = \frac{(2k+1)\pi}{n} + \arccos\left(\frac{c_1}{R}\right) \text{ and } \eta''_k = \frac{(2k+1)\pi}{n} - \arccos\left(\frac{c_1}{R}\right),$$

modulo 2π, $0 \le k < n$. Then, we collect these angles η'_k and η''_k in an ascending order sequence (η_i) as in (3). Next, we gather into a first class $\mathcal{C}_{R,\nu+1}$ the points R_i with i even and into a second class $\mathcal{C}_{R,\nu+2}$ the other R_i with i odd (or vice versa). In this way, we may extend the numbering g to include the $2n$ points at infinity, first the "even" points, and next the others, respectively indexed by

$$g(R_{2i}) = n\nu + i \text{ and } g(R_{2i-1}) = n(\nu+1) + i,\ 1 \le i \le n.$$

This explains the appearance in the lower right corner of (4) of the adjacency matrix of the subgraph consisting of points at infinity, that is $\begin{pmatrix} 0_n & P^T + I_n \\ P + I_n & 0_n \end{pmatrix}$.

For any index $k \in \{0, \dots, n-1\}$, we define R'_k and R''_k as the two points among the collection $\{R_1, \dots, R_{2n}\}$ that belong to $\mathcal{L}_k$. Let us remark that when n is odd, all straight lines $\mathcal{L}_i$ and $\mathcal{L}_j$ intersect, so that all circular angles $\widehat{R'_k, R''_k}$ contain one and only one point R'_j or R''_j for any other index $j \ne k$. Hence, in this particular case, the preceding numbering satisfies $g(R''_k) = g(R'_k) + n$ for all indices k.

Now we are interested in the closest neigbours of each vertex $z_{k,\ell}$. In this respect, we introduce for each index j, $0 \le j \le n-1$, the linear form

$$\xi_j(x, y) = -xs_{2j+1} + yc_{2j+1}.$$

This mapping describes the abscissa along the straight line $\mathcal{L}_j$ computed in an orthogonal frame. What is important here is the fact that ξ_j is injective on $\mathcal{L}_j$. For each $k \in \{0, \dots, n-1\}$, we get

$$\xi_k(z_{k,\ell}) = \frac{c_1 c_{k-\ell}}{s_{k-\ell}},\ \forall \ell \in \{0, \dots, n-1\},\ \ell \ne k,\ |k - \ell| \ne \frac{n}{2}. \tag{9}$$

For sake of conciseness, let us denote $\tau_j = \frac{c_1 s_j}{c_j}$ for all integers j such that $j \neq \frac{n}{2} \bmod n$. The increasing re-arrangement of the sequence $(\xi_k(z_{k,\ell}))$ gives rise to the antisymmetric sequence of length 2ν

$$(\tau_{-\nu}, \ldots, \tau_{-1}, \tau_1, \ldots, \tau_\nu) = (-\tau_\nu, \ldots, -\tau_1, \tau_1, \ldots, \tau_\nu).$$

We notice that the sequence of abscissas along $\mathcal{L}_k$ does not depend on k. Now, for each $j \in \{2, \ldots, \nu - 1\}$, the closest points of τ_j in this sequence are τ_{j-1} and τ_{j+1}. This amounts to saying that in the geometric graph $\mathcal{R}(n)^b$, the closest neighbours of $z_{k,\ell}$ on $\mathcal{L}_k$ are thus $z_{k,\ell-1}$ and $z_{k,\ell+1}$. So we are led to the crucial fact:

Lemma 3.1. *Let k, ℓ two integers in $\{0, \ldots, n-1\}$ such that $k \neq \ell$ and $|k - \ell| \neq \frac{n}{2}$, then the four closest neighbours of $z_{k,\ell}$, if applicable, are $z_{k-1,\ell}$, $z_{k+1,\ell}$, $z_{k,\ell-1}$ and $z_{k,\ell+1}$.*

Proof. Indeed, the indices p, q of a neighbour $z_{p,q}$ of a given point $z_{k,\ell}$ must satisfy $\{k, \ell\} \cap \{p, q\} \neq \emptyset$ and the requirement that abscissas $\xi_j(z_{k,\ell})$ and $\xi_j(z_{p,q})$ along the straight line $\mathcal{L}_j$ containing $z_{k,\ell}$ and $z_{p,q}$ cannot be intertwinned by any other value $\xi_k(z_{s,t})$. Since this second condition is equivalent to $\max(|k - \ell|, |p - q|) = 1$, the result of the lemma holds.

If $|k - \ell| = m$, the closest neighbours of the point $z_{k,\ell}$ lie on one of the two circles $\mathcal{C}_{m-1}$ and $\mathcal{C}_{m+1}$. It amounts to saying that two adjacent points $z_{k,\ell}$ and $z_{p,q}$ in the geometric graph $\mathcal{R}(n)^b$ lie on two successive circles. In the (k, ℓ)-representation, the geometric graph $\mathcal{R}(n)^b$ is mapped to a "diamond lattice" (see Figure 2). The opposite vertical sides correspond to the half-lines $\theta = 0$ and $\theta = 2\pi$ and must be identified as usual to catch $\mathbb{R}^* \times [0, 2\pi]/\sim$ as $\mathbb{R}^2 - \{(0,0)\}$. This explains the distinct oblique crossing edges.

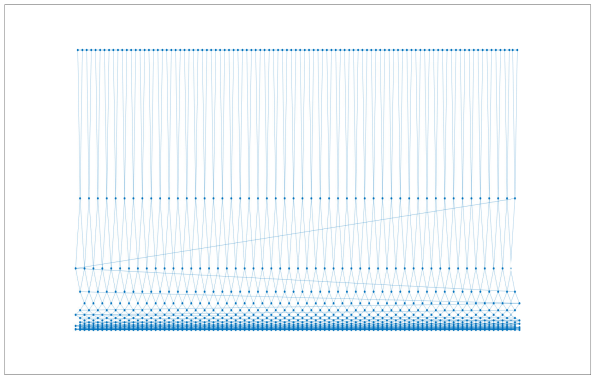

FIGURE 2. "Diamond lattice" associated to $\mathcal{R}(50)$

Let us give three consequences of the Lemma 3.1. First, the adjacency matrix contains two upper and lower triangular arrays of size $n(\nu - 1)$ consisting of zero blocks 0_n located at entries (i, j) with $2 \leq |i - j| \leq \nu$. Second, if two points $z_{k,\ell}$ and $z_{p,q}$ lie on the same circle, i.e. $|z_{k,\ell}| = |z_{p,q}| = r_m$ with $m \geq 2$, they are not adjacent. In other words, two points $z_{k,\ell}$ and $z_{p,q}$,

neither at infinity nor on the unit circle, such that $|k-\ell| \neq |p-q|$ and which are connected, do not lie on a same circle $\mathcal{C}_m$. Indeed, otherwise, we would have for instance $|k-\ell| = |p-q| = m$, and $\{k,\ell\} \cap \{p,q\} \neq \emptyset$ which is impossible. As a consequence, the adjacency matrix (4) has zero blocks 0_n at entries (i,i) with $2 \leq i \leq \nu$. Third and last result, since each point $z_{k,\ell}$ lying on some circle $\mathcal{C}_m$ $(2 \leq m \leq \nu-1)$ has valency equal to 4, we obtain the blocks $P+I_n$ and P^T+I_n in (4) respectively located at positions $(m,m+1)$ and $(m-1,m)$, which highlights the connectivity between the various four neighbours.

So far, we have specified the locations of the different intersection points of the arrangement and discussed about their closest neighbours. At this point, the structure of the adjacency matrix A defined by (4) has been totally detailed and explained. The next step consists in characterizing the chambers generated by the arrangement and, in order to do this, we shall repetitively use the following trick:

Scholia 3.1. *To prove that a convex subset $\mathcal{Y}$ of $\mathcal{A}^b$ is a chamber, we proceed as follows. We give the finite sequence of vertices $(v_1, v_2, \ldots, v_p, v_1)$ of $\mathcal{Y}$ then, help to the crucial lemma 3.1 and its consequences, we verify that $\forall j$, $\{v_j, v_{j+1}\}$ shares one index k so that $\{v_j, v_{j+1}\} \subset \mathcal{L}_k$, and lastly that no straight line $\mathcal{L}_i$ cuts $\mathcal{Y}$.*

Let $k \in \{0,1,\ldots,n\}$, then the triangle with vertices $\{z_k, z_{k+1}, z_{k-1,k+1}\}$ is a chamber. Indeed, this triangle exists, that is to say all segments of the boundary are included in straight lines $\mathcal{L}_j$. Because of the indices of its vertices, the convex hull $\mathrm{Conv}(\{z_k, z_{k+1}, z_{k-1,k+1}\})$ does not contain any auxiliary vertex, so that no straightline $\mathcal{L}_j$ cuts this triangle. Then we obtain, by rotation, n compact triangles as chambers of $\mathcal{R}(n)^b$.

Let k,ℓ be two integers such that $k,\ell \in \{0,\ldots,n-1\}$, $k \neq \ell$ and $|k-\ell| \neq \frac{n}{2}$ then the quadrilateral $\mathrm{Conv}(\{z_{k,\ell}, z_{k+1,\ell}, z_{k+1,\ell-1}, z_{k,\ell-1}\})$ is a chamber. Indeed, this quadrilateral exists in $\mathcal{R}(n)$ and no straightline $\mathcal{L}_j$ cuts this quadrilateral because of the indices of its vertices. While the requirement $|k-\ell| \neq \frac{n}{2}$ avoids singular system when determining $\mathcal{L}_k \cap \mathcal{L}_\ell$, the two following conditions $|k-\ell| \geq 1$ and $|(k+1)-(\ell-1)| \leq \nu$ ensure the existence of the four connected edges that characterize the quadrilateral. The first condition being obviously satisfied, we easily prove that the number of those quadrilaterals is equal to $n(\nu-2)$.

Let $k,\ell \in \{0,\ldots,n-1\}$ such that $|k-\ell| = \nu$, then $z_{k,\ell} \in \mathcal{C}_\nu$. According to previous notation, let $\mathcal{C}_R \cap \mathcal{L}_k = \{R'_k, R''_k\}$ for all k. Straightforward computations show that the coordinates of these two points are equal to

$$\left(c_1 c_{2k+1} + \varepsilon s_{2k+1}\sqrt{R^2 - c_1^2}, c_1 s_{2k+1} - \varepsilon c_{2k+1}\sqrt{R^2 - c_1^2}\right),$$

where $\varepsilon \in \{-1,1\}$. We define R'_k as the closest neighbour of $z_{k,\ell}$ on $\mathcal{C}_R$ along $\mathcal{L}_k$, that is to say

$$|\xi_k(R'_k) - \xi(z_{k,\ell})| < |\xi_k(R''_k) - \xi(z_{k,\ell})|, \tag{10}$$

where $\xi(z_{k,\ell})$ is given by (9), and we choose R'_ℓ similarly. We remind that R'_k and R'_ℓ are two points that belong necessarily to $\{R_1,\ldots,R_{2n}\}$. Since

$R > c_1$, there exists $\beta \in [0, \frac{n}{2}]$ such that $\frac{c_1}{R} = c_\beta$. We may prove that

$$\forall R > \frac{c_1}{c_\nu},\ 0 < \beta - \nu = \frac{n}{\pi} \arccos\left(\frac{c_1}{R}\right) - \left[\frac{n-1}{2}\right] < 1. \tag{11}$$

This results from applying the cosinus mapping to $c_{\nu+1} < 0 < \frac{c_1}{R} < c_\nu$. Next we may state that for any couple of indices (k, ℓ) such that $|k - \ell| = \nu$, the euclidean distance $d_{k,\varepsilon}$ between $z_{k,\ell}$ and R'_k or R''_k may be expressed as

$$d^2_{k,\varepsilon} = \left(\frac{c_1}{c_\nu}\right)^2 + R^2 - 2\left(\frac{c_1}{c_\nu}\right) R c_{\beta+\varepsilon\nu} = \left(\frac{R}{c_\nu}\right)^2 s^2_{\beta+\varepsilon\nu},$$

while the distance $d_{\varepsilon,\ell}$ between $z_{k,\ell}$ and R'_ℓ or R''_ℓ writes as

$$d^2_{\varepsilon,\ell} = \left(\frac{c_1}{c_\nu}\right)^2 + R^2 - 2\left(\frac{c_1}{c_\nu}\right) R c_{\beta-\varepsilon\nu} = \left(\frac{R}{c_\nu}\right)^2 s^2_{\beta-\varepsilon\nu} = d^2_{\ell,-\varepsilon}.$$

Since $\beta\frac{\pi}{n}$ and $\nu\frac{\pi}{n}$ belong to $[0, \frac{\pi}{2}]$, $s_{\beta+\nu} - s_{\beta-\nu} = 2c_\beta s_\nu > 0$. Because of (11), $s_{\beta-\nu} > 0$ and thus, $d^2_{k,1} > d^2_{k,-1}$. It occurs that the distances $d_{k,\varepsilon}$ and $d_{\varepsilon,\ell}$ are minimal if and only if ε is equal to -1 and 1 respectively. Therefore,

$$R'_k = \begin{pmatrix} c_1 c_{2k+1} - s_{2k+1}\sqrt{R^2 - c_1^2} \\ c_1 s_{2k+1} + c_{2k+1}\sqrt{R^2 - c_1^2} \end{pmatrix} = R \begin{pmatrix} c_{\beta+(2k+1)} \\ s_{\beta+(2k+1)} \end{pmatrix} = Re^{I(\beta+(2k+1))\frac{\pi}{n}}$$

and

$$R''_k = \begin{pmatrix} c_1 c_{2k+1} + s_{2k+1}\sqrt{R^2 - c_1^2} \\ c_1 s_{2k+1} - c_{2k+1}\sqrt{R^2 - c_1^2} \end{pmatrix} = R \begin{pmatrix} c_{\beta-(2k+1)} \\ s_{\beta-(2k+1)} \end{pmatrix} = Re^{I(\beta-(2k+1))\frac{\pi}{n}}.$$

The non-compact triangle with vertices $z_{k,\ell}$, R'_k and R'_ℓ exists and is a chamber. Indeed, if a straight line $\mathcal{L}_j$ cut this triangle, we would have an intersection point beyond $\mathcal{C}_\nu$ and this is impossible. So R'_k and R'_ℓ are closest neighbours on the circle $\mathcal{C}_R$ and the polar angles of the two points R'_k and R'_ℓ are consecutive terms of the sequence $(\eta_1, \ldots, \eta_{2n})$. At this point, since $k \in \{0, \ldots, n-1\}$ determines ℓ, we obtain n such non-compact triangles.

Next, let us consider $z_{k,\ell}$ such that $|k - \ell| = \nu - 1$, then we know that $z_{k+1,\ell}$ and $z_{k,\ell-1}$ belong to $\mathcal{C}_\nu$. As before, we define $R'_{k+1} \in \mathcal{C}_R \cap \mathcal{L}_{k+1}$ and $R'_{\ell-1} \in \mathcal{C}_R \cap \mathcal{L}_{\ell-1}$ as the closest neighbours of $z_{k+1,\ell}$ and $z_{k,\ell-1}$ on $\mathcal{C}_R$ along $\mathcal{L}_{k+1}$ and $\mathcal{L}_{\ell-1}$ respectively. Then, the non-compact pentagon

$$\mathrm{Conv}(\{z_{k,\ell}, z_{k+1,\ell}, \widehat{R'_{k+1}R'_{\ell-1}}, z_{k,\ell-1}\})$$

is a chamber. Indeed, by the crucial fact mentioned above, no straight line $\mathcal{L}_j$ cuts this pentagonal region. Therefore, R'_{k+1} and $R'_{\ell-1}$ are closest neighbours on $\mathcal{C}_R$ and lie in separate classes $\mathcal{C}_{R,\nu+1}$ or $\mathcal{C}_{R,\nu+2}$. So, since n vertices lie on $\mathcal{C}_{\nu-1}$, we obtain n non-compact pentagonal regions.

At last, taking into account the central n-gon, we see that the enumeration of chambers is complete. Let us note to conclude that Robert's formula (2) allows to write that

$$1 + n + \binom{n}{2} - 0 - n\left(\frac{n-1}{2} - \nu\right) = n(\nu + 1) + 1,$$

which states a mathematical equivalence between the number of connected components of $\mathbb{R}^2 - \mathcal{R}(n)$ and the amount of chambers given in the table of Theorem 3.1.

Remark 3.1. *Although the previous theorem holds only for $n \geq 5$, we may state some results for $n = 3$ and $n = 4$ by invoking simple geometric considerations.*

- $\mathbb{R}^2 - \mathcal{R}(3)$ *consists in one compact triangle, 3 non-compact triangles and 3 non-compact quadrilaterals.*
- $\mathbb{R}^2 - \mathcal{R}(4)$ *consists in one compact quadrilateral, 4 non-compact triangles and 4 non-compact quadrilaterals.*

Remark 3.2. *Let $s_n = n(\nu + 1) + 1$ be the total number of chambers of $\mathcal{R}(n)^b$. This sequence is referred as A249333 in* [6] *and is attributed to Richard Stanley. In contrast, the specific sequence which counts the number of quadrilaterals in $\mathcal{R}(n)^b$ is not referenced in OEIS.*

Remark 3.3. *The angular diameters of the circular arcs bounding "at infinity" any non-compact triangle and any non-compact pentagon of $\mathcal{R}(n)$ are respectively equal to*

$$\beta_T = \widehat{R'_{k+1}OR_\ell} = \frac{2\pi}{n}(\nu - \beta) \text{ and } \beta_P = \widehat{R'_{\ell-1}OR'_{k+1}} = \frac{2\pi}{n}(\nu + 1 - \beta).$$

We may then observe that the sequence of angles $(\eta_{j+1} - \eta_j)$ is a 2-periodic sequence with terms alternatively equal to β_T and β_P. More accurately, we have

$$\eta_{2p} - \eta_{2p-1} = \begin{cases} \beta_P & \text{if } n \equiv 0, 3 \\ \beta_T & \text{if } n \equiv 1, 2 \end{cases} \text{ and } \eta_{2p+1} - \eta_{2p} = \begin{cases} \beta_T & \text{if } n \equiv 0, 3 \\ \beta_P & \text{if } n \equiv 1, 2 \end{cases}$$

for all integers n computed modulo 4 and for all convenient indices p.

Remark 3.4. *Instead of using the adjacency matrix A, we could use a boolean mapping $B : V(\mathcal{R}(n)^b)^2 \to \{0, 1\}$ detecting the connectivity of the pairs of vertices in the geometric graph $\mathcal{R}(n)^b$. For the "finite" points, we would have*

$$B((\ell - 1, \ell); (\ell' - 1, \ell')) = 1 \text{ iff } |\ell - \ell'| = 1,$$
$$B((k, \ell); (k', \ell')) = 1 \text{ iff } |k - k'| + |\ell - \ell'| = 1, \ |k - \ell| \geq 1, \ |k' - \ell'| \geq 1.$$

Unfortunately, the $2n$ points at infinity are difficult to handle because they involve transcendental extraneous conditions and not only diophantine ones.

4. About the areas of the chambers

In this last section, we give the formulas for the areas of the chambers of the space $\mathcal{R}(n)^b$. Let $A_{n,0} = \frac{n}{2}s_2$ be the area of the central regular n-gon. Let us denote by $A_{n,T}$ the area of a compact triangle, by $A_{n,Q}(m)$ the area of a compact quadrilateral of which two vertices lie on $\mathcal{C}_m$, and by $A_{n,P}$ and $A_{n,S}$ the respective areas of the non-compact pentagonal and non-compact triangular chambers. Invoking the distribution given in Theorem 3.1, we have

$$\pi R^2 = A_{n,0} + nA_{n,T} + n\sum_{m=2}^{\nu} A_{n,Q}(m) + n(A_{n,P} + A_{n,S}).$$

The two hand-sides of this equation are polynomials of degree 2 w.r.t. R, provided $R > r_\nu$, while $A_{n,P}$ and $A_{n,S}$ are algebraic functions of R. In

the following, we repetitively use the rotation ρ to deduce from one particular calculus, the area of congruent chambers of $\mathcal{R}(n)^b$. Using geometric considerations, we may provide the following interesting result

Theorem 4.1. *For all integers $n \geq 5$, we have*

$$\sum_{m=2}^{\nu-1} \frac{\sin\left(\frac{m\pi}{n}\right)}{\cos\left((m-1)\frac{\pi}{n}\right)\cos\left(m\frac{\pi}{n}\right)\cos\left((m+1)\frac{\pi}{n}\right)}$$

$$= \frac{1}{2\sin\left(\frac{\pi}{n}\right)}\left(\frac{1}{\cos\left(\frac{(\nu-1)\pi}{n}\right)\cos\left(\frac{\nu\pi}{n}\right)} - \frac{1}{\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)}\right). \tag{12}$$

Proof. The proof proceeds from the exact computation of the explicit values of the areas found in the regular n-gon homothetic to the central regular n-gon of ratio $r_\nu = \frac{c_1}{c_\nu}$. This regular n-gon is partitioned in a series of convex polygons with pairwise disjoint interiors, including the central regular n-gon, the compact triangles, the whole sequence of compact quadrilaterals, as well as the triangular parts $A_{n,P,1}$ extracted from the non-compact pentagonal chambers and characterized exclusively by some vertices $z_{k,\ell}$. All these terms will be defined shortly and allow to state

$$A_{n,0}r_\nu^2 = A_{n,0} + nA_{n,T} + n\sum_{m=2}^{\nu-1} A_{n,Q}(m) + nA_{n,P,1}, \tag{13}$$

or what amounts to the same thing

$$\sum_{m=2}^{\nu-1} A_{n,Q}(m) = \frac{s_2}{2}(r_\nu^2 - 1) - A_{n,T} - A_{n,P,1}. \tag{14}$$

As will be explained below, this formula is the exact replica of the result mentioned in the theorem up to a multiplicative factor $\frac{2}{ns_2^2}$.

So, to begin, the area of a triangle is given as $\frac{1}{2}|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$ when the coordinates of all vertices are known. We consider the triangle $\mathrm{Conv}(\{z_k, z_{k+1}, z_{k-1,k+1}\})$, see Figure 3, and we deduce from its area, the area of any compact triangle

$$A_{n,T} = \frac{s_1^2 s_2}{c_2}. \tag{15}$$

Now, let m be an integer such that $2 \leq m \leq \nu - 1$, and let us consider the quadrilateral in $\mathcal{R}(n)^b$ whose vertices $z_{k,\ell}$, $z_{k+1,\ell}$, $z_{k,\ell-1}$, $z_{k+1,\ell-1}$ are such that $|k + 1 - \ell| = m$, see Figure 4.

As seen previously, two vertices lie on $\mathcal{C}_m$, one is on $\mathcal{C}_{m-1}$ and the last is on $\mathcal{C}_{m+1}$. Since the polar angles of $z_{k,\ell}$ and $z_{k+1,\ell-1}$ are equal, the straight line Δ' joining $z_{k,\ell}$ and $z_{k+1,\ell-1}$ passes through the origin. Furthermore, the straight line Δ'' from $z_{k+1,\ell}$ to $z_{k,\ell-1}$ is orthogonal to Δ'. The orthogonal symmetry w.r.t the straight line Δ'' shows that the area $A_{n,Q}(m)$ is the half of the area of the rectangle with two sides parallel to straight line Δ'' and having on its boundary the four vertices $z_{k,\ell}$, $z_{k+1,\ell}$, $z_{k,\ell-1}$, $z_{k+1,\ell-1}$. By

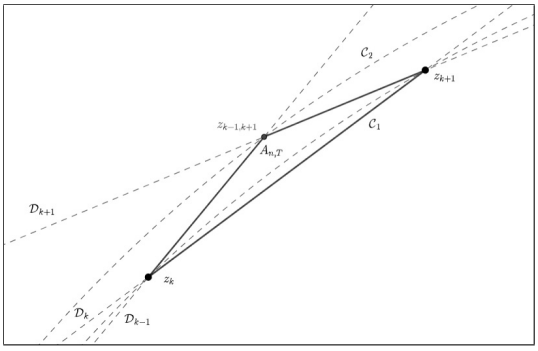

FIGURE 3. Area $A_{n,T}$ of a compact triangle

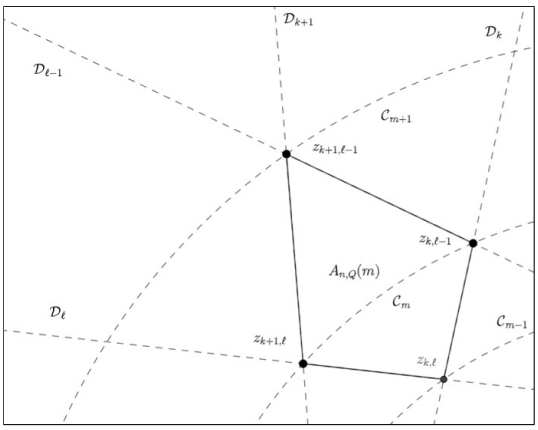

FIGURE 4. Area $A_{n,Q}(m)$ of a compact quadrilateral

the way, the length of one side of this rectangle is $r_{m+1} - r_{m-1} = \frac{s_2 s_m}{c_{m+1}c_{m-1}}$ while the length of the other side is given by the distance between $z_{k+1,\ell}$ and $z_{k,\ell-1}$ that is to say $\frac{s_2}{c_m}$. Hence we get the area

$$A_{n,Q}(m) = \frac{s_2^2 s_m}{2c_{m-1}c_m c_{m+1}}, \tag{16}$$

with the usual restriction on the integer m characterizing the three circles $\mathcal{C}_{m-1}$, $\mathcal{C}_m$, $\mathcal{C}_{m+1}$, i.e. $m, m \pm 1 \neq \frac{n}{2}$. Next, we may state that the area of $\mathrm{Conv}(\{z_{\ell+\nu-1,\ell}, z_{\ell+\nu,\ell}, z_{\ell+\nu-1,\ell-1}\})$, see Figure 5, is equal to

$$A_{n,P,1} = \frac{c_1^2 s_1^2 s_{\nu-1}}{c_\nu^2 c_{\nu-1}}. \tag{17}$$

Gathering together (14), (15), (16) and (17), we get

$$\frac{s_2^2}{2}\sum_{m=2}^{\nu-1}\frac{s_m}{c_{m-1}c_m c_{m+1}} = \frac{s_2}{2}\left(\frac{c_1^2}{c_\nu^2} - 1\right) - \frac{s_1^2 s_2}{c_2} - \frac{c_1^2 s_1^2 s_{\nu-1}}{c_\nu^2 c_{\nu-1}}$$

or equivalently

$$\sum_{m=2}^{\nu-1}\frac{s_m}{c_{m-1}c_m c_{m+1}} = \frac{c_1^2}{s_2 c_\nu^2} - \frac{1}{s_2} - \frac{2s_1^2}{s_2 c_2} - \frac{2c_1^2 s_1^2 s_{\nu-1}}{s_2^2 c_\nu^2 c_{\nu-1}} = \frac{c_1^2}{s_2 c_\nu^2} - \frac{1}{s_2 c_2} - \frac{c_1 s_1 s_{\nu-1}}{s_2 c_\nu^2 c_{\nu-1}}$$

which provides after a few simplification the formula (12).

Remark 4.1. *The area* $Q_n = \sum_{m=2}^{\nu} A_{n,Q}(m)$ *of the whole cohort of quadrilaterals may be, as a function of* n*, expressed as an interesting algebraic number in the cyclotomic field* $\mathbb{Q}\left(\exp\left(\frac{2I\pi}{n}\right)\right)$*. For* $n = 5$ *and* $n = 6$*, the two sums are void and thus are equal to 0. One obtains also for example*

$$Q_8 = \frac{1}{2}\sqrt{2},\ Q_{12} = \frac{\sqrt{3}}{6} + \left(\frac{5}{6}\sqrt{6} - \frac{\sqrt{2}}{2}\right)\cos\frac{\pi}{12},$$

while Q_{16} *is an explicit rational expression involving surds and* $\cos\frac{\pi}{8}$*.*

We provide now a synopsis of the areas of the chambers of $\mathcal{R}(n)^b$.

Theorem 4.2. *Notation being as above, the areas of the different chambers of* $\mathcal{R}(n)^b$ *and their asymptotics as* n *tends to* $+\infty$ *are given by*

Polygon	*Area of the polygon*	*Asymptotic equivalent*
$A_{n,0}$	$\frac{n}{2}s_2$	π
$A_{n,T}$	$\frac{s_1^2 s_2}{c_2}$	$\frac{2\pi^3}{n^3}$
$A_{n,Q}(m)$	$\frac{s_2^2 s_m}{2c_{m-1}c_m c_{m+1}}$	$\frac{2m\pi^3}{n^3}$ *(m fixed)*
Q_n	$\frac{s_2^2}{4s_1}\left(\frac{1}{c_{\nu-1}c_\nu} - \frac{1}{c_1 c_2}\right)$	$\frac{n}{2\pi}$ *if n is even* $\frac{4n}{3\pi}$ *if n is odd*
$A_{n,P,1}$	$\frac{c_1^2 s_1^2 s_{\nu-1}}{c_\nu^2 c_{\nu-1}}$	$\frac{n}{2\pi}$ *if n is even* $\frac{8n}{3\pi}$ *if n is odd*
$A_{n,P,2}$	$\left(Rs_{\nu+1-\beta} + \frac{c_1 s_1}{c_\nu}\right) R\frac{s_{\nu+1}s_{\beta-\nu}}{c_\nu}$	$2R - \frac{2n}{\pi}$ *if n is even* $\frac{\pi R^2}{2n} - \frac{6n}{\pi} + 2R$ *if n is odd*
$A_{n,P,3}$	$R^2(\nu+1-\beta)\frac{\pi}{n} - \frac{R^2}{2}s_{2(\nu+1-\beta)}$	$\frac{4}{3R}$ *if n is even* $\frac{4}{3}R^2\left(\frac{\pi}{n}\right)^3\left(\frac{1}{2} + \frac{n}{\pi R}\right)^3$ *if n is odd*
$A_{n,S,1}$	$R^2 s_{\beta-\nu}^2 \frac{s_\nu}{c_\nu}$	$\frac{\pi R^2}{n} - 2R + \frac{n}{\pi}$ *if n is even* $\frac{\pi R^2}{2n} - 2R + \frac{2n}{\pi}$ *if n is odd*
$A_{n,S,2}$	$R^2(\beta-\nu)\frac{\pi}{n} - \frac{R^2}{2}s_{2(\beta-\nu)}$	$\frac{4}{3}R^2\left(\frac{\pi}{n}\right)^3\left(1 - \frac{n}{\pi R}\right)^3$ *if n is even* $\frac{4}{3}R^2\left(\frac{\pi}{n}\right)^3\left(\frac{1}{2} - \frac{n}{\pi R}\right)^3$ *if n is odd*

TABLE 2. Areas of the chambers of $\mathcal{R}(n)^b$ and their asymptotics as n tends to $+\infty$

Proof. The first half of the results presented in this array has already been proved in the previous theorem. Before embarking on the asymptotic expansions of these areas, it remains to focus on the areas of the non-compact chambers which are characterized by vertices lying on the two last orbits $\mathcal{C}_{\nu-1}$, $\mathcal{C}_\nu$ and the circle $\mathcal{C}_R$ and that we formalize below through the relationships $k = \ell + \nu - 1$, $0 \le \ell < \nu$, $k > \ell$.

In order to determine the area $A_{n,P}$ of any pentagon, all others being congruent modulo ρ and thus having the same area, we split it into

$$\begin{cases} \text{a triangle} & \mathrm{Conv}(\{z_{\ell+\nu-1,\ell}, z_{\ell+\nu,\ell}, z_{\ell+\nu-1,\ell-1}\}), \\ \text{a quadrilateral} & \mathrm{Conv}(\{z_{\ell+\nu,\ell}, z_{\ell+\nu-1,\ell-1}, R'_{\ell-1}, R'_{\ell+\nu}\}), \\ \text{a disk segment} & R'_{\ell+\nu} \frown R'_{\ell-1}. \end{cases}$$

The area $A_{n,P,1}$ of $\mathrm{Conv}(\{z_{\ell+\nu-1,\ell}, z_{\ell+\nu,\ell}, z_{\ell+\nu-1,\ell-1}\})$, see Figure 5, has already been computed and is given in (17). Let us remind next that the intersection points $R'_{\ell-1}$ and $R'_{\ell+\nu}$ obtained by intersecting $\mathcal{C}_R$ with the straight lines $\mathcal{L}_{\ell-1}$ and $\mathcal{L}_{\ell+\nu}$ have the following coordinates

$$R'_{\ell-1} = R(c_{2\ell-1+\beta}, s_{2\ell-1+\beta}) \text{ and } R'_{\ell+\nu} = R(c_{2\ell+2\nu+1-\beta}, s_{2\ell+2\nu+1-\beta}).$$

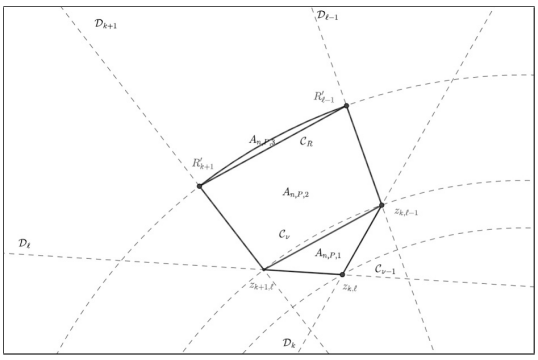

FIGURE 5. Area $A_{n,P}$ of a non-compact pentagon

We remark next that the quadrilateral $\mathrm{Conv}(\{z_{\ell+\nu,\ell}, z_{\ell+\nu-1,\ell-1}, R'_{\ell-1}, R'_{\nu+\ell}\})$ is a trapezoid since the straight lines $(z_{\ell+\nu,\ell}z_{\ell+\nu-1,\ell-1})$ and $(R'_{\ell-1}R'_{\ell+\nu})$ are parallel, their slope being equal to $-\frac{c_{2\ell+\nu}}{s_{2\ell+\nu}}$. In order to determine the height between these two bases, we note that the origin O of the coordinate system and the respective middles P_1 and P_2 of the sides $z_{\ell+\nu,\ell}z_{\ell+\nu-1,\ell-1}$ and $R'_{\ell-1}R'_{\ell+\nu}$ are colinear. Indeed,

$$P_1 = \frac{c_1^2}{c_\nu}(c_{2\ell+\nu}, s_{2\ell+\nu}) \text{ and } P_2 = Rc_{\nu+1-\beta}(c_{2\ell+\nu}, s_{2\ell+\nu}).$$

Therefore, the height of the trapezoid is simply equal to the euclidean distance between P_1 and P_2 that is to say

$$0 < \mathrm{dist}(P_1, P_2) = \frac{c_1^2}{c_\nu} - Rc_{\nu+1-\beta} = R\frac{s_{\nu+1}s_{\beta-\nu}}{c_\nu},$$

the positive sign being obtained using (11). We may then compute the lengths of the two bases $z_{\ell+\nu,\ell}z_{\ell+\nu-1,\ell-1}$ and $R'_{\ell-1}R'_{\ell+\nu}$ which are respectively equal to $2\frac{c_1 s_1}{c_\nu}$ and $2(c_1 s_{\nu+1} - c_{\nu+1}\sqrt{R^2 - c_1^2}) = 2Rs_{\nu+1-\beta}$. We deduce that

$$A_{n,P,2} = \left(Rs_{\nu+1-\beta} + \frac{c_1 s_1}{c_\nu}\right) R\frac{s_{\nu+1}s_{\beta-\nu}}{c_\nu}.$$

In order to determine the area of the circular segment $R'_{\ell+\nu} \frown R'_{\ell-1}$, we use Al-Kashi's law of cosines which states that the distance between $R'_{\ell-1}$

and $R'_{\ell+\nu}$ checks the relationship

$$\operatorname{dist}(R'_{\ell-1}, R'_{\ell+\nu})^2 = 4R^2 s^2_{\nu+1-\beta} = 2R^2 - 2R^2 \cos \widehat{R'_{\ell+\nu} O R'_{\ell-1}}$$

which yields

$$\widehat{R'_{\ell+\nu} O R'_{\ell-1}} = \arccos\left(1 - 2s^2_{\nu+1-\beta}\right) = \arccos c_{2(\nu+1-\beta)} = 2(\nu+1-\beta)\frac{\pi}{n}.$$

Then,

$$A_{n,P,3} = R^2(\nu+1-\beta)\frac{\pi}{n} - \frac{R^2}{2} s_{2(\nu+1-\beta)}.$$

Using the distinct areas presented above, we find

(18)

$$A_{n,P} = \frac{c_1^2 s_1^2 s_{\nu-1}}{c_\nu^2 c_{\nu-1}} + R^2\left(\left(s_{\nu+1-\beta} + \frac{c_\beta s_1}{c_\nu}\right)\frac{s_{\nu+1} s_{\beta-\nu}}{c_\nu} + \left((\nu+1-\beta) - \frac{1}{2} s_{2(\nu+1-\beta)}\right)\right).$$

The last step consists in computing the areas of the non-compact triangles which are all identical and may identified to $\operatorname{Conv}(\{z_{\ell+\nu,\ell}, R'_\ell, R'_{\ell+\nu}\})$, see Figure 6.

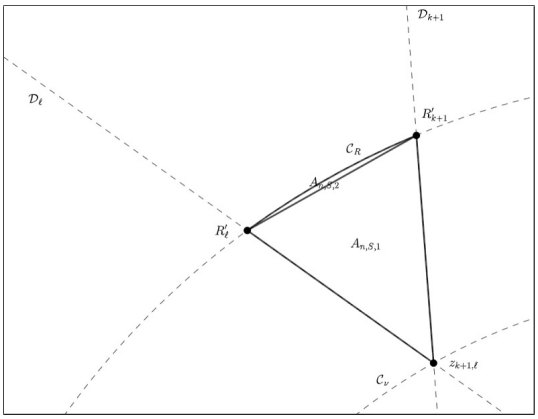

FIGURE 6. Area $A_{n,S}$ of a non-compact triangle

By introducing the middle point $P = Rc_{\nu-\beta}(c_{2\ell+\nu+1}, s_{2l+\nu+1})$ of the side $R'_\ell R'_{\ell+\nu}$, it is obvious that O, $z_{\ell+\nu,\ell} = \frac{c_1}{c_\nu}(c_{2\ell+\nu+1}, s_{2l+\nu+1})$ and P are colinear, which means that the area $A_{n,S,1}$ of the triangular part of $\operatorname{Conv}(\{z_{\ell+\nu,\ell}, R'_\ell, R'_{\ell+\nu}\})$ is twice the area of the triangle $\operatorname{Conv}(\{z_{\ell+\nu,\ell}, R'_\ell, P\})$. We deduce easily that

$$A_{n,S,1} = \frac{1}{2}(2Rs_{\beta-\nu})\left(\frac{R}{c_\nu} s_{\beta-\nu} s_\nu\right) = R^2 s^2_{\beta-\nu}\frac{s_\nu}{c_\nu}.$$

At last, we proceed as before to determine the area of the circular segment $R'_{\ell+\nu} \frown R'_\ell$, i.e.

$$A_{n,S,2} = R^2(\beta-\nu)\frac{\pi}{n} - \frac{R^2}{2} s_{2(\beta-\nu)}.$$

Therefore,

$$A_{n,S} = R^2 s^2_{\beta-\nu}\frac{s_\nu}{c_\nu} + R^2(\beta-\nu)\frac{\pi}{n} - \frac{R^2}{2} s_{2(\beta-\nu)}. \tag{19}$$

It remains then to prove the asymptotics. First we have $R > \frac{n}{\pi}$ since $R > \frac{c_1}{c_\nu}$, then when n tends to $+\infty$, so does R. Next we use the following simple results

Quantities	n even	n odd
ν	$\frac{n}{2}-1$	$\frac{n}{2}-1$
c_ν	$\frac{\pi}{n}$	$\frac{\pi}{2n}$
r_ν	$\frac{n}{\pi}$	$\frac{2n}{\pi}$
$\beta-\nu$	$1-\frac{n}{\pi R}$	$\frac{1}{2}-\frac{n}{\pi R}$
$s_{\beta-\nu}$	$1-\frac{n}{\pi R}$	$\frac{1}{2}-\frac{n}{\pi R}$
$\nu+1-\beta$	$\frac{n}{\pi R}$	$\frac{1}{2}+\frac{n}{\pi R}$
$s_{\nu+1-\beta}$	$\frac{n}{\pi R}$	$\frac{1}{2}+\frac{n}{\pi R}$

TABLE 3. Asymptotics as n tends to $+\infty$ of several parameters

In this way we obtain the asymptotics of all the areas except $A_{n,P,3}$ and $A_{n,S,2}$. To obtain these last ones, we use the asymptotic expansion $R^2(\theta - \frac{1}{2}\sin(2\theta)) \sim \frac{4}{3}R^2\theta^3$. Here, θ tends to 0, R tends to $+\infty$ and θ is chosen as $(\beta-\nu)\frac{\pi}{n}$ or $(\nu+1-\beta)\frac{\pi}{n}$, these two angles being positive and less than $\frac{\pi}{n}$.

Remark 4.2. *Let us consider the ratios of areas in formula (13) w.r.t. the total area πr_ν^2. An interesting feature arises from the asymptotics. The subsequence with n odd or even of the preceding sequence of ratios have the following behaviour*

$$\frac{1}{\pi r_\nu^2}\sum_{m=2}^{\nu-1} A_{n,Q}(m) \underset{+\infty}{\to} \begin{cases} \frac{1}{2}, n \equiv 0 \ mod\ 2 \\ \frac{1}{3}, n \equiv 1 \ mod\ 2 \end{cases}, \ \frac{1}{\pi r_\nu^2} n A_{n,P,1} \underset{+\infty}{\to} \begin{cases} \frac{1}{2}, n \equiv 0 \ mod\ 2 \\ \frac{2}{3}, n \equiv 1 \ mod\ 2 \end{cases}.$$

Remark 4.3. *Let us consider the function $f(t) = \frac{\sin(t)}{\cos(t)^3}$ on $\left[0, \frac{\pi}{2}\right[$ whose integral on this interval is divergent. Let us denote by S_n the left Riemann sum of f over the interval $\left[0, \frac{\pi}{2}\right[$. Using the relationship $s_{m-1}s_{m+1} - s_m^2 = -s_1^2 < 0$, for all $2 \le m \le \nu$, we easily get $s_2^2 S_n \le nQ_{2n}$. Comparison theorem between integrals and Riemann sums for monotonic functions implies that the sums $P_n = \frac{2n}{\pi}S_n$ lie in $\mathcal{O}(n^2)$ as n tends to $+\infty$. It would be interesting to study the link between P_n and $\sum_{m=2}^{\nu-1}\frac{c_m}{s_{m-1}s_m s_{m+1}}$ as well as to obtain the explicit value of the limit of the sequence $\left(\frac{P_n}{n^2}\right)$.*

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