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BEYOND 100 CHARACTERIZATIONS OF TANGENTIAL QUADRILATERALS

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Abstract. This is the fourth part in our extensive study of new characterizations of tangential quadrilaterals, preceded by [7, 8, 9]. Here we shall prove 27 more necessary and sufficient conditions for when a convex quadrilateral can have an incircle, making it 127 known such characterizations.

1. Introduction

There has been a dramatic increase in the number of known characterizations of tangential quadrilaterals in the last decade, as indicated by the diagram in Figure 1. A steady growth culminated with a doubling in 2021, according to statistics from Table 1 in [9].

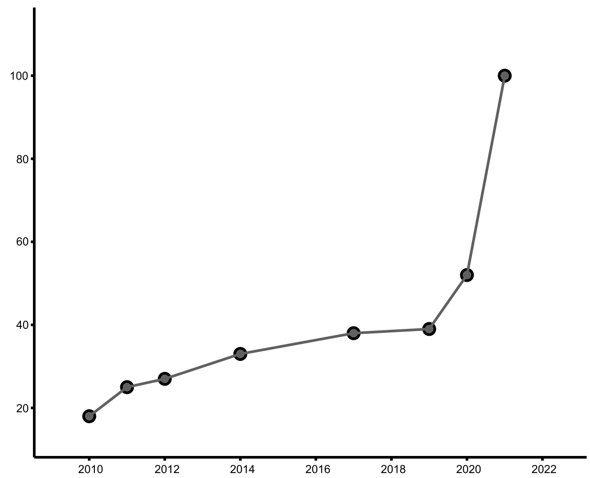

Figure 1. Number of known characterizations the last decade

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In this paper we will continue to prove more new characterizations of *tangential quadrilaterals* (also called circumscribed quadrilaterals), that is, necessary and sufficient conditions for when a convex quadrilateral can have an incircle (an internal circle tangent to all four sides, see Figure 2). Even though there are already 100 characterizations known (see [9]), there does not seem to be an end to the possibility to discover and prove more such conditions. We begin with one that is very basic, and it surprised us that it was not included among those previously known.

Theorem 1.1. *Suppose Q is an arbitrary point inside a convex quadrilateral $ABCD$. Denoting by $\mathscr{P}(ABQ)$ the perimeter of triangle ABQ, then*

$$\mathscr{P}(ABQ) + \mathscr{P}(CDQ) = \mathscr{P}(BCQ) + \mathscr{P}(DAQ)$$

holds if and only if $ABCD$ is a tangential quadrilateral.

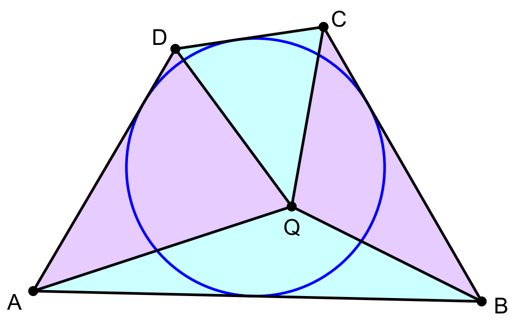

Figure 2. Arbitrary point Q in a tangential quadrilateral

Proof. It is well known that $ABCD$ is tangential if and only if the sides satisfy *Pitot's theorem* (regarding different proofs, see [6])

$$AB + CD = BC + DA. \tag{1}$$

Adding the quantity $AQ + BQ + CQ + DQ$ to both sides (see Figure 2), we get

$$(AB+AQ+BQ)+(CD+CQ+DQ) = (BC+BQ+CQ)+(DA+AQ+DQ)$$

which is equivalent to

$$\mathscr{P}(ABQ) + \mathscr{P}(CDQ) = \mathscr{P}(BCQ) + \mathscr{P}(DAQ)$$

concluding the proof. □

We note that there is nothing preventing the point Q from being an arbitrary external point, so the theorem holds in that case as well. The corresponding condition with area instead of perimeter was proved by V. Pop and I. Gavrea in 1999, but then the point Q cannot be arbitrary but must be the intersection of two opposite internal angle bisectors, which was proved in [11, pp. 134–135].

2. Tangent circles

Theorem 3.1 in [7] and Theorems 4.1 and 4.2 in [9] gave different characterizations of tangential quadrilaterals concerning tangent circles. In this section we prove three more. The first one is so simple that is has previously been overlooked.

Theorem 2.1. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. Then the incircles in triangles ADJ and ABK are tangent to each other on AB if and only if $ABCD$ is a tangential quadrilateral.*

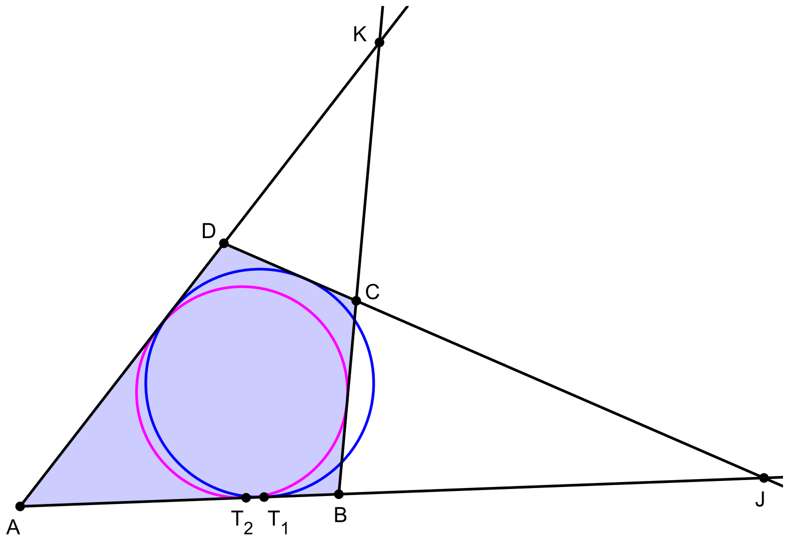

Figure 3. $ABCD$ is tangential $\Leftrightarrow$ $T_1T_2 = 0$

Proof. Suppose the incircles in triangles ADJ and ABK are tangent to AB at T_1 and T_2 respectively (see Figure 3). Using well-known triangle formulas (see Lemma 2.1 in [9], and if they are unknown to the reader, we encourage him/her to prove them before continuing), we have

$$2AT_1 = AJ - JD + DA, \qquad 2AT_2 = AB - BK + KA.$$

The two incircles are tangent at the same point on AB if and only if $T_1 \equiv T_2$, that is $2AT_1 = 2AT_2$, which is equivalent to

$$AJ - JD + DA = AB - BK + KA.$$

This in turn is equivalent to

$$AB + BJ - DJ + DA = AB - BK + KD + DA$$

which is simplified into

$$BJ + BK = DJ + DK. \tag{2}$$

This is *Grossman's second characterization* of a tangential quadrilateral, published in [3] (he calls it $(4')$) and uses other notations. $\square$

Of course there is a similar criterion for tangency on side AD. When $T_1T_2 = 0$ in Figure 3, then the two triangle incircles coincide, and this common incircle is also the incircle in $ABCD$.

Theorem 2.2. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. The incircles in triangles ACJ and ACK are tangent to each other if and only if $ABCD$ is a tangential quadrilateral.*

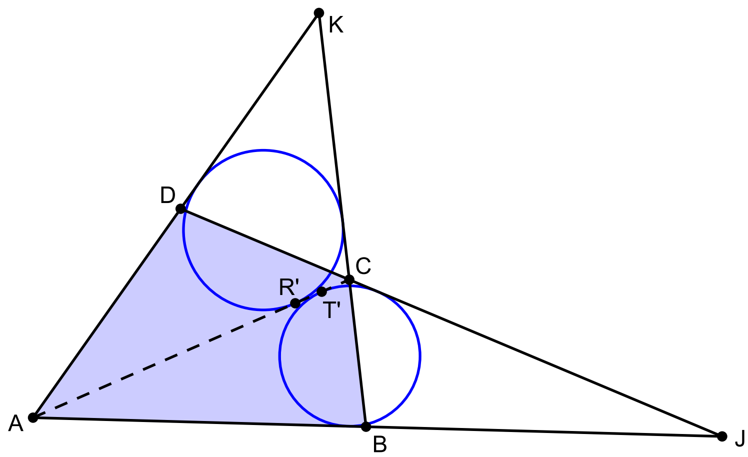

FIGURE 4. $ABCD$ is tangential $\Leftrightarrow$ $R'T' = 0$

Proof. Using well-known triangle formulas, we have (see Figure 4)

$$2AT' = AC + AJ - CJ, \qquad 2AR' = AC + AK - CK$$

where the incircles in triangles ACK and ACJ are tangent to AC at R' and T' respectively, so

$$2(AT' - AR') = AJ - CJ - AK + CK.$$

Hence

$$AT' = AR' \quad \Leftrightarrow \quad AJ - AK = CJ - CK \tag{3}$$

where the equality to the right is *Grossman's first characterization* of a tangential quadrilateral, published in [3] (he calls it (3′)). This proves that the incircles in triangles ACJ and ACK are tangent to each other if and only if $ABCD$ is a tangential quadrilateral. □

The incircles in the previous theorem can be exchanged for the excircles outside of diagonal AC.

Theorem 2.3. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. The excircles to triangles ACJ and ACK that are outside of AC are tangent to each other on AC if and only if $ABCD$ is a tangential quadrilateral.*

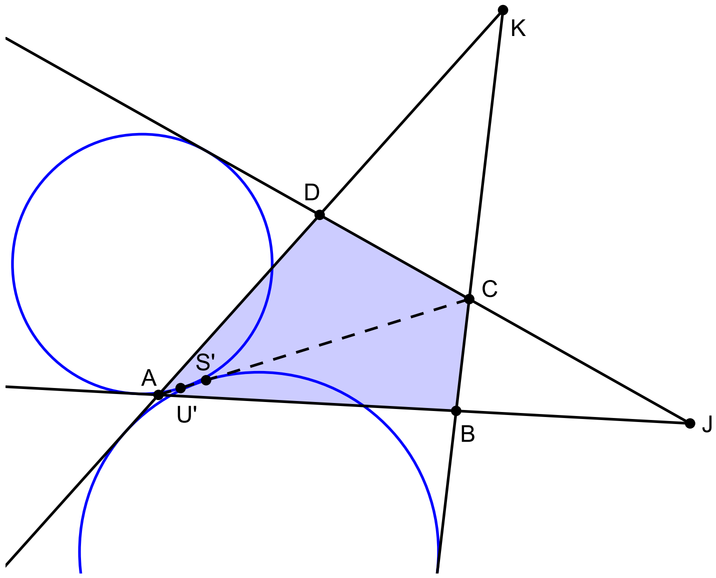

FIGURE 5. $ABCD$ is tangential $\Leftrightarrow$ $S'U' = 0$

Proof. Suppose the excircles to ACK and ACJ are tangent to AC at S' and U' respectively. Using well-known triangle formulas (Lemma 2.1 in [9]), we have (see Figure 5)

$$2AU' = -AJ + CJ + AC, \qquad 2AS' = -AK + CK + AC$$

so

$$2(AU' - AS') = -AJ + CJ + AK - CK.$$

Hence

$$AU' = AS' \quad \Leftrightarrow \quad AJ - AK = CJ - CK$$

where the equality to the right is Grossman's first characterization of a tangential quadrilateral (3). □

3. Equal line segments

In this section we prove nine characterizations for when a convex quadrilateral is tangential expressed in terms of two equal line segments.

Theorem 3.1. *In a convex quadrilateral $ABCD$, let the excircle to triangle ABC that is outside of AC be tangent to the extensions of AB and BC at A_2 and D_2 respectively. If the incircles in triangles ABD and BCD are tangent to AB and BC at B_2 and C_2 respectively, the $A_2B_2 = C_2D_2$ if and only if $ABCD$ is a tangential quadrilateral.*

Proof. Using well-known triangle formulas, we have (see Figure 6)

$$2AB_2 = AB - BD + DA, \qquad 2AA_2 = -AB + BC + CA$$

so

$$2A_2B_2 = 2AA_2 + 2AB_2 = BC + CA - BD + DA.$$

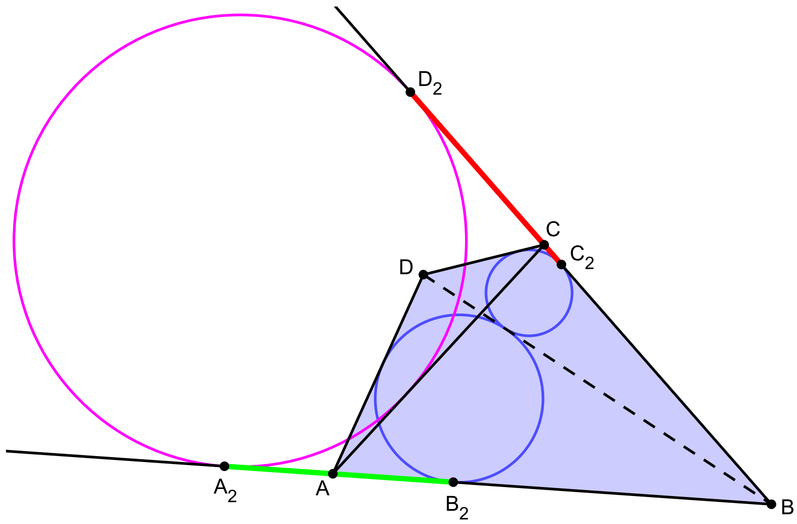

FIGURE 6. $ABCD$ is tangential $\quad\Leftrightarrow\quad A_2B_2 = C_2D_2$

In the same way,

$$2C_2D_2 = AB - DB + CA + CD$$

and we get that

$$A_2B_2 = C_2D_2 \quad \Leftrightarrow \quad BC + DA = AB + CD$$

where the equality to the right is Pitot's theorem (1). □

In the following theorem we have three triangle excircles.

Theorem 3.2. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of D. If the excircles to triangles CJK and BCJ that are outside of CK and BC are tangent to the extension of CD at E_2 and F_2 respectively, and the excircles to triangles AJK and BCJ that are outside of AK and BC are tangent to AB or its extension at H_2 and G_2 respectively, then $E_2F_2 = G_2H_2$ if and only if $ABCD$ is tangential.*

Proof. Suppose the excircles to triangles AJK and CJK are tangent to the extension of JK at O' and O'' respectively, and assume without loss of generality that $KO' > KO''$ when $ABCD$ is not tangential (see Figure 7). Then we have

$$G_2H_2 - E_2F_2 = JG_2 - JH_2 - (JF_2 - JE_2) = JE_2 - JH_2 = JO' - JO''$$

since $JF_2 = JG_2$ according to the *two tangent theorem*[1]. It holds that $JO' = JO''$ if and only if $ABCD$ is tangential according to Theorem 4.1 in [9], which proves this characterization. □

Next we have a theorem about four different incircles.

[1]Two tangents to a circle through an external point have equal lengths.

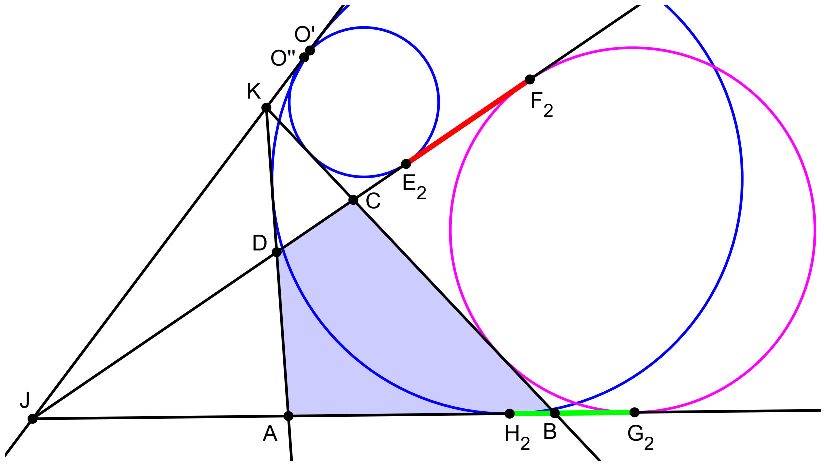

FIGURE 7. $ABCD$ is tangential $\quad\Leftrightarrow\quad E_2F_2 = G_2H_2$

Theorem 3.3. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the incircles in triangles ABC and AJC are tangent to AJ at A_1 and E_1 respectively, and the incircles in triangles ADC and AKC are tangent to AK at D_1 and H_1 respectively, then $A_1E_1 = D_1H_1$ if and only if $ABCD$ is a tangential quadrilateral.*

Proof. With notations as in Figure 8, it holds that

$$2AE_1 = AJ - JC + CA, \qquad 2AA_1 = AB - BC + CA$$

so

$$2A_1E_1 = 2AE_1 - 2AA_1 = AJ - JC - AB + BC = BJ - JC + BC > 0$$

where the inequality is due to the triangle inequality. In the same way

$$2D_1H_1 = DK - KC + CD > 0$$

so we get

$$2(A_1E_1 - D_1H_1) = BJ - JC + BC - DK + KC - CD = BJ - JD - DK + KB.$$

Hence

$$A_1E_1 = D_1H_1 \quad\Leftrightarrow\quad JB + BK = KD + DJ$$

where the equality to the right is Grossman's second characterization of a tangential quadrilateral (2). □

We also have the following characterization regarding the same four incircles.

Theorem 3.4. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the incircles in triangles ABC and AKC are tangent to BK at B_1 and F_1 respectively, and the incircles in triangles ADC and AJC are tangent to DJ at C_1 and G_1 respectively, then $B_1F_1 = C_1G_1$ if and only if $ABCD$ is a tangential quadrilateral.*

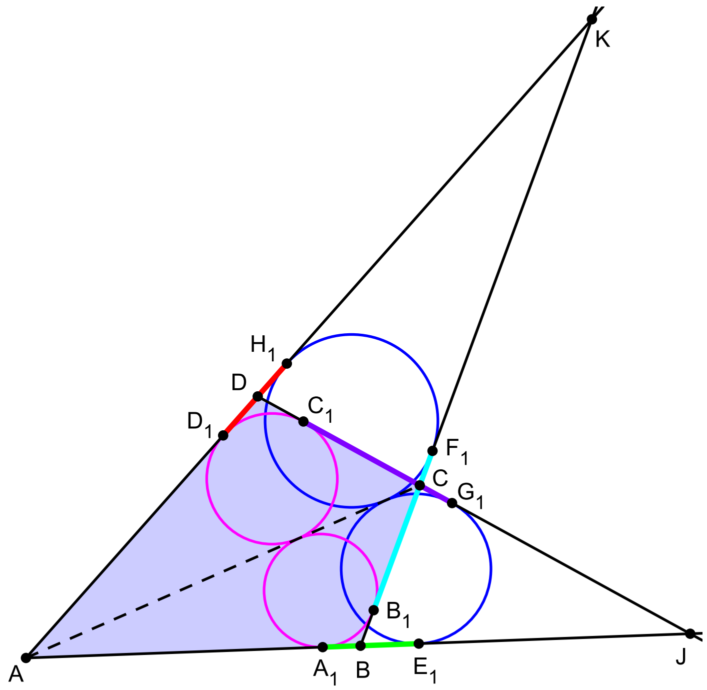

FIGURE 8. $A_1E_1 = D_1H_1 \quad \Leftrightarrow \quad ABCD$ is tangential $\quad \Leftrightarrow \quad B_1F_1 = C_1G_1$

Proof. We get (see Figure 8)

$$2B_1F_1 = 2B_1C + 2CF_1 = -AB + BC + CA + CK - KA + AC$$

and in the same way

$$2C_1G_1 = -AJ + JC + CA - AD + CA + CD.$$

Then

$$\begin{aligned} 2(B_1F_1 - C_1G_1) &= -AB + BC + CK - KA + AJ - JC + AD - CD \\ &= BK - DK + BJ - JD \end{aligned}$$

so

$$B_1F_1 = C_1G_1 \quad \Leftrightarrow \quad BJ + BK = DK + DJ$$

where the right equality is Grossman's second characterization (2). □

In Figures 6 and 8 we see that the incircles in the two subtriangles created by one diagonal are tangent to each other. This is another characterization regarding tangent circles that has been known since 2001, as discussed in [9, p. 38]. An English proof was given in [4, pp. 66–67].

Next we state the corresponding two characterizations with excircles instead of incircles. Their proofs are more or less identical to those of the previous two theorems and are therefor left as exercises for the reader. These theorems are illustrated in Figure 9.

Theorem 3.5. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the excircles to triangles ABC and AKC*

that are outside of AC are tangent to BK or its extension at L_1 and S_1 respectively, and the excircles to triangles ADC and AJC that are outside of AC are tangent to DJ or its extension at M_1 and T_1 respectively, then $L_1S_1 = M_1T_1$ if and only if $ABCD$ is a tangential quadrilateral.

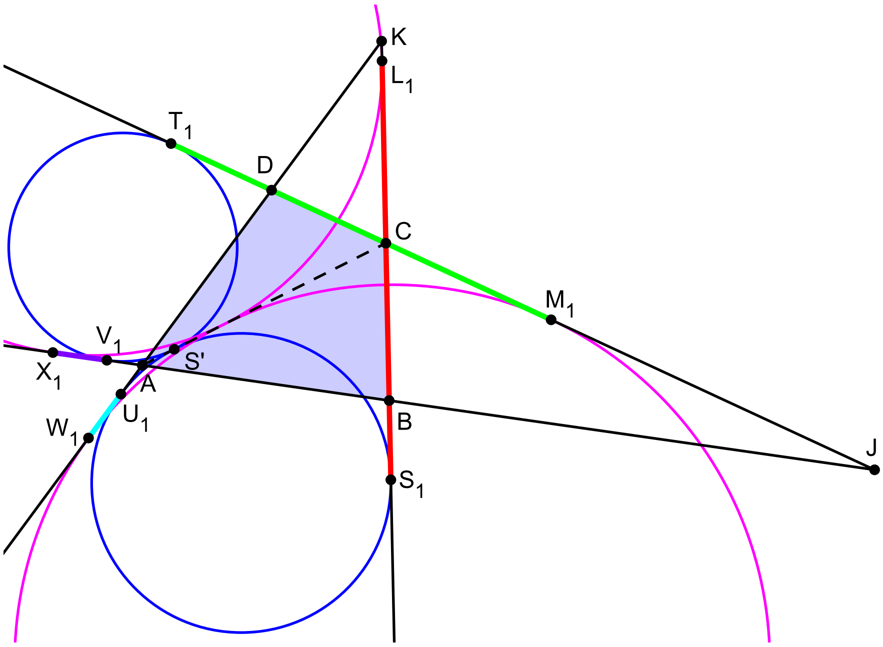

FIGURE 9. $L_1S_1 = M_1T_1 \quad \Leftrightarrow \quad ABCD$ is tangential $\quad \Leftrightarrow \quad U_1W_1 = V_1X_1$

Theorem 3.6. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the excircles to triangles ABC and AJC that are outside of AC are tangent to AJ or its extension at X_1 and V_1 respectively, and the excircles to triangles ADC and AKC that are outside of AC are tangent to AK or its extension at W_1 and U_1 respectively, then $U_1W_1 = V_1X_1$ if and only if $ABCD$ is a tangential quadrilateral.*

In the following theorem we return to the configuration of Theorem 1.2 in [9].

Theorem 3.7. *In a convex quadrilateral $ABCD$ where the angle bisectors at A and C intersect at the interior point I, let E, F, G, H be the projections of I on the sides AB, BC, CD, DA respectively. Suppose that EG and FH intersect at J, and that points K and L are on the sides AD and BC respectively such that KL is perpendicular to IJ. Then $KJ = JL$ if and only if $ABCD$ is a tangential quadrilateral.*

Proof. Applying the Pythagorean theorem four times in a convex quadrilateral $ABCD$, we get (see Figure 10)

$$\begin{aligned} KJ^2 - JL^2 &= KI^2 - IJ^2 - (LI^2 - IJ^2) \\ &= KI^2 - LI^2 \\ &= KH^2 + HI^2 - LF^2 - FI^2. \end{aligned} \tag{4}$$

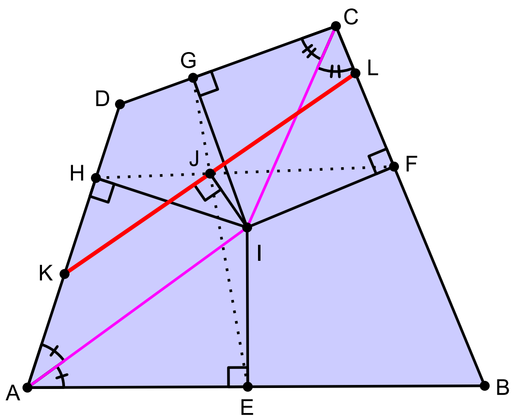

FIGURE 10. $ABCD$ is tangential $\quad \Leftrightarrow \quad KJ = JL$

($\Rightarrow$) In a tangential quadrilateral $ABCD$, $KH = LF$ according to Theorem 1.2 in [9] and $HI = FI$ according to Theorem 1.1 in [9]. Then (4) is simplified to

$$KJ^2 - JL^2 = 0$$

so $KJ = JL$.

($\Leftarrow$) We do a contrapositive proof of the converse. If the quadrilateral is not tangential, then we can assume without loss of generality that $HI > FI$ according to Theorem 1.1 in [9]. In the proof of Theorem 1.2 in [9] is was shown that

$$\frac{KH}{LF} = \frac{HI}{FI}$$

in all convex quadrilaterals. Together with $HI > FI$, this implies that $KH > FL$. Then, by (4), we have

$$KJ^2 - JL^2 = (KH^2 - LF^2) + (HI^2 - FI^2) > 0,$$

so $KJ > JL$, completing the proof of the converse. □

In [5, p. 71], a circle tangent to one side of a quadrilateral and the extensions of the two adjacent sides was called an *escribed circle* to distinguish it from an excircle (tangent to the extensions of all four sides), which there is at most one outside a quadrilateral (characterizations of this latter circle were studied in the same paper). Here we shall prove a characterization of tangential quadrilaterals that is about internal tangents to the four escribed circles. To prove that this equality is a necessary condition in a tangential quadrilateral was problem O535 in [10], proposed by Waldemar Pompe from Poland. We present our own proof.

Theorem 3.8. *In a convex quadrilateral, the internal tangents to the two pairs of opposite escribed circles have the same lengths if and only if the quadrilateral is tangential.*

Proof. According to the two tangent theorem, $QT = QN$ with notations as in Figure 11, and also (as a consequence of the same theorem) $ST = ZX$.

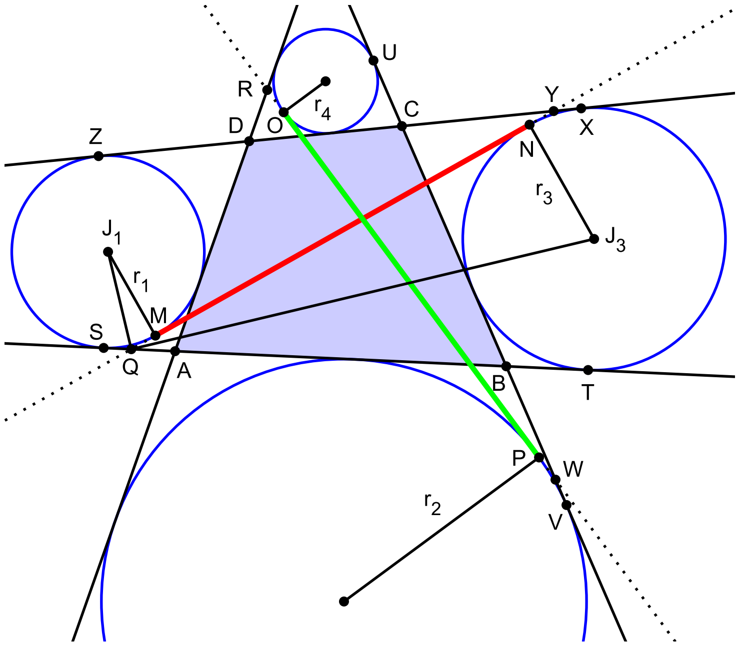

FIGURE 11. $ABCD$ is tangential $\Leftrightarrow$ $MN = OP$

Then

$$SQ + QT = ZY + YX \quad \Rightarrow \quad QM + QM + MN = MN + NY + NY$$

and we get $QM = NY$. Thus

$$ST = SQ + QT = QM + QN = NY + QN = QY.$$

By repeated use of the two tangent theorem, it is not difficult to see that $ST = UV$ holds in all convex quadrilaterals, which yields

$$QY = RW \quad \Rightarrow \quad 2QM + MN = 2RO + OP. \tag{5}$$

The equalities so far hold in all convex quadrilaterals.

Next note that triangles J_1MQ and QNJ_3 are similar (due to AA) since $J_1Q \perp QJ_3$. Then

$$\frac{r_1}{QN} = \frac{QM}{r_3},$$

so $r_1r_3 = QM \cdot QN$. In the same way $r_2r_4 = RO \cdot RP$. According to Theorem 5 in [5], $ABCD$ is tangential if and only if $r_1r_3 = r_2r_4$, so it is tangential if and only if $QM \cdot QN = RO \cdot RP$, that is, if and only if

$$QM(QM + MN) = RO(RO + OP). \tag{6}$$

From (5), we get $QM = \frac{1}{2}(2RO + OP - MN)$ (valid in all convex quadrilaterals). Substituting it into (6) yields, after multiplying both sides by 4, the characterization

$$(2RO + OP - MN)(2RO + OP + MN) = 4RO(RO + OP)$$

which is expanded and simplified to

$$OP^2 - MN^2 = 0.$$

Since we still have equivalence, $OP = MN$ is a necessary and sufficient condition for $ABCD$ to be tangential. □

Corollary 3.1. *In a convex quadrilateral, consider two internal tangents to different pairs of opposite escribed circles. The distance along one tangent from a tangent point to where that tangent intersect the extension of one side of the quadrilateral that is tangent to the same circle is equal to the corresponding distance for an adjacent escribed circle if and only if the quadrilateral is tangential.*

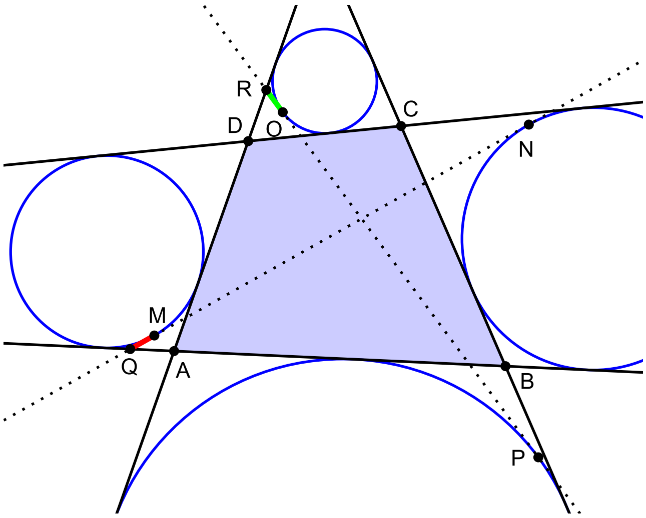

FIGURE 12. $ABCD$ is tangential $\Leftrightarrow$ $QM = RO$

Proof. With notations as in Figure 12, $2QM + MN = 2RO + OP$ holds in all convex quadrilaterals according to (5), so

$$OP = MN \quad \Leftrightarrow \quad QM = RO.$$

Since the equality to the left is a characterization of tangential quadrilaterals, then so is the equality to the right. □

We note that the formulation of the corollary could be interpreted as $MY = OW$ with notations as in Figure 11, but that would also be a valid characterization since $QY = RW$, which holds in all convex quadrilaterals, can be rewritten as $QM + MY = RO + OW$, so

$$QM = RO \quad \Leftrightarrow \quad MY = OW.$$

Since these are only different interpretations, we do not count them as two different characterizations.

4. CONCURRENCES

Here we prove four different necessary and sufficient conditions for when a convex quadrilateral can have an incircle that are related to concurrent lines or circles. Eight others were proved in [8, 9].

The first is a generalization of Theorem 1 in [4], which states that *a convex quadrilateral is tangential if and only if the incircles in the two triangles formed by a diagonal are tangent to each other*. To prove that two circles in a tangential quadrilateral satisfy the property included in the following theorem was Problem 5.4.8 solved in [2, pp. 271–272], but it was worded a little differently.

Theorem 4.1. *In a convex quadrilateral $ABCD$, let a circle that is tangent to sides AB and BC intersect diagonal AC at points V_2 and W_2. Another circle, tangent to sides CD and DA that goes through V_2, intersect AC a second time at W_2 if and only if $ABCD$ is a tangential quadrilateral.*

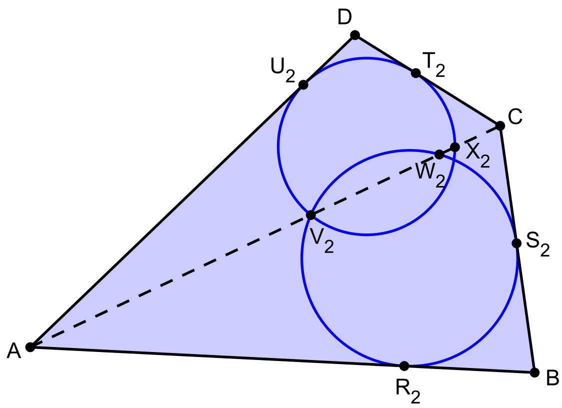

FIGURE 13. $ABCD$ is tangential $\Leftrightarrow$ $W_2 \equiv X_2$

Proof. Suppose the second circle intersect AC a second time at X_2 and that the two circles are tangent to AB, BC, CD, DA at R_2, S_2, T_2, U_2 respectively. Applying the Tangent-secant theorem four times, we have using notations as in Figure 13

$$(AR_2)^2 = AV_2 \cdot AW_2, \qquad (AU_2)^2 = AV_2 \cdot AX_2 \tag{7}$$

and

$$(CS_2)^2 = CV_2 \cdot CW_2, \qquad (CT_2)^2 = CV_2 \cdot CX_2. \tag{8}$$

($\Rightarrow$) If $W_2 = X_2$, then $AR_2 = AU_2$ and $CS_2 = CT_2$ according to (7) and (8), and using the two tangent theorem, we have $BR_2 = BS_2$ and $DT_2 = DU_2$. Then

$$AB+CD = AR_2+R_2B+CT_2+T_2D = AU_2+BS_2+CS_2+DU_2 = BC+DA$$

so $ABCD$ is tangential since it satisfies Pitot's theorem (1).

($\Leftarrow$) We do a contrapositive proof of the converse. When $W_2 \neq X_2$, assuming without loss of generality that $AW_2 < AX_2$, then we get $AR_2 < AU_2$ from (7), and $CW_2 > CX_2$ implies $CS_2 > CT_2$ according to (8). Hence

$$AB+CD = AR_2+BR_2+CT_2+DT_2 < AU_2+BS_2+CS_2+DU_2 = AD+BC$$

which proves that $ABCD$ is not tangential. □

There is also the following similar characterization with circles tangent to the extensions of opposite sides instead of adjacent sides, which is a generalization of Theorems 2.2 and 2.3.

Theorem 4.2. *In a convex quadrilateral $ABCD$, let a circle that is tangent to AB and CD or their extensions intersect diagonal AC at points V_3 and W_3. Another circle, tangent to sides BC and DA or their extensions that goes through W_3, intersect AC a second time at V_3 if and only if $ABCD$ is a tangential quadrilateral.*

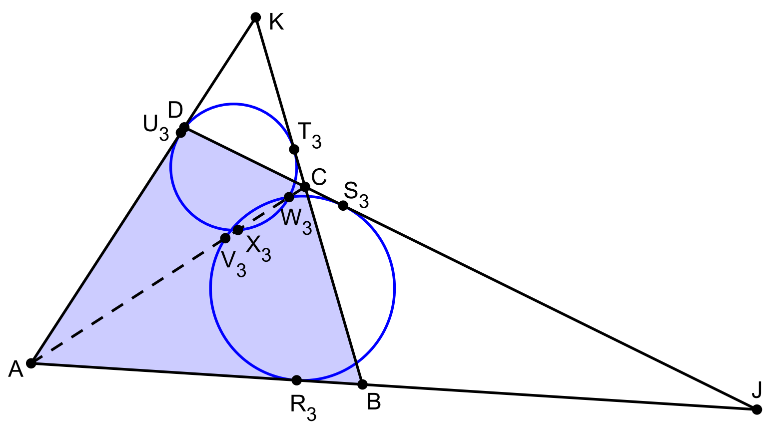

FIGURE 14. $ABCD$ is tangential $\Leftrightarrow$ $V_3 \equiv X_3$

Proof. Suppose the second circle intersect AC a second time at X_3 and that the two circles are tangent to AB, CD, BC, DA at R_3, S_3, T_3, U_3 respectively. Applying the Tangent-secant theorem four times, we have using notations as in Figure 14

$$(AR_3)^2 = AV_3 \cdot AW_3, \qquad (AU_3)^2 = AX_3 \cdot AW_3 \tag{9}$$

and

$$(CS_3)^2 = CV_3 \cdot CW_3, \qquad (CT_3)^2 = CX_3 \cdot CW_3. \tag{10}$$

($\Rightarrow$) If $V_3 = X_3$, then $AR_3 = AU_3$ and $CS_3 = CT_3$ according to (9) and (10), and using the two tangent theorem, we have $JR_3 = JS_3$ and $KT_3 = KU_3$. Then

$$\begin{aligned} AJ - AK &= AR_3 + R_3J - AU_3 - U_3K \\ &= JR_3 - KU_3 = JS_3 - KT_3 \\ &= JS_3 + S_3C - CT_3 - T_3K = CJ - CK \end{aligned}$$

so $ABCD$ is tangential according to Grossman's first characterization (3).

($\Leftarrow$) We do a contrapositive proof of the converse. When $V_3 \neq X_3$, assuming without loss of generality that $AV_3 < AX_3$, then we get $AR_3 < AU_3$ from (9), and $CV_3 > CX_3$ implies $CS_3 > CT_3$ according to (10). Hence

$$\begin{aligned} AJ - AK &= AR_3 + R_3J - AU_3 - U_3K \\ &< JR_3 - KU_3 = JS_3 - KT_3 \\ &< JS_3 + S_3C - CT_3 - T_3K = CJ - CK \end{aligned}$$

which proves that $ABCD$ is not tangential. □

Next we have the first of two theorems regarding two extended chords that intersect on a diagonal if and only if the quadrilateral is tangential.

Theorem 4.3. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the incircles in triangles ACJ and ACK are tangent to AB, BC, CD, DA or their the extensions at E_1, F_1, G_1, H_1 respectively, then the lines E_1G_1, F_1H_1 and AC are concurrent if and only if $ABCD$ is a tangential quadrilateral.*

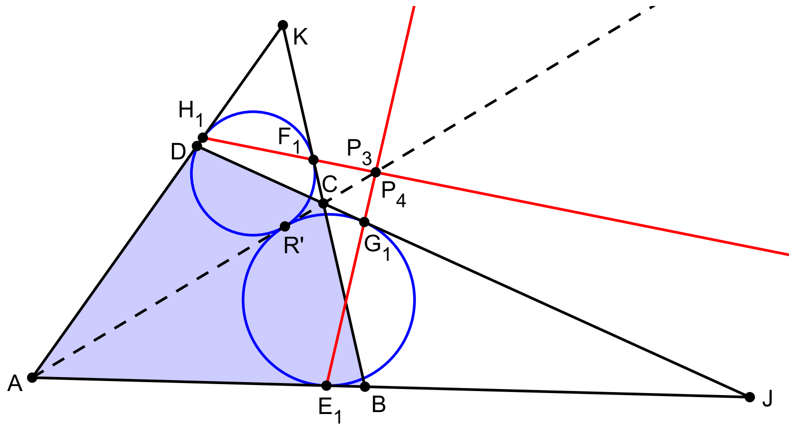

FIGURE 15. $ABCD$ is tangential ⇔ E_1G_1, F_1H_1 and AC are concurrent

Proof. (⇒) Let $P_3 = E_1G_1 \cap AC$ and $P_4 = H_1F_1 \cap AC$ (see Figure 15). Applying Menelaus' theorem (with non-directed distances) in triangles ACJ and ACK with transversals E_1G_1 and F_1H_1 respectively yields

$$\frac{AP_3}{P_3C} \cdot \frac{CG_1}{G_1J} \cdot \frac{JE_1}{E_1A} = 1 = \frac{AP_4}{P_4C} \cdot \frac{CF_1}{F_1K} \cdot \frac{KH_1}{H_1A}. \tag{11}$$

Here $JE_1 = JG_1$ and $KF_1 = KH_1$ according to the two tangent theorem. When $ABCD$ is tangential, then $CG_1 = CR' = CF_1$ and $AE_1 = AR' = AH_1$ (according to Theorem 2.2), so (11) is simplified to

$$\frac{AP_3}{P_3C} = \frac{AP_4}{P_4C}$$

and it follows that $P_3 \equiv P_4$ since these two points divide AC in the same ratio.

(⇐) We do a contrapositive proof of the converse. When $ABCD$ is not tangential, assume without loss of generality that

$$CG_1 = CR' > CT' = CF_1 \tag{12}$$

and thus

$$AE_1 = AR' < AT' = AH_1 \tag{13}$$

where R' and T' are the points where the incircles in triangles ACJ and ACK are tangent to AC respectively (see Figure 4). From Menelaus' theorem we get, after eliminating equal tangents and using the inequalities (12) and (13), that

$$\frac{AP_3}{P_3C}\cdot\frac{CF_1}{H_1A}<\frac{AP_3}{P_3C}\cdot\frac{CG_1}{E_1A}=\frac{AP_4}{P_4C}\cdot\frac{CF_1}{H_1A}.$$

Hence

$$\frac{AP_3}{P_3C}<\frac{AP_4}{P_4C}$$

which proves that E_1G_1, F_1H_1 and AC are not concurrent. □

There is the following excircle version of the previous theorem:

Theorem 4.4. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB\cap CD$ and $K = BC\cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the excircles to triangles ACJ and ACK that are outside of AC are tangent to AB, BC, CD, DA or their the extensions at V_1, S_1, T_1, U_1 respectively, then the lines S_1U_1, T_1V_1 and AC are concurrent if and only if $ABCD$ is a tangential quadrilateral.*

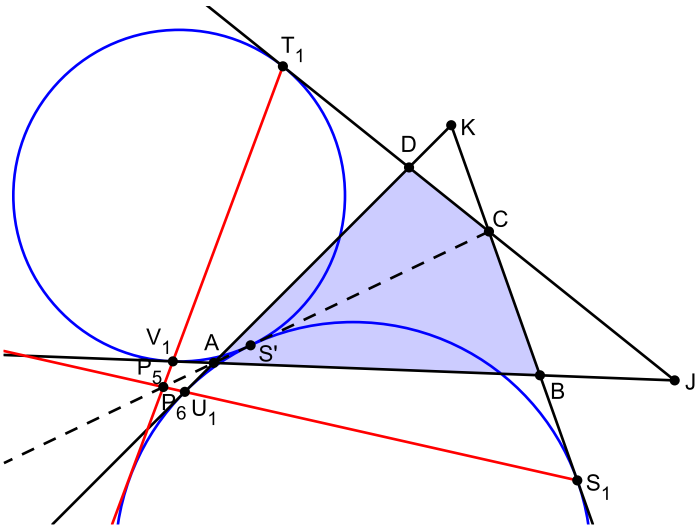

FIGURE 16. $ABCD$ is tangential ⇔ S_1U_1, T_1V_1 and AC are concurrent

Proof. (⇒) Let $P_5 = S_1U_1 \cap AC$ and $P_6 = T_1V_1 \cap AC$ (see Figure 16). Applying Menelaus' theorem in triangles ACJ and ACK with transversals S_1U_1 and T_1V_1 respectively yields

$$\frac{AP_5}{P_5C}\cdot\frac{CT_1}{T_1J}\cdot\frac{JV_1}{V_1A}=1=\frac{AP_6}{P_6C}\cdot\frac{CS_1}{S_1K}\cdot\frac{KU_1}{U_1A}. \quad (14)$$

Here $JT_1 = JV_1$ and $KS_1 = KU_1$ according to the two tangent theorem. When $ABCD$ is tangential, then $CS_1 = CS' = CT_1$ and $AU_1 = AS' = AV_1$, so (14) is simplified to

$$\frac{AP_5}{P_5C}=\frac{AP_6}{P_6C}$$

and it follows that $P_5 \equiv P_6$.

($\Leftarrow$) We do a contrapositive proof of the converse. When $ABCD$ is not tangential, assume without loss of generality that $CS_1 = CU' > CS' = CT_1$ and thus $AV_1 = AU' < AS' = AU_1$ where U' and S' are the points where the excircles to triangles ACJ and ACK are tangent to AC respectively (see Figure 5). From Menelaus' theorem we get, after eliminating equal tangents and applying relevant inequalities, that

$$\frac{AP_5}{P_5C}\cdot\frac{CS_1}{U_1A} < \frac{AP_5}{P_5C}\cdot\frac{CT_1}{V_1A} = \frac{AP_6}{P_6C}\cdot\frac{CS_1}{U_1A}.$$

Hence

$$\frac{AP_5}{P_5C} < \frac{AP_6}{P_6C}$$

which proves that S_1U_1, T_1V_1 and AC are not concurrent. □

5. Cyclic quadrilaterals

We have proved a total of eight characterizations of tangential quadrilaterals regarding cyclic quadrilaterals in [7, 8, 9]. Here we prove six more, but we start with a necessary and sufficient condition about an isosceles trapezoid (which is a special case of a cyclic quadrilateral).

Theorem 5.1. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap AD$. Assume the vertices are labeled such that JK lies outside of D. Suppose the excircles to triangle AJK outside of AJ and AK are tangent to AK or its extension at I_2 and L_2 respectively, and that the excircles to triangle CJK outside of CJ and CK are tangent to CK or its extension at J_2 and K_2 respectively. Then $I_2J_2K_2L_2$ is an isosceles trapezoid if and only if $ABCD$ is a tangential quadrilateral.*

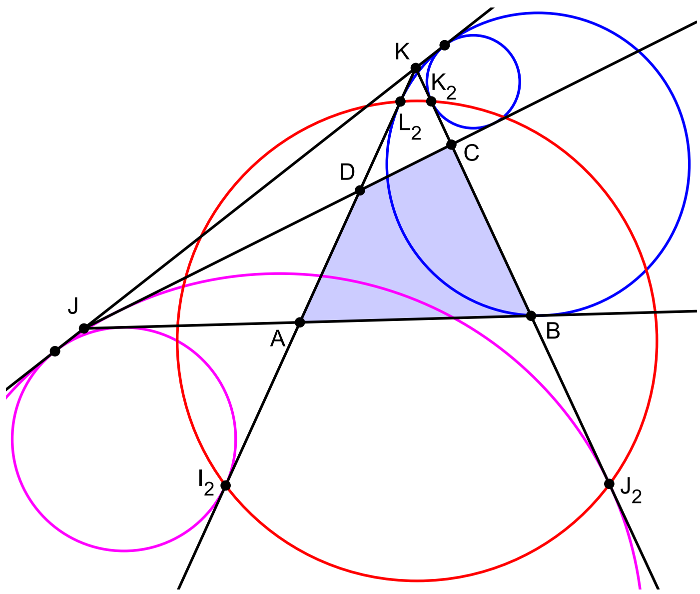

Figure 17. $ABCD$ is tangential $\Leftrightarrow$ $I_2J_2K_2L_2$ is an isosceles trapezoid

Proof. According to Lemma 2.1 in [9], we have (see Figure 17)

$$2KJ_2 = CJ + JK + KC, \qquad 2KK_2 = CJ - JK + KC.$$

Then

$$2K_2J_2 = 2KJ_2 - 2KK_2 = 2JK$$

so $K_2J_2 = JK$ and in the same way $L_2I_2 = JK$. Thus $K_2J_2 = L_2I_2$ and this is true in all convex quadrilaterals. It holds that $I_2J_2K_2L_2$ is an isosceles trapezoid if and only if $KK_2 = KL_2$, which is equivalent to that the two blue excircles are tangent to each other on the extension of JK according to the two tangent theorem. This in turn is equivalent to that $ABCD$ is a tangential quadrilateral according to Theorem 4.1 in [9]. □

There is another version of the theorem concerning the tangency points of the same four excircles on the extensions of AJ and CJ instead.

The two blue excircles in Figure 17 also appear in the next theorem.

Theorem 5.2. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap AD$. Assume the vertices are labeled such that JK lies outside of D. Suppose the excircle to triangle AJK outside of AK is tangent to AK at L_2 and the extension of AJ at M_2 respectively, and that the excircle to triangle CJK outside of CK is tangent to CK at K_2 and the extension of CJ at E_2 respectively. Then $E_2K_2L_2M_2$ is a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.*

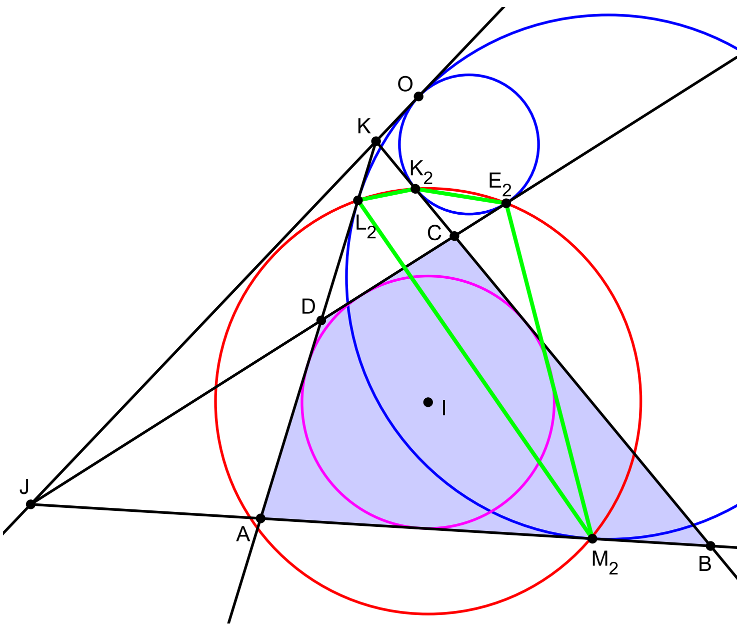

FIGURE 18. $ABCD$ is tangential $\Rightarrow$ $E_2K_2L_2M_2$ is cyclic

Proof. ($\Rightarrow$) The two excircles are tangent to each other (at O in Figure 18) when $ABCD$ is tangential according to Theorem 4.1 in [9]. Then $JM_2 = JO = JE_2$ according to the two tangent theorem, so

$$\angle JE_2M_2 = \tfrac{1}{2}(\pi - \angle BJC) = \tfrac{1}{2}(\pi - (\pi - \angle B - \angle C)) = \tfrac{1}{2}(\angle B + \angle C).$$

Since $\angle JE_2K_2 = \frac{1}{2}(\pi - \angle C)$, this implies that

$$\angle M_2E_2K_2 = \angle JE_2M_2 + \angle JE_2K_2 = \tfrac{1}{2}(\pi + \angle B).$$

We further have $\angle AL_2M_2 = \frac{1}{2}(\pi - \angle A)$ and $\angle AKB = \pi - \angle A - \angle B$, so

$$\angle KL_2K_2 = \tfrac{1}{2}(\pi - \angle AKB) = \tfrac{1}{2}(\angle A + \angle B)$$

since triangle KL_2K_2 is isosceles when $ABCD$ is tangential ($KL_2 = KO = KK_2$). Hence

$$\angle M_2L_2K_2 = \pi - \angle AL_2M_2 - \angle KL_2K_2 = \pi - \tfrac{1}{2}(\pi - \angle B)$$

so

$$\angle M_2E_2K_2 + \angle M_2L_2K_2 = \pi$$

confirming that $E_2K_2L_2M_2$ is cyclic.

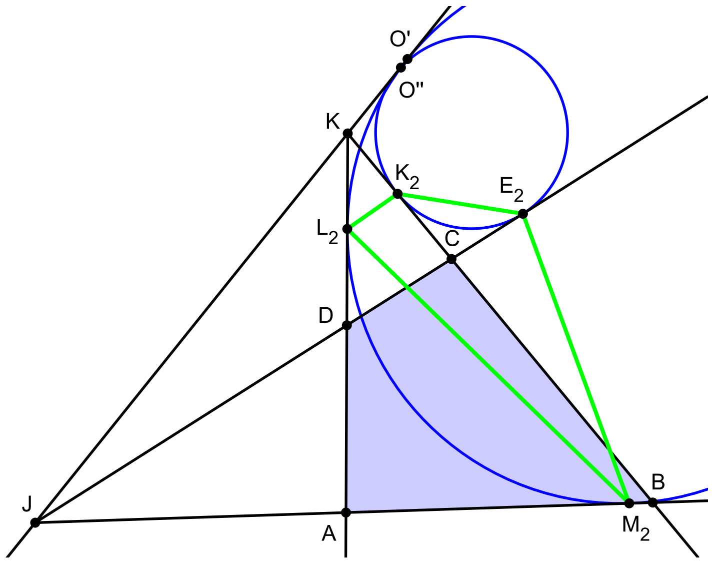

FIGURE 19. $ABCD$ is not tangential $\Rightarrow$ $E_2K_2L_2M_2$ is not cyclic

($\Leftarrow$) We do a contrapositive proof of the converse. When $ABCD$ is not tangential, assume without loss of generality that the two excircles to AJK and CJK are tangent to the extension of JK at O' and O'' respectively such that $KO' > KO''$ (see Figure 19). Then triangles JM_2E_2 and KL_2K_2 are no longer isosceles. Instead we have $JM_2 = JO' > JO'' = JE_2$, so

$$\angle JE_2M_2 > \tfrac{1}{2}(\pi - \angle BJC) = \tfrac{1}{2}(\pi - (\pi - \angle B - \angle C)) = \tfrac{1}{2}(\angle B + \angle C)$$

since a longer side in a triangle is opposite a larger angle, and $KL_2 = KO' > KO'' = KK_2$, so

$$\angle KL_2K_2 < \tfrac{1}{2}(\pi - \angle AKB) = \tfrac{1}{2}(\angle A + \angle B).$$

Hence

$$\angle M_2E_2K_2 = \angle JE_2M_2 + \angle JE_2K_2 > \tfrac{1}{2}(\pi + \angle B)$$

($\angle JE_2K_2 = \frac{1}{2}(\pi - \angle C)$ still holds since triangle CE_2K_2 is still isosceles), and

$$\angle M_2L_2K_2 = \pi - \angle AL_2M_2 - \angle KL_2K_2 > \pi - \tfrac{1}{2}(\pi - \angle B)$$

(triangle AM_2L_2 is also still isosceles) so

$$\angle M_2E_2K_2 + \angle M_2L_2K_2 > \pi$$

proving that $E_2K_2L_2M_2$ is not cyclic. □

In Figure 18 we might suspect that the circumcircle to $E_2K_2L_2M_2$ has the same center as the incircle in $ABCD$. This is true and the reason is that the perpendicular bisectors of M_2L_2 and K_2E_2 are also the angle bisectors of vertex angles A and C, so both pairs intersect at the incenter I.

The next theorem is a variant of Theorem 5.2 in [7].

Theorem 5.3. *In a convex quadrilateral $ABCD$, let the incircles in triangles ABC and CDA be tangent to AB, BC, CD, DA at A_1, B_1, C_1, D_1 respectively, and the excircles to the same triangles that are outside of AC be tangent to the extensions of the same sides at K_1, L_1, M_1, N_1 respectively. If $O_1 = A_1D_1 \cap K_1L_1$, $P_1 = B_1C_1 \cap K_1L_1$, $Q_1 = B_1C_1 \cap M_1N_1$, and $R_1 = A_1D_1 \cap M_1N_1$, then $O_1P_1Q_1R_1$ is a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.*

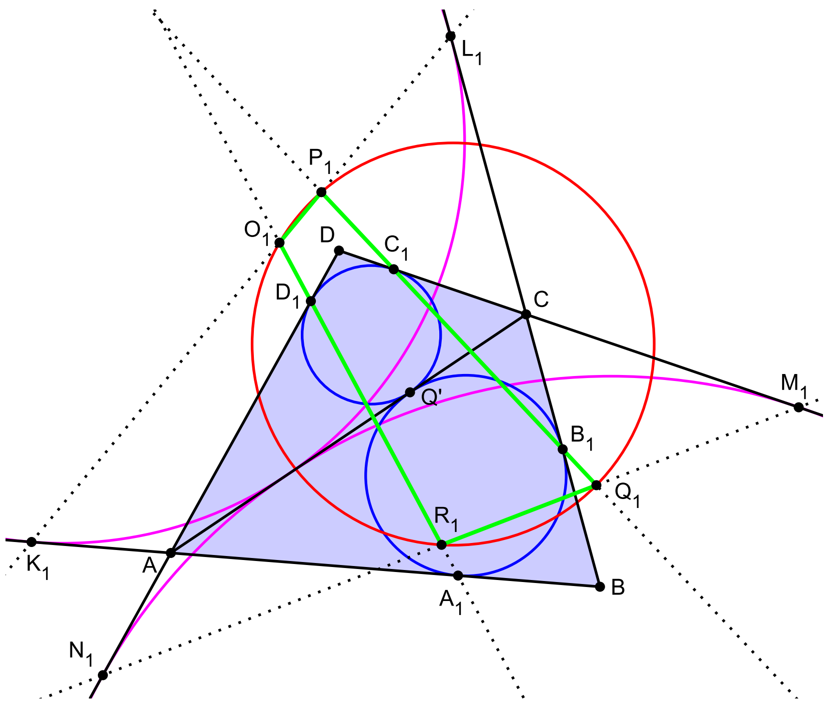

FIGURE 20. $ABCD$ is tangential $\Leftrightarrow$ $O_1P_1Q_1R_1$ is cyclic

Proof. ($\Rightarrow$) In all convex quadrilaterals, we have (see Figure 20)

$$\angle A_1K_1O_1 = \angle BK_1L_1 = \tfrac{1}{2}(\pi - \angle B) \qquad (15)$$

and when $ABCD$ is tangential, the two incircles are tangent to each other at a point Q' according to Theorem 1 in [4], so

$$\angle K_1A_1O_1 = \angle AA_1D_1 = \tfrac{1}{2}(\pi - \angle A)$$

since $AA_1 = AQ' = AD_1$. Then, by the exterior angle theorem,

$$\angle P_1O_1R_1 = \angle A_1K_1O_1 + \angle K_1A_1O_1 = \pi - \tfrac{1}{2}(\angle A + \angle B).$$

In the same way we get

$$\angle P_1Q_1R_1 = \pi - \tfrac{1}{2}(\angle C + \angle D).$$

Hence, by the angle sum of a quadrilateral, we have

$$\angle P_1O_1R_1 + \angle P_1Q_1R_1 = 2\pi - \tfrac{1}{2}(\angle A + \angle B + \angle C + \angle D) = \pi$$

confirming that $O_1P_1Q_1R_1$ is cyclic.

($\Leftarrow$) We prove the converse with a contrapositive proof. In a quadrilateral that is not tangential, (15) still holds. Assume without loss of generality that $AA_1 > AD_1$ (since now the two incircles are not tangent to each other). This implies

$$\angle K_1A_1O_1 < \tfrac{1}{2}(\pi - \angle A)$$

and since $CC_1 > CB_1$, we get

$$\angle CC_1B_1 < \tfrac{1}{2}(\pi - \angle C).$$

Hence

$$\angle P_1O_1R_1 < \pi - \tfrac{1}{2}(\angle A + \angle B), \qquad \angle P_1Q_1R_1 < \pi - \tfrac{1}{2}(\angle C + \angle D).$$

Then

$$\angle P_1O_1R_1 + \angle P_1Q_1R_1 < 2\pi - \tfrac{1}{2}(\angle A + \angle B + \angle C + \angle D) = \pi$$

which proves that $O_1P_1Q_1R_1$ is not cyclic. □

The following characterization is related to the configuration in Theorem 4.2 in [9].

Theorem 5.4. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap AD$. Assume the vertices are labeled such that JK lies outside of D. Suppose the excircle to triangle BJK outside of JK is tangent to the extensions of AB and BC at N_2 and O_2 respectively, and the excircle to triangle DJK outside of JK is tangent to the extensions of DJ and DK at P_2 and Q_2 respectively. Then $N_2O_2Q_2P_2$ is a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.*

Proof. In convex quadrilateral $ABCD$, we have $\angle N_2O_2B = \frac{1}{2}(\pi - \angle B)$, $\angle DP_2Q_2 = \frac{1}{2}(\pi - \angle D)$, $\angle AKB = \pi - \angle A - \angle B$ and $\angle BJC = \pi - \angle B - \angle C$, see Figure 21. When $ABCD$ is also tangential, $KO_2 = KQ'' = KQ_2$ where Q'' is the point on JK where the two excircles are tangent to each other according to Theorem 4.2 in [9]. Then

$$\angle BO_2Q_2 = \tfrac{1}{2}(\pi - \angle O_2KQ_2) = \tfrac{1}{2}(\pi - \angle AKB) = \tfrac{1}{2}(\angle A + \angle B)$$

and similarly

$$\angle JP_2N_2 = \tfrac{1}{2}(\pi - \angle BJC) = \tfrac{1}{2}(\angle B + \angle C).$$

Hence

$$\angle Q_2P_2N_2 = \angle JP_2N_2 - \angle DP_2Q_2 = \tfrac{1}{2}(\angle B + \angle C + \angle D - \pi) = \tfrac{1}{2}(\pi - \angle A)$$

and

$$\angle N_2O_2Q_2 = \angle N_2O_2B + \angle BO_2Q_2 = \tfrac{1}{2}(\pi + \angle A)$$

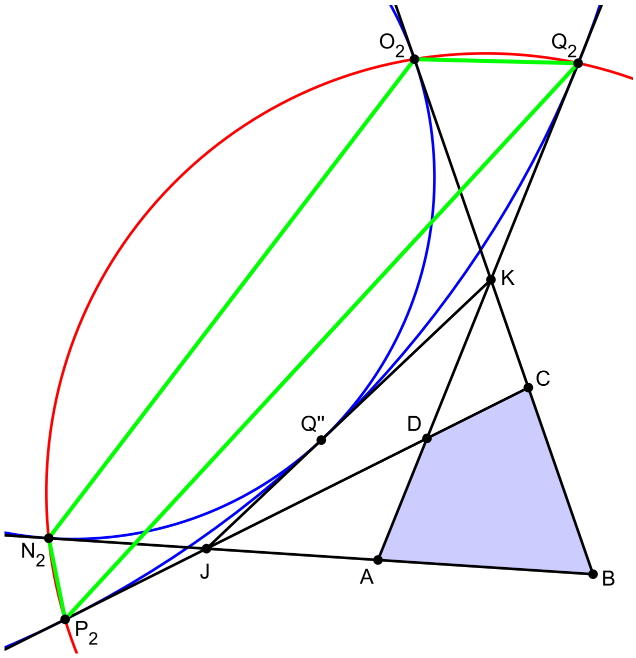

FIGURE 21. $ABCD$ is tangential $\Leftrightarrow$ $N_2O_2Q_2P_2$ is cyclic

so

$$\angle Q_2P_2N_2 + \angle N_2O_2Q_2 = \pi$$

confirming that $N_2O_2Q_2P_2$ is cyclic.

The converse can be proved with a contrapositive proof in the same way as in the last two theorems. We leave this part as an exercise for the reader. □

Next we have the fourth characterization regarding a cyclic quadrilateral.

Theorem 5.5. *In a convex quadrilateral $ABCD$ where the angle bisectors at A and C intersect at an internal point I, let E, F, G, H be the projections of I on AB, BC, CD, DA respectively. If A' and C' are the midpoints of HE and FG respectively, then $AA'C'C$ is a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.*

Proof. Triangles EIA' and AEI are similar (due to AA, see Figure 22), so

$$\frac{x}{AI} = \frac{A'I}{x}$$

and we get $x^2 = AI \cdot A'I$. In the same way $y^2 = CI \cdot C'I$. It holds that $AA'C'C$ is cyclic if and only if $AI \cdot A'I = CI \cdot C'I$ according to the external case of the intersecting chords theorem, which is equivalent to $x^2 = y^2$, that is $x = y$. This is a characterization of a tangential quadrilateral according to Theorem 1.1 in [9]. □

The following theorem is related to the configuration in Theorem 2.2.

Theorem 5.6. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the incircles in triangles ACJ and ACK are*

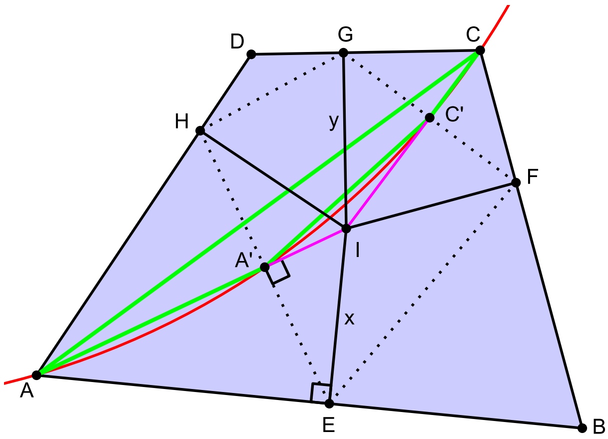

Figure 22. $ABCD$ is tangential $\Leftrightarrow$ $AA'C'C$ is cyclic

tangent to AB, BC, CD, DA or their the extensions at E_1, F_1, G_1, H_1 respectively, then $E_1G_1F_1H_1$ is a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.

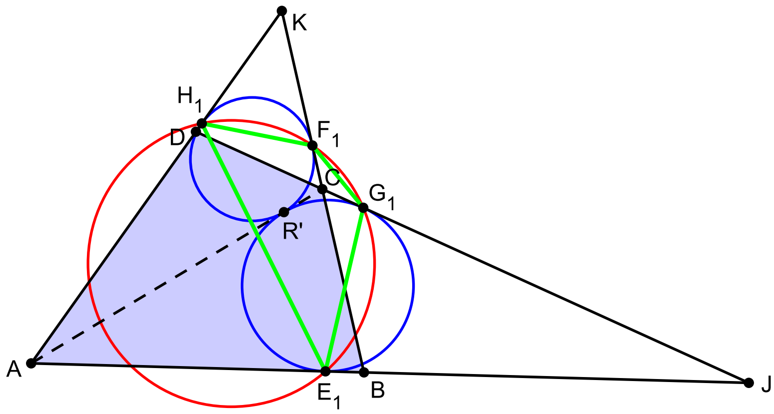

Figure 23. $ABCD$ is tangential $\Leftrightarrow$ $E_1G_1F_1H_1$ is cyclic

Proof. ($\Rightarrow$) In all convex quadrilaterals, we have $\angle AJD = \pi - \angle A - \angle D$ (see Figure 23), so

$$\angle JE_1G_1 = \tfrac{1}{2}(\pi - (\pi - \angle A - \angle D)) = \tfrac{1}{2}(\angle A + \angle D).$$

When $ABCD$ is cyclic, the incircles in triangles ACJ and ACK are tangent to each other at a point R' on AC according to Theorem 2.2, so $AE_1 = AR' = AH_1$. Thus $\angle AE_1H_1 = \frac{1}{2}(\pi - \angle A)$ and we get

$$\angle G_1E_1H_1 = \pi - \tfrac{1}{2}(\angle A + \angle D) - \tfrac{1}{2}(\pi - \angle A) = \tfrac{1}{2}(\pi - \angle D).$$

In all convex quadrilaterals, we also have $\angle AKB = \pi - \angle A - \angle B$, so $\angle KF_1H_1 = \frac{1}{2}(\angle A + \angle B)$. When $ABCD$ is cyclic, $CF_1 = CR' = CG_1$, so $\angle CF_1G_1 = \frac{1}{2}(\pi - \angle C)$. Thus

$$\angle G_1F_1H_1 = \pi - \tfrac{1}{2}(\angle A + \angle B) + \tfrac{1}{2}(\pi - \angle C) = \tfrac{1}{2}(\pi + \angle D)$$

where we applied the angle sum of a quadrilateral. Hence for two opposite angles in $E_1G_1F_1H_1$, we have

$$\angle G_1E_1H_1 + \angle G_1F_1H_1 = \tfrac{1}{2}(\pi - \angle D) + \tfrac{1}{2}(\pi + \angle D) = \pi$$

confirming that $E_1G_1F_1H_1$ is cyclic.

($\Leftarrow$) We prove the converse with a contrapositive proof. When $ABCD$ is not cyclic, suppose without loss of generality that the incircles in triangles ACJ and ACK are tangent to AC at T' and R' respectively such that $AR' < AT'$ (see Figure 4). Then $\angle AE_1H_1 < \frac{1}{2}(\pi - \angle A)$ since a shorter side in a triangle is opposite a smaller angle, so

$$\angle G_1E_1H_1 > \pi - \tfrac{1}{2}(\angle A + \angle D) - \tfrac{1}{2}(\pi - \angle A) = \tfrac{1}{2}(\pi - \angle D).$$

Since now $\angle CF_1G_1 < \frac{1}{2}(\pi - \angle C)$, we also have

$$\angle G_1F_1H_1 > \pi - \tfrac{1}{2}(\angle A + \angle B) + \tfrac{1}{2}(\pi - \angle C) = \tfrac{1}{2}(\pi + \angle D).$$

Hence

$$\angle G_1E_1H_1 + \angle G_1F_1H_1 > \tfrac{1}{2}(\pi - \angle D) + \tfrac{1}{2}(\pi + \angle D) = \pi$$

so $E_1G_1F_1H_1$ is not cyclic. □

There is an excircle version of the previous theorem which is also related to the configuration in Theorem 2.3:

Theorem 5.7. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of C. If the excircles to triangles ACJ and ACK that are outside of AC are tangent to the extensions of AB, BC, CD, DA at V_1, S_1, T_1, U_1 respectively, then $S_1T_1V_1U_1$ is a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.*

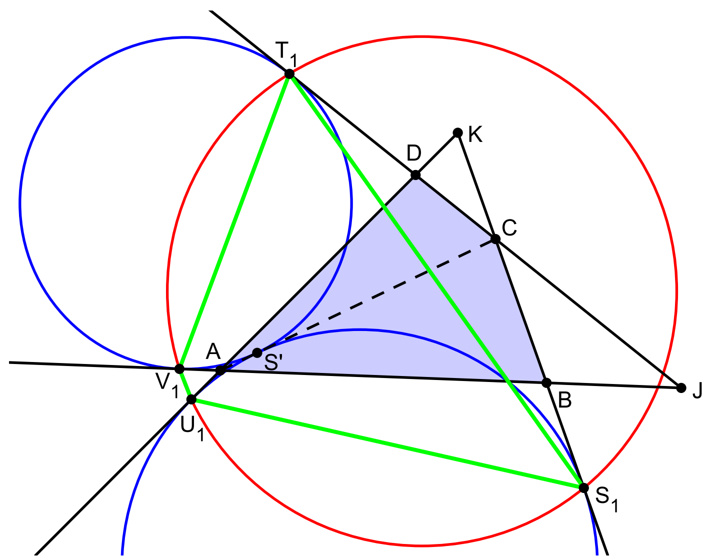

FIGURE 24. $ABCD$ is tangential $\Leftrightarrow$ $S_1T_1V_1U_1$ is cyclic

The proof is identical to that of the previous theorem, so we leave it as an exercise for the reader to write down the proof. The theorem is illustrated in Figure 24.

6. Miscellaneous characterizations

In this section we prove three characterizations of tangential quadrilaterals that have no common theme.

Theorem 6.1. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. If the excircles to triangles ABK, BCJ, CDK, DAJ, that are not tangent to the sides AB, BC, CD, DA, are tangent to the extensions of these sides at $\{T, U\}$, $\{R, S\}$, $\{X, Y\}$, $\{V, W\}$ respectively, then*

$$RS + TU = VW + XY$$

if and only if $ABCD$ is a tangential quadrilateral.

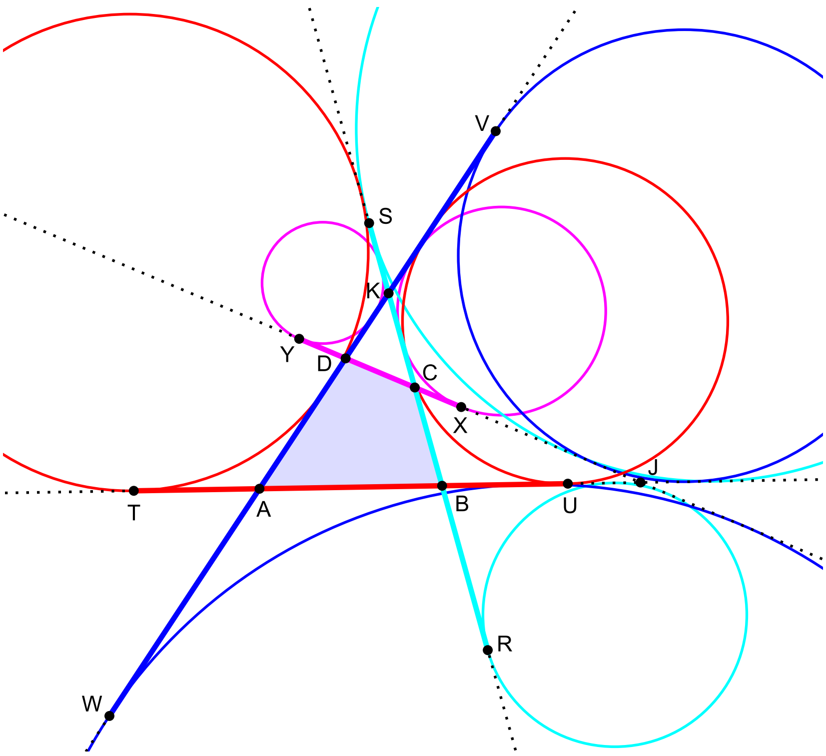

Figure 25. $ABCD$ is tangential $\Leftrightarrow$ $RS + TU = VW + XY$

Proof. We have (see Figure 25)

$$2TA = -AB + BK + KA = 2BU$$

so

$$2TU = 2TA + 2AB + 2BU = 2(BK + KA)$$

that is $TU = AK + KB$ and similarly $RS = BJ + JC$. We get

$$TU + RS = AD + DK + BC + CK + BJ + CJ$$

and since also $XY = CK + DK$ and $VW = AJ + JD$, then

$$XY + VW = DK + CK + AB + BJ + CJ + CD.$$

Hence

$$TU + RS = XY + VW$$

is equivalent to

$$AD + DK + BC + CK + BJ + CJ = DK + CK + AB + BJ + CJ + CD$$

which in turn is equivalent to

$$BC + DA = AB + CD.$$

This concludes the proof according to Pitot's theorem. □

The following angle equality characterizes tangential quadrilaterals:

Theorem 6.2. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Assume the vertices are labeled such that JK lies outside of D, and that the angle bisectors at A and C intersect at an internal point I. Denoting the incenters of triangles ADJ and CDK by J_1 and J_4 respectively, we have that $\angle AIJ_1 = \angle CIJ_4$ if and only if $ABCD$ is a tangential quadrilateral.*

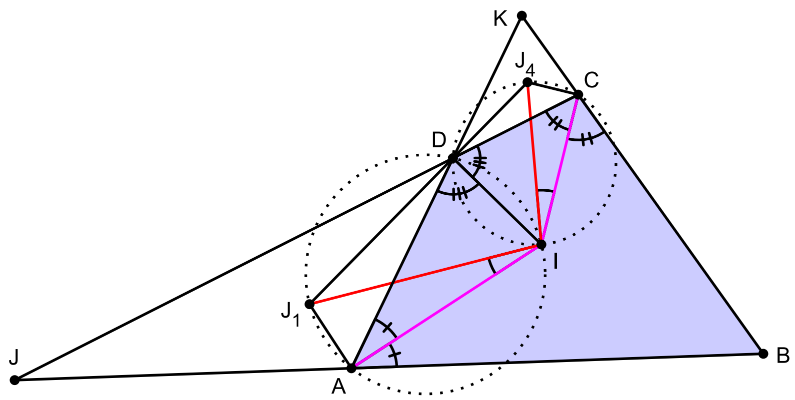

FIGURE 26. $ABCD$ is tangential $\Rightarrow$ $\angle AIJ_1 = \angle CIJ_4$

Proof. ($\Rightarrow$) We have that $AI \perp AJ_1$ and $DI \perp DJ_1$ when $ABCD$ is tangential, since an internal and an external angle bisector are orthogonal (see Figure 26). Then quadrilaterals $AIDJ_1$ and $CIDJ_4$ are cyclic, so

$$\angle AIJ_1 = \angle ADJ_1 = \angle KDJ_4 = \angle CDJ_4 = \angle CIJ_4.$$

($\Leftarrow$) We do a contrapositive proof of the converse. If $ABCD$ is not tangential, extend the angle bisector at D to intersect the circumcircle of ADJ_1 at G' and let the angle bisector at C intersect the circumcircle of CDJ_4 at H' (see Figure 27). Then

$$\angle AIJ_1 > \angle AG'J_1 = \angle ADJ_1 = \angle CDJ_4 = \angle CH'J_4 > \angle CIJ_4$$

so $\angle AIJ_1 > \angle CIJ_4$, concluding the proof of the converse. □

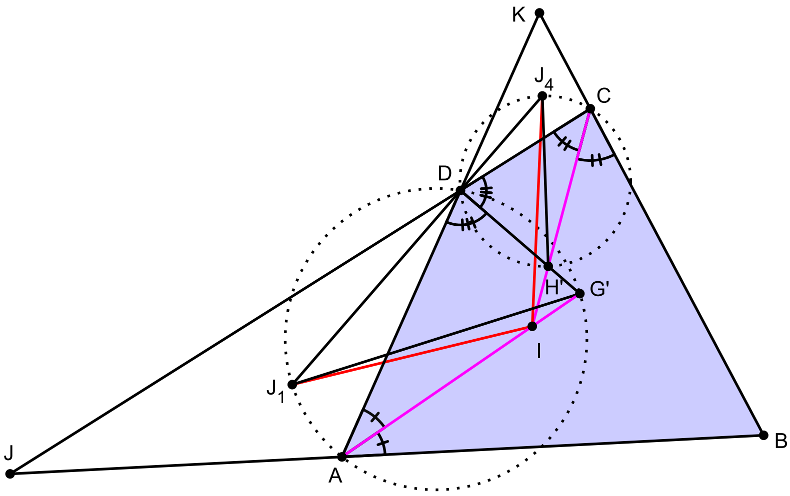

FIGURE 27. $ABCD$ is not tangential $\Rightarrow$ $\angle AIJ_1 \neq \angle CIJ_4$

The last theorem is a generalization of Problem 5.4.7 in [2] in such a way that we also prove a converse. To prove the necessary condition was Problem 3 for grade 11 on the 1999 All-Russian Mathematical Olympiad [1]. We will use more or less the same proof given in [2] except a minor change that will allow us to also prove the converse with very little extra effort. The proof is based on two lemmas. For the first we will give a different argument than in [2] and the second is reformulated to fit the proof of the converse of the main theorem.

Lemma 6.1. *If C is an external point to a circle and A and B are points on the circle such that CA and CB are tangents, then the incenter of triangle ABC lies on this circle.*

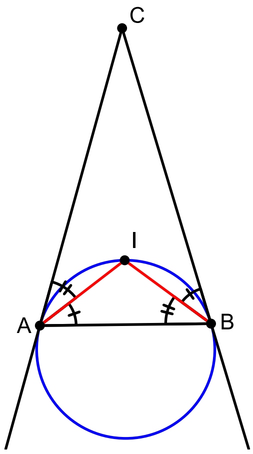

FIGURE 28. I is the incenter of triangle ABC

Proof. Denote the midpoint of minor arc AB by I. According to the Tangent-chord theorem, we have $\angle CBI = \angle IAB$ and $\angle CAI = \angle IBA$ (see

Figure 28). Triangle ABI is isosceles by symmetry, so $\angle IAB = \angle IBA$. Hence $\angle CAI = \angle IAB$ and $\angle CBI = \angle IBA$, confirming that I is the incenter of triangle ABC. □

Lemma 6.2. *In a convex quadrilateral $ABCD$, suppose k_1, k_2, k_3, k_4 are circles inscribed in angles DAB, ABC, BCD, CDA respectively. Denote the common external tangent lines of k_1 and k_2, of k_2 and k_3, of k_3 and k_4, and of k_4 and k_1, different from the sides of $ABCD$, by t_{12}, t_{23}, t_{34}, t_{41} respectively. Let $P = t_{41} \cap t_{12}$, $Q = t_{12} \cap t_{23}$, $R = t_{23} \cap t_{34}$ and $S = t_{34} \cap t_{41}$. Then $ABCD$ is tangential if and only if $PQRS$ is tangential.*

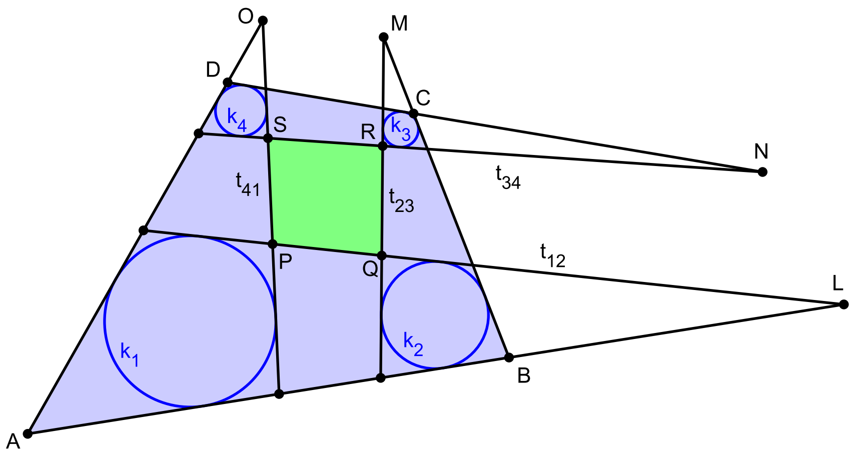

FIGURE 29. $ABCD$ is tangential ⇔ $PQRS$ is tangential

Proof. Suppose the tangent lines intersect at L, N, M, O as indicated in Figure 29. Then

$$\begin{aligned} SR + PQ &= SN - RN + PL - QL \\ &= (ND + DO - SO) - (RM + CN - MC) \\ &\quad + (AL + PO - AO) - (LB + BM - MQ) \\ &= (ND - NC) + (DO - AO) + (PO - SO) \\ &\quad + (MQ - RM) + (MC - MB) + (AL - LB) \\ &= DC - AD + PS + QR - BC + AB \end{aligned}$$

where we in the second equality applied Grossman's first and second characterizations two times each. Hence

$$SR + PQ - PS - QR = AB + CD - BC - DA.$$

According to Pitot's theorem applied in quadrilaterals $PQRS$ and $ABCD$, one of them is tangential if and only if the other one is tangential. □

Now let us state and prove the last characterization in this paper.

Theorem 6.3. *In a convex quadrilateral $ABCD$ where the angle bisectors at A and C intersect at the interior point I, let E, F, G, H be the projections of I on the sides AB, BC, CD, DA respectively, and let k_1, k_2, k_3, k_4 be* *the incircles in triangles AEH, BFE, CGF, DHG respectively. Denote the common external tangent lines of k_1 and k_2, of k_2 and k_3, of k_3 and k_4, and of k_4 and k_1, different from the sides of $ABCD$, by t_{12}, t_{23}, t_{34}, t_{41} respectively. Let $P = t_{41} \cap t_{12}$, $Q = t_{12} \cap t_{23}$, $R = t_{23} \cap t_{34}$ and $S = t_{34} \cap t_{41}$. Then $PQRS$ is a rhombus if and only if $ABCD$ is a tangential quadrilateral.*

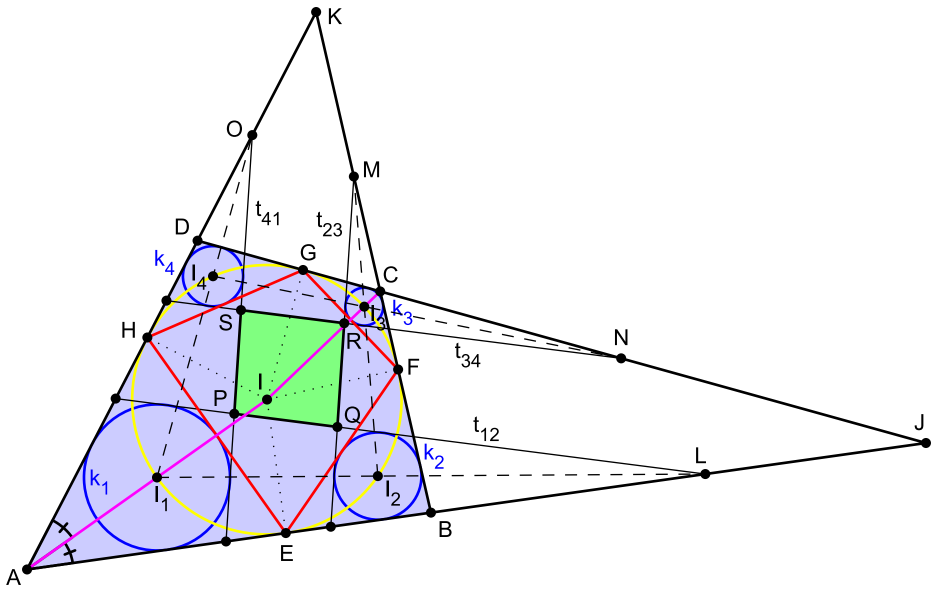

FIGURE 30. $ABCD$ is tangential $\Leftrightarrow$ $PQRS$ is a rhombus

Proof. Let $J = AB \cap CD$, $K = AD \cap BC$, $L = t_{12} \cap AB$, $M = t_{23} \cap BC$, $N = t_{34} \cap CD$, and $O = t_{41} \cap AD$ (see Figure 30).

($\Rightarrow$) When $ABCD$ is tangential, then its incircle k is tangent to the sides AB, BC, CD, DA at E, F, G, H respectively. It follows that quadrilateral $I_1I_2I_3I_4$ is inscribed in circle k according to Lemma 6.1.

Next we prove that opposite sides of $PQRS$ are parallel. In order to prove that $PQ \parallel SR$, we show the equivalent assertion $\angle PLA + \angle SND = \angle AJD$. To this end, we have

$$\begin{aligned}
\angle PLA + \angle SND &= 2\angle I_1LE + 2\angle I_4NG \\
&= 2\angle I_1EA - 2\angle EI_1I_2 + 2\angle I_4GD - 2\angle GI_4I_3 \\
&= 2\angle I_1EA - 2\angle EFI_2 + 2\angle I_4GD - 2\angle GFI_3 \\
&= \angle AEH - \angle EFB + \angle DGH - \angle GFC \\
&= \frac{\pi - \angle A}{2} - \frac{\pi - \angle B}{2} + \frac{\pi - \angle D}{2} - \frac{\pi - \angle C}{2} \\
&= \frac{\angle B + \angle C}{2} - \frac{\angle A + \angle D}{2} \\
&= \frac{2\pi - (\angle A + \angle D)}{2} - \frac{\angle A + \angle D}{2} \\
&= \pi - \angle A - \angle D \\
&= \angle AJD
\end{aligned}$$

confirming that $PQ \parallel SR$. In the same way it is proved that $QR \parallel PS$, so we now know that $PQRS$ is a parallelogram. Thus $PQ = SR$ and $QR = PS$. But $PQRS$ is also tangential according to Lemma 6.2, so $PQ + SR = QR + PS$. It follows that $PQ = RS = QR = PS$, so $PQRS$ is indeed a rhombus.

($\Leftarrow$) For the converse, we start with a convex quadrilateral $ABCD$ such that $PQRS$ is a rhombus. Since a rhombus is a special case of a tangential quadrilateral, the converse in Lemma 6.2 directly yields that $ABCD$ is tangential, concluding the proof. □

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