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COUNTING PERPENDICULARLY INSCRIBED POLYGONS THAT INTERSECT A GIVEN SIDE IN AN ODD SIDED REGULAR POLYGON

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Abstract. A polygonal chain P defined by the sequence $(A_1, A_2, \dots, A_k)$ is said to be perpendicularly inscribed in a simple polygon Q if the line segment A_iA_{i+1} is perpendicular to the side of Q in which A_i lies. If P closes after n steps, then P is a perpendicularly inscribed polygon in P. The goal of this paper is to determine the number of perpendicularly inscribed polygons that intersect a given side of a regular polygon with an odd number of sides. This is done using circular permutations with repetition and partitions of an integer, and some special cases are calculated via circulant matrices and the Binomial Theorem. A method for finding such polygons, based on Banach's Fixed Point Theorem, is also developed.

1. Introduction

A polygonal chain P defined by the sequence $(A_1, A_2, \dots, A_k)$ is *inscribed* in a simple polygon Q if every vertex of P lies in a side of Q. Moreover, we will say that P is *perpendicularly inscribed* in Q if the line segment A_iA_{i+1} is perpendicular to the side of Q in which A_i lies. We also refer to P as an *orbit*.

This paper focuses in the case in which P closes after n steps (i.e., $A_{n+1} = A_1$), so P is a polygon perpendicularly inscribed in the simple polygon Q. As before, we also refer to such a polygon as a *periodic orbit* with period n, or a n-periodic. See Figure 1.

The problem of finding a polygon inscribed in some kind of geometric object has been studied in the context of billiards, for example. In elliptic billiards [5, 13, 12, 3, 1], this is related to Poncelet's Porism [8], whereas for polygonal billiards this is a harder problem [4]. For triangular billiards, the problem has been solved in some special cases [2, 6]. Non-billiard papers, which are more closely related to this one, appear in the works of Lowry [9] and Pamfilos [11].

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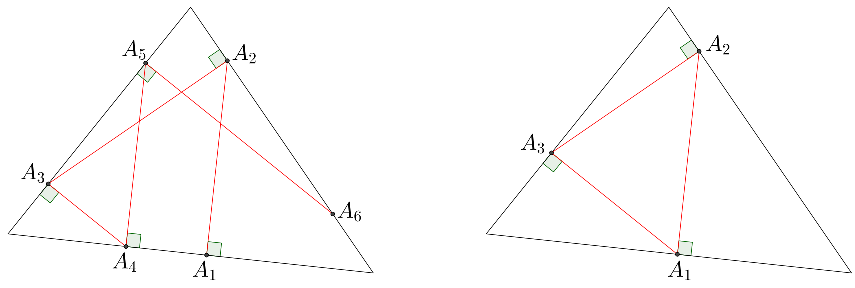

FIGURE 1. On the left, a polygonal chain with 6 points perpendicularly inscribed in a triangle. On the right, a triangle perpendicularly inscribed in another triangle.

The goal of this paper is to determine the number of perpendicularly inscribed polygons that intersect a given side of an odd sided regular polygon. Regular polygons with an even number of sides are not considered since the periodic orbits produced will always have period 2.

In Section 2, an argument based on Banach's Fixed Point Theorem that allows us to find the periodic orbits is presented. Section 3 develops the main result, counting the periodic orbits using circular permutations with repetition [10]. Some special cases can be easily calculated using circulant matrices [7]. This is done in Section 4. Section 5 concludes the work with applications of the formula deduced in Section 3.

2. A method to produce periodic orbits

Consider an odd sided regular polygon Q. Its sides are labeled 1, 2, ..., $2k+1$ counterclockwise as in Figure 2. This is done simply to distinguish the orbits by the sequence of sides that are visited by the polygonal chain, starting at side 1.

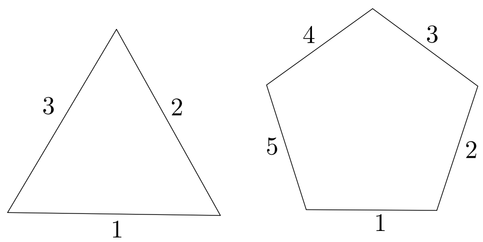

FIGURE 2. The labels of the regular polygon's sides.

Choose two points P_1 and Q_1 on a given side i of Q such that the lines perpendicular to i through P_1 and Q_1 intersect the triangle on points P_2 and Q_2, respectively, that also lie on the same side. Let θ be the measure of the internal angles of Q, and let x_1 and x_2 be the measures of the segments P_1Q_1 and P_2Q_2, respectively. Then, as Figure 3 indicates,

$$x_2 = \frac{x_1}{\sin(\theta/2)}. \tag{1}$$

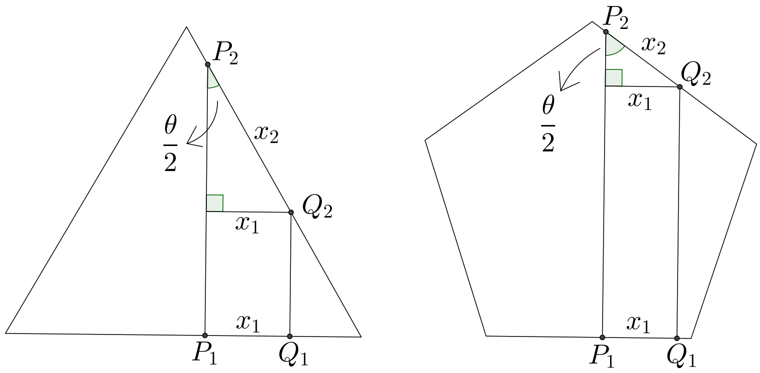

FIGURE 3. Figures for (1).

Repeating this argument, if the orbits of P_1 and Q_1 return to their original side after n iterations, then the segment defined by the returning points P_n and Q_n has length x_n satisfying

$$x_n = \frac{x_1}{\sin^n(\theta/2)}. \tag{2}$$

Hence, the backward orbit map with domain P_nQ_n and side i as codomain is a contraction. The backward orbit of any point between P_n and Q_n converges to an unique fixed point by Banach's Fixed Point Theorem. If the chosen sequence of sides visits only two of Q's sides, then this fixed point is a vertex of Q (equivalently, there are no 2-periodics in odd sided regular polygons). If not, then it is a vertex of a periodic orbit with period n.

3. The number of perpendicularly inscribed polygons that visit a given side

Let Q be a regular polygon with $2k+1$ sides. We wish to count all periodic orbits with period n that pass through side 1. Let $O_{2k+1}(n)$ denote this number. Without loss of generality, we can consider side 1 to be the first side visited by the orbit and let

$$\alpha = (1, \alpha_2, \ldots, \alpha_n)$$

be the sequence of sides it visits before returning to side 1. Also, suppose that more than two different sides appear in α.

Notice that the lines perpendicular to side i can only intersect either side $i+k$ or side $i+k+1$, where additions are performed modulo $2k+1$. Consider a graph G_{2k+1} with $2k+1$ vertices such that on vertex i there are only edges adjoining it to vertices $i+k$ and $i+k+1$ modulo $2k+1$ and let A_{2k+1} be its adjacency matrix. The entry $(1,1)$ in A_{2k+1}^n counts the number of sequences of n sides that start in side 1 and also end in side 1.

Besides the sequences that involve only two of Q's sides, there are other sequences that prevent this entry from being our desired number. For example, if α is such that $\alpha_i = 1$ for some $i \in \{3, \dots, n-1\}$, then α is counted more than once, i.e., periodic orbits produced from sequences such as 123132 and 132123 are the same, only the starting point on side 1 is changed. The same happens whenever a sequence is a cyclic permutation of another.

Hence, we break α into smaller pieces, each with 1 appearing only as the first element. We call these *pure orbits* (of length r). The number of pure orbits of length r in Q, which will be denoted by $P_{2k+1}(r)$, can be calculated as follows. Consider a graph H_{2k+1} obtained from G_{2k+1} by deleting both edges that incide on vertex 1 and let B_{2k+1} be its adjacency matrix. $P_{2k+1}(r)$ is given by the sum of entries of the submatrix of B_{2k+1}^{r-2} given by rows and columns in the set $\{k+1, k+2\}$. Notice that we are considering $P_{2k+1}(2) = O_{2k+1}(2) = 2$ for every k.

Therefore, we have to consider partitions of n in parts that are at least 2, since it is necessary to reach at least one other side before returning to side 1. Every part r in the partition represents a pure orbit of length r. We first consider partitions with at least two distinct parts.

Let p be a partition of n into parts $p_1, p_2, \dots, p_m$ that are all greater than 2. The number $F(p)$ of partitions that are not cyclical permutations of p corresponds to the number of circular permutations of m parts out of r different kinds, with β_1 of the first kind, β_2 of the second and so forth. Let $CP(\beta_1, \dots, \beta_r)$ and $LP(\beta_1, \dots, \beta_r)$ be the numbers of circular and linear permutations of these k objects. By [10], $F(p)$ is given by

$$CP(\beta_1, \dots, \beta_r) = \frac{1}{m} \sum_{d|m} \phi(d) LP\left(\frac{N}{d}\beta_1', \dots, \frac{N}{d}\beta_r'\right) \tag{3}$$

where $N = \gcd(\beta_1, \dots, \beta_r)$, $\beta_i' = \beta_i / N$ and ϕ is Euler's totient function.

Moreover, there are $P_{2k+1}(p_i)$ pure orbits, for $i \in \{1, \dots, m\}$. Hence, the number of sequences of n sides induced by partitions of n that have at least two different parts is

$$\sum_{p \in S_1(n)} F(p) \prod_{i=1}^{m} P_{2k+1}(p_i), \tag{4}$$

where $S_1(n)$ is the set of partitions of n with parts greater than 1 and, at least, two distinct ones among them.

On the other hand, partitions of n with equal parts consist of a divisor $d \neq 1$ of n added n/d times. For every part, we must choose one pure d-periodic among the $\ell_d = P_{2k+1}(d)$ possibilities. Hence, if β_1, ..., β_{ℓ_d} are the numbers of pure d-periodics of each kind, then

$$\sum_{i=1}^{\ell_d} \beta_i = \frac{n}{d}. \tag{5}$$

Consequently, if $S_2(n/d)$ is the set of nonnegative integer solutions of (5) and $x \in S_2(n/d)$, then the number of sequences of n sides induced by these partitions is

$$\sum_{\substack{d|n \\ d\neq 1}} \sum_{x\in S_2(n/d)} CP(\hat{x}), \tag{6}$$

where $\hat{x}$ denotes the nonzero terms in x.

Finally, notice that for every divisor $d \neq 1$ of n, the d-periodics traveled n/d times are counted above, so we need to subtract

$$\sum_{\substack{d|n \\ d\neq 1}} O_{2k+1}(d). \tag{7}$$

Notice that when we subtract $O_{2k+1}(2) = 2$ we are disregarding the sequences of sides that visit only two of them. Combining (4), (6) and (7) we get

$$\begin{aligned} O_{2k+1}(n) = &\sum_{p\in S_1(n)} F(p) \prod_{i=1}^{m} P_{2k+1}(p_i) \\ &+ \sum_{\substack{d|n \\ d\neq 1}} \left(\sum_{x\in S_2(n/d)} CP(\hat{x}) - O_{2k+1}(d) \right). \end{aligned} \tag{8}$$

Some examples of the formula above will be discussed in Section 5.

4. Results regarding some special cases

The i-th line of the adjacency matrix A_{2k+1}, defined in the previous Section, has nonzero elements (which are ones) in columns $i+k$ and $i+k+1$ modulo $2k+1$. It follows that A_{2k+1} is a circulant matrix. To easily compute powers of a circunlant matrix, we follow [7] and consider the matrix C_{2k+1} such that its i-th line consists entirely of zeros, except for a 1 in column $i+1$.

A_{2k+1} can be written, then, as

$$A_{2k+1} = C_{2k+1}^{k} + C_{2k+1}^{k+1}. \tag{9}$$

Notice, also, that $C_{2k+1}^{2k+1} = I_{2k+1}$, the identity matrix. We now prove some interesting results.

Proposition 4.1. *$O_{2k+1}(2k+1) = 2$ for every $k \geq 1$.*

Proof. The paths $1 \to k+1 \to 2k+1 \to \ldots \to (2k+1)k+1$ and $1 \to (k+1)+1 \to 2(k+1)+1 \to \ldots \to (2k+1)(k+1)+1$ (modulo $2k+1$) are $(2k+1)$-periodics. We show that these are the only ones. In order to calculate the $(1,1)$ entry of A_{2k+1}^{2k+1}, we use the Binomial Theorem and write

$$\begin{aligned} A_{2k+1}^{2k+1} &= \left(C_{2k+1}^{k} + C_{2k+1}^{k+1}\right)^{2k+1} \\ &= \sum_{j=0}^{2k+1} \binom{2k+1}{j} C_{2k+1}^{kj} C_{2k+1}^{(k+1)(2k+1-j)}. \end{aligned}$$

The exponent of C_{2k+1} in the last expression can be simplified to

$$(k+1)(2k+1) - j.$$

Since we are looking for the $(1,1)$ entry (actually all entries in the diagonal are the same), we want the coefficient of $C_{2k+1}^{2k+1} = I_{2k+1}$ in this expansion. The exponent is a multiple of $2k+1$ if $j=0$ or if $j = 2k+1$, so the desired coefficient is

$$\binom{2k+1}{0} + \binom{2k+1}{2k+1} = 2.$$

Hence, there are only two ways to start in side 1 and return to it after $2k+1$ steps. Illustrations for $O_3(3)$ and $O_5(5)$ are provided in Figure 4.

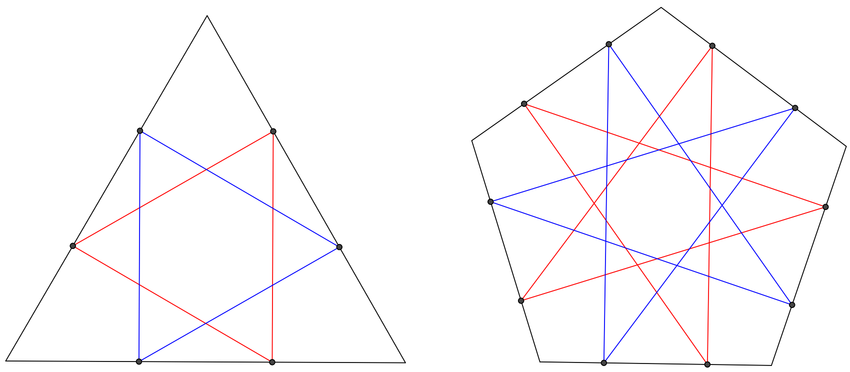

FIGURE 4. Illustrations for $O_{2k+1}(2k+1) = 2$ for $k=1$ (triangle) and $k=2$ (pentagon).

Proposition 4.2. *$O_{2k+1}(2t+1) = 0$ if $1 \leq t < k$.*

Proof. We now have

$$\begin{aligned} A_{2k+1}^{2t+1} &= \left(C_{2k+1}^{k} + C_{2k+1}^{k+1}\right)^{2t+1} \\ &= \sum_{j=0}^{2t+1} \binom{2t+1}{j} C_{2k+1}^{kj} C_{2k+1}^{(k+1)(2t+1-j)}. \end{aligned}$$

Now, a simplification of the exponent yields

$$(2t+1)(k+1) - j.$$

Since $j \in \{0, 1, \ldots, 2k+1\}$, we have

$$(2t+1)k = (2t+1)(k+1) - (2t+1) \leq (2t+1)(k+1) - j \leq (2t+1)(k+1),$$

so the exponent cannot equal any multiple of $2k+1$, hence the coefficient of the identity matrix in the expansion is zero. In particular, there is no $(2t+1)$-periodic in a regular polygon with $2k+1$ sides for $t < k$.

Proposition 4.3. *$O_{2k+1}(4) = 3$ for all $k \geq 1$.*

Proof. As before, we write

$$\begin{aligned} A_{2k+1}^4 &= \left(C_{2k+1}^k + C_{2k+1}^{k+1}\right)^4 \\ &= \sum_{j=0}^{2t+1} \binom{4}{j} C_{2k+1}^{kj} C_{2k+1}^{(k+1)(4-j)}. \end{aligned}$$

The exponent of C_{2k+1} is

$$4k + 4 - j,$$

which is a multiple of $2k+1$ only for $j = 2$. The coefficient in the binomial expansion is $\binom{4}{2} = 6$. These include the sequences $(1, k+1, 1, k+1)$ and $(1, k+2, 1, k+2)$, which do not produce periodic orbits, and the sequences $(1, k+1, 1, k+2)$ and $(1, k+2, 1, k+1)$, which produce the same periodic orbit. Hence, out of the 6 walks that arise from the $(1,1)$ entry of A_{2k+1}^4, only 3 remain: $(1, k+1, 2k+1, k+1)$, $(1, k+2, 2, k+2)$ and $(1, k+1, 1, k+2)$. Figure 5 illustrates the cases of $O_3(4)$ and $O_5(4)$.

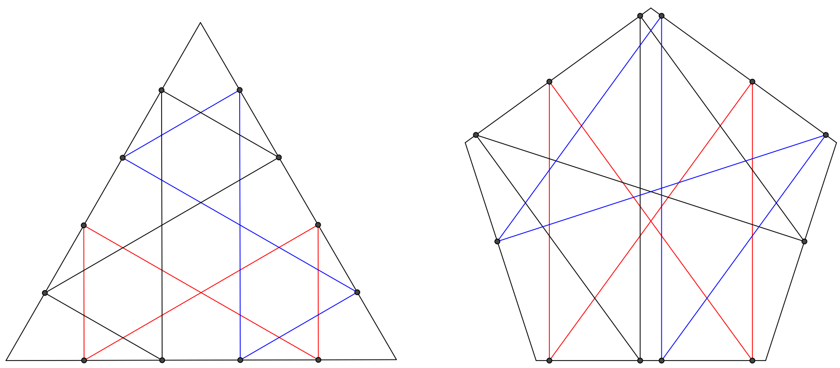

FIGURE 5. Illustrations for $O_{2k+1}(4) = 3$ for $k = 1$ (triangle) and $k = 2$ (pentagon).

Proposition 4.4. *$O_{2k+1}(2k+3) = 4k+2$ for all $k \geq 1$.*

Proof. Again, write

$$\begin{aligned} A_{2k+1}^{2k+3} &= \left(C_{2k+1}^k + C_{2k+1}^{k+1}\right)^{2k+3} \\ &= \sum_{j=0}^{2t+1} \binom{2k+3}{j} C_{2k+1}^{kj} C_{2k+1}^{(k+1)(2k+3-j)}. \end{aligned}$$

The exponent simplifies to

$$(k+2)(2k+1) + 1 - j,$$

which is a multiple of $2k+1$ only for $j = 1$ or $j = 2k+2$. Hence, the $(1,1)$ entry of A_{2k+1}^{2k+3} is

$$\binom{2k+3}{1} + \binom{2k+3}{2k+2} = 4k + 6.$$

The only remaining partitions of $2k+3$ are $2+(2k+1)$ and $2k+3$ itself, since every partition must involve at least one odd number and we have seen in Proposition 4.2 that there are no pure sequences of sides with odd length smaller than $2k+1$. Hence, the periodic orbits in the partition $2+(2k+1)$ are being counted twice. Since $P_{2k+1}(2) = 2$ and $P_{2k+1}(2k+1) = 2$ (by Proposition 4.1), then

$$O_{2k+1}(2k+3) = 4k+6-2\cdot 2 = 4k+2.$$

An illustration for $O_3(5) = 6$ is given in Figure 6.

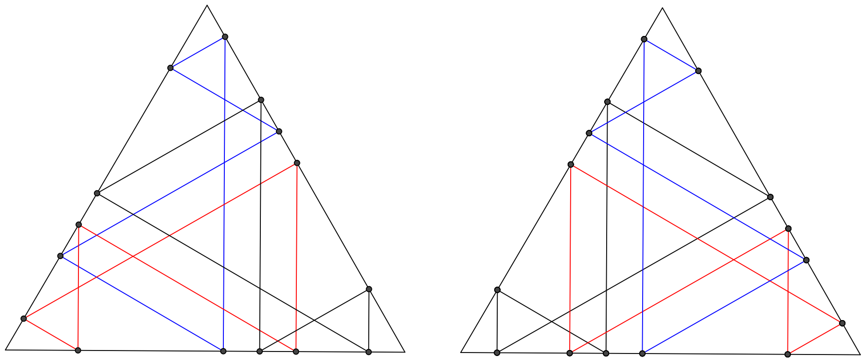

FIGURE 6. Illustrations for $O_{2k+1}(2k+3) = 4k+2$ for $k=1$ (triangle).

To finish this Section we note that all the periodic orbits appearing in Figures 4, 5 and 6 can be transformed into one another by a symmetry of the regular polygon. This does not happen every time, which can be seen, for example, in the case of the 6-periodics originating from the sequences $(1,2,3,2,1,3)$ and $(1,2,1,3,1,2)$ in an equilateral triangle, shown in Figure 7.

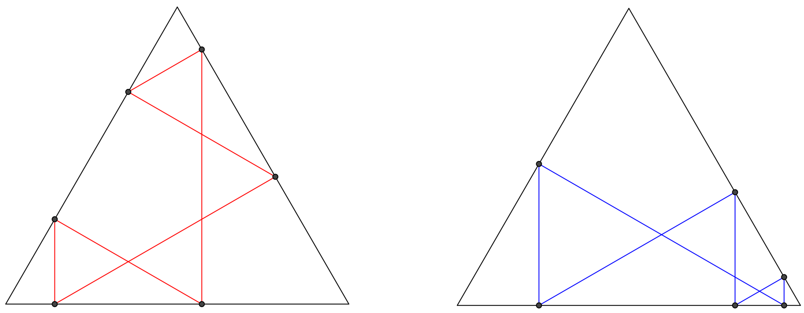

FIGURE 7. Two periodic orbits that cannot be transformed into each other by a symmetry of the regular polygon.

5. APPLICATIONS

Example 5.1. $O_3(8) = 30$.

Proof. It is easy to see that, in an equilateral triangle, there are exactly 2 pure r-periodics for every $r \geq 2$, so

$$\prod_{i=1}^{m} P_{2k+1}(p_i) = 2^m$$

in (4). The partitions with at least two distinct parts are listed in Table 1.

Table 1. Partitions of 8 with at least two distinct parts.

p	m	$F(p)$	$2^m F(p)$
$2+6$	2	$CP(1,1)=1$	4
$3+5$	2	$CP(1,1)=1$	4
$2+2+4$	3	$CP(2,1)=1$	8
$2+3+3$	3	$CP(1,2)=1$	8
Total			24

The trivial partition produces 2 (pure) 8-periodics. The other partitions with equal parts are $4+4$ and $2+2+2$ (with n/d equal to 2 and 4, respectively). They are displayed in Table 2.

Table 2. The $4+4$ and $2+2+2+2$ partitions of 8.

$4+4$	(α_1,α_2)	$CP(\alpha_1,\alpha_2)$	$2+2+2+2$	(α_1,α_2)	$CP(\alpha_1,\alpha_2)$
	$(2,0)$	1		$(4,0)$	1
	$(1,1)$	1		$(3,1)$	1
	$(0,2)$	1		$(2,2)$	2
Total		3		$(1,3)$	1
				$(0,4)$	1
			Total		6

By (8) and Proposition 4.3,

$$O_3(8) = 24 + 2 + 3 + 6 - O_3(2) - O_3(4) = 35 - 2 - 3 = 30.$$

Example 5.2. $O_5(6) = 10$.

Proof. The partitions of 6 are $2+2+2$, $2+4$, $3+3$ and 6 itself. By Proposition 4.2 there are no pure 3-periodics, so the $3+3$ partition does not need to be analyzed. As in Section 3, we consider the matrix

$$B_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The numbers of pure orbits with length 2 is 2, whereas for 4 and 6 they are given by the sum of entries in the following submatrices of B_5^2 and B_5^4, respectively:

$$B_5^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \boxed{1} & \boxed{0} & 0 \\ 0 & 0 & \boxed{0} & \boxed{1} & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad B_5^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 & 0 \\ 0 & 3 & \boxed{2} & \boxed{0} & 0 \\ 0 & 0 & \boxed{0} & \boxed{2} & 3 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix}.$$

Hence, $P_5(4) = 2$ and $P_5(6) = 4$. The partition $p = 2 + 4$ has $F(p) = CP(1, 1) = 1$, so it produces $F(p)P_5(2)P_5(4) = 4$ periodic orbits.

Now let $p = 2 + 2 + 2$. There are two kinds of pure 2 orbits and $n/2 = 3$, so there are

$$CP(3, 0) + CP(2, 1) + CP(1, 2) + CP(0, 3) = 4$$

periodic orbits.

Finally, we conclude that

$$O_5(6) = 4 + 4 + 4 - O_5(2) - O_5(3) = 10.$$

6. Acknowledgements

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