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MINIMAL NUMBER OF POINTS ON A GRID FORMING LINE SEGMENTS OF EQUAL LENGTH

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Abstract. We consider the minimal number of points on a regular grid on the plane that generates n line segments of points of exactly length k. We illustrate how this is related to the n-queens problem on the toroidal chessboard and show that this number is upper bounded by $kn/3$ and approaches $kn/4$ as $n \to \infty$ when $k+1$ is coprime with 6 or when k is large.

1. Introduction

We consider points on a regular grid on the plane which form horizontal, vertical or diagonal line segments of exactly k points [1]. For example, the set of 12 points in Fig. 1 form many line segments and form exactly 3 (overlapping) line segments of length 5. Note that since a line segment of length k consists of exactly k points and no more[2], the set of points in Fig. 1 contains 4 line segments of length 2 and does not contain any line segments of length 4 or of length 3. Our motivation for studying this problem is the Bingo-4 problem proposed by Sun et al. and described in OEIS [6] sequence A273916 where the case $k = 4$ is considered. This problem can be considered a type of orchard-planting problem [3] restricted to a grid.

Let $a_k(n)$ denote the minimal number of points needed to form n line segments of length k. Fig. 1 shows that $a_5(3) = 12$ as any constellation of 11 points will not generate 3 segments of length 5. Note that the constellation of points achieving $a_k(n)$ is typically not unique. Finding the exact value of $a_k(n)$ appears to be difficult and currently not feasible for large n. The purpose of this paper is to provide an analysis on the asymptotic behavior of $a_k(n)$.

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[1] We use the convention that an isolated point corresponds to 4 line segments of length 1; a horizontal, a vertical and 2 diagonal line segments.

[2] This implies that two line segments of the same orientation (horizontal, vertical or diagonal) must not overlap.

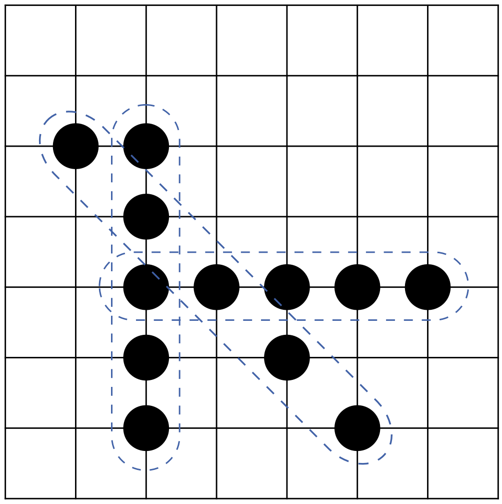
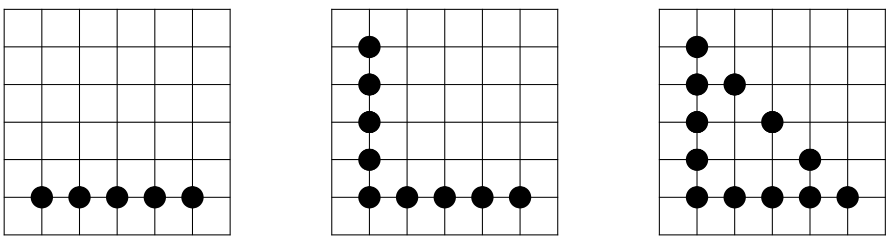


Figure 1. A constellation of 12 points on a grid. Among the line segments formed by these points are 3 (overlapping) line segments of length 5 as shown by the dashed boundaries.

2. Bounds and asymptotic behavior of $a_k(n)$

It is easy to show that $a_k(1) = k$, $a_k(2) = 2k - 1$ and $a_k(3) = 3(k - 1)$ as 2 line segments overlap in at most one point and 3 line segments overlaps in at most 3 points, as illustrated in Fig. 2 for $k = 5$. Note that $a_k(3)$ can be obtained with points forming a right isosceles triangle.

Lemma 2.1 (Fekete's subadditive Lemma [4])**.** *If the sequence $a(n)$ is subadditive, i.e. $a(n + m) \leq a(n) + a(m)$, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\inf_n \frac{a_n}{n}$.*



(A) $a_5(1) = 5$ (B) $a_5(2) = 9$ (C) $a_5(3) = 12$

Figure 2. Sets of points illustrating $a_k(n)$ for $n = 1, 2, 3$.

Theorem 2.1. *For all k, $a_k(n)$ is subadditive, and $f(k) = \lim_{n\to\infty} \frac{a_k(n)}{n}$ exists and satisfies $\frac{k}{4} \leq f(k) \leq \frac{k}{3}$.*

Proof. Since each line segment takes k points and each point can be part of at most 4 line segments (horizontal, vertical or diagonal), $a_k(n) \geq \frac{kn}{4}$. Since the set of points for $a_k(n)$ and $a_k(m)$ separated apart leads to $m+n$ line segments of length k, it is clear that $a_k(n)$ is subadditive. Lemma 2.1 implies that $f(k)$ exists and is equal to $\inf_n \frac{a_k(n)}{n}$. Consider a k by m rectangular array of points with $k \leq m$. There are m vertical line segments and $m-k+1$ diagonal line segments of each orientation and thus there are $3m-2k+2$ length k line segments. This shows that $a_k(3m-2k+2) \leq km$ which implies that $\frac{k}{4} \leq f(k) \leq \frac{k}{3}$. □

3. Constellations where each point is part of 4 different line segments

The upper bound $\frac{k}{3}$ on $f(k)$ in Theorem 2.1 shows that for large n we can construct a constellation of n points such that most points are part of 3 different line segments. Is it possible to construct a constellation such that most points are part of 4 different line segments (a horizontal, a vertical and two diagonal line segments) and thus achieve the lower bound $\frac{k}{4}$? The case $k = 1$ is simple. Since $a_1(4n) = n$ as exhibited by the constellation of n isolated points, this implies that $f(1) = \frac{1}{4}$.

Let $\sigma \in S_{k+1}$ be a permutation on the integers $\{0, 1, \cdots, k\}$. Consider a $k+1$ by $k+1$ square grid and place a point on each position (i, j) except when it is of the form $(i, \sigma(i))$. It is clear that tiling this grid on the plane results in a constellation where every point is part of a horizontal and a vertical line segment of length k. The shear maps $(i, j) \to (i, i+j)$ and $(i, j) \to (i, i-j)$ map the two diagonal line segments to a vertical line segment. Thus in order to also have every point be part of two diagonal line segments of exactly k points, we want $\{i + \sigma(i) \mod (k+1)\}$ and $\{i - \sigma(i) \mod (k+1)\}$ to be permutations of $\{0, 1, \cdots, k\}$ as well. If this is the case, consider a N by N subgrid of this tiling and let n be the number of points in this subgrid. Except for points near the edges which is on the order of $kN \propto k\sqrt{n}$, all points belong to 4 line segments of length k. Thus we have proved the following:

Theorem 3.1. *If there is a permutation σ of the numbers $\{0, 1, \cdots, k\}$ such that $\sigma_1 = \{i + \sigma(i) \mod (k+1)\}$ and $\sigma_2 = \{i - \sigma(i) \mod (k+1)\}$ are both permutations, then $f(k) = \frac{k}{4}$. In particular, $\frac{a_k(n)}{n}$ converges to $f(k)$ on the order of $O\left(\frac{1}{\sqrt{n}}\right)$.*

If σ satisfies the conditions of Theorem 3.1, then so does σ^{-1}. For a fixed integer m, the permutation $\sigma(i) + m \mod (k+1)$ also satisfies these conditions. We will use this to partition the set of admissible permutations into equivalent classes. More specifically,

Definition 3.1. *Let S_{k+1} be the set of permutations on $\{0, 1, \cdots, k\}$. $T_{k+1} \subset S_{k+1}$ is defined as the set of permutations σ such that $\{i+\sigma(i) \mod (k+1)\}$*

and $\{i - \sigma(i) \mod (k+1)\}$ *are in* S_{k+1}*. The equivalence relation* $\sim$ *is defined on* T_{k+1} *as follows. If* $\sigma, \tau \in T_{k+1}$*, then* $\sigma \sim \tau$ *if* $\tau = \sigma^{-1}$ *or there exist an integer* m *such that* $\sigma(i) = \tau(i) + m \mod (k+1)$ *for all* i*.*

Thus Theorem 3.1 implies that if $T_{k+1} \neq \varnothing$, then $f(k) = \frac{k}{4}$.

4. Modular n-queens problem

In this section we show that the above constellation is related to an n-queens problem on a toroidal chessboard. The n-queens problem asks whether n nonattacking queens can be placed on an n by n chessboard. The answer is yes and is first shown by Pauls [7, 1]. Next consider a toroidal n by n chessboard, where the top edge is connected to the bottom edge and the left edge is connected to the right edge. The corresponding n-queens problem is called a *modular* n-queens problem. For the $k+1$ by $k+1$ square grid above, if we put a queen on each position $(i, \sigma(i))$, then it is easy to see that $\sigma \in T_{k+1}$ if and only if it provides a solution to the modular $(k+1)$-queens problem. For instance, for $k = 4$, consider the permutation $\sigma = (0, 2, 4, 1, 3)$. Figure 3 shows a 5 by 5 grid where all the points are part of 4 line segments if the grid tiles the plane (or equivalently, the grid lives on a torus). This means that each point in the center of a finite tiling are part of 4 line segments. If we put a queen on each of the 5 empty locations, we obtain a solution to the modular 5-queens problem.

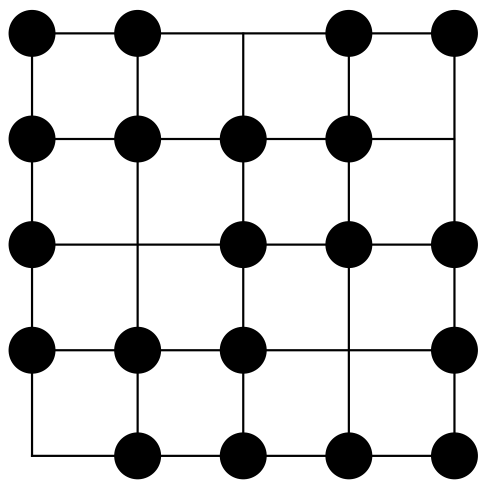


Figure 3. Points where the empty locations are of coordinates $(i, \sigma(i))$. Putting a queen at each empty location results in a solution to the modular 5-queen problem.

Pólya [8] showed that a solution to the modular n-queens problem exists if and only if n is coprime with 6. Thus Pólya's result is equivalent to the following:

Theorem 4.1. *$T_{k+1} \neq \varnothing$ if and only if $k+1$ is coprime with 6.*

Corollary 4.1. *If $k+1$ is coprime with 6, then $f(k) = \frac{k}{4}$.*

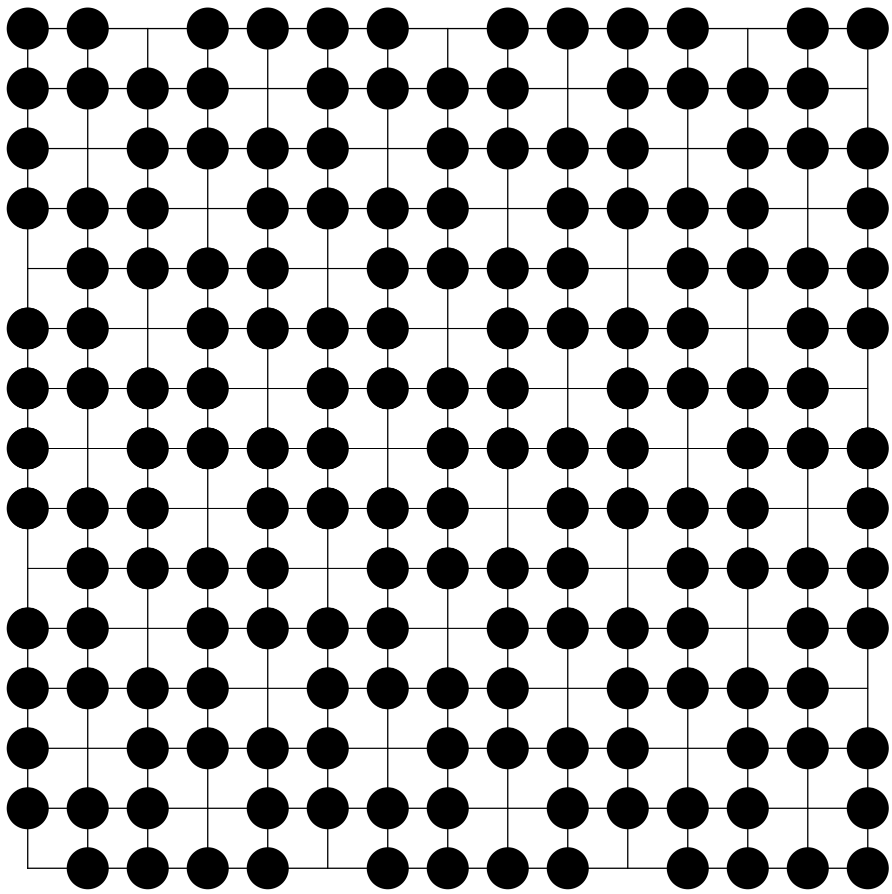


FIGURE 4. A lattice constellation. Points in the center of the grid are part of 4 different patterns, showing that $\frac{a_4(n)}{n} \to 1$ as $n \to \infty$.

Monsky [5] shows that $n-2$ nonattacking queens can be placed on an n by n toroidal chess board and $n-1$ queens can be placed if n is not divisible by 3 or 4. This implies the following which shows that for k large, $f(k)$ approaches the lower bound $\frac{k}{4}$:

Theorem 4.2. $f(k) \leq \frac{k(k+1)+2}{4(k-1)}$. *If $k+1$ is not divisible by 3 or 4, then* $f(k) \leq \frac{k(k+1)+1}{4k}$.

Proof. Consider a $k+1$ by $k+1$ array with $k+1-r$ nonattacking queens. By placing a point only on the locations where there are no queens we obtain a constellation with $(k+1)^2 - (k+1-r)$ points. Each point then is part of 4 line segments of length k. Thus when this array is tiled, we get for a large number of points a ratio $\frac{a_k(n)}{n}$ approaching $\frac{(k+1)^2-(k+1-r)}{4(k+1-r)} = \frac{k(k+1)+r}{4(k+1-r)}$. The conclusion follows by setting $r = 1$ or $r = 2$. □

Corollary 4.2. $\lim_{k\to\infty} \frac{f(k)}{k} = \frac{1}{4}$.

4.1. **Lattice construction.** As in the n-queens problem, we can construct permutations in T_{k+1} via a lattice construction.

Definition 4.1. *Given two vectors v_1 and v_2, the lattice construction is defined as a constellation of points such that a point is on the grid if and only if the point is not a linear combination of v_1 and v_2.*

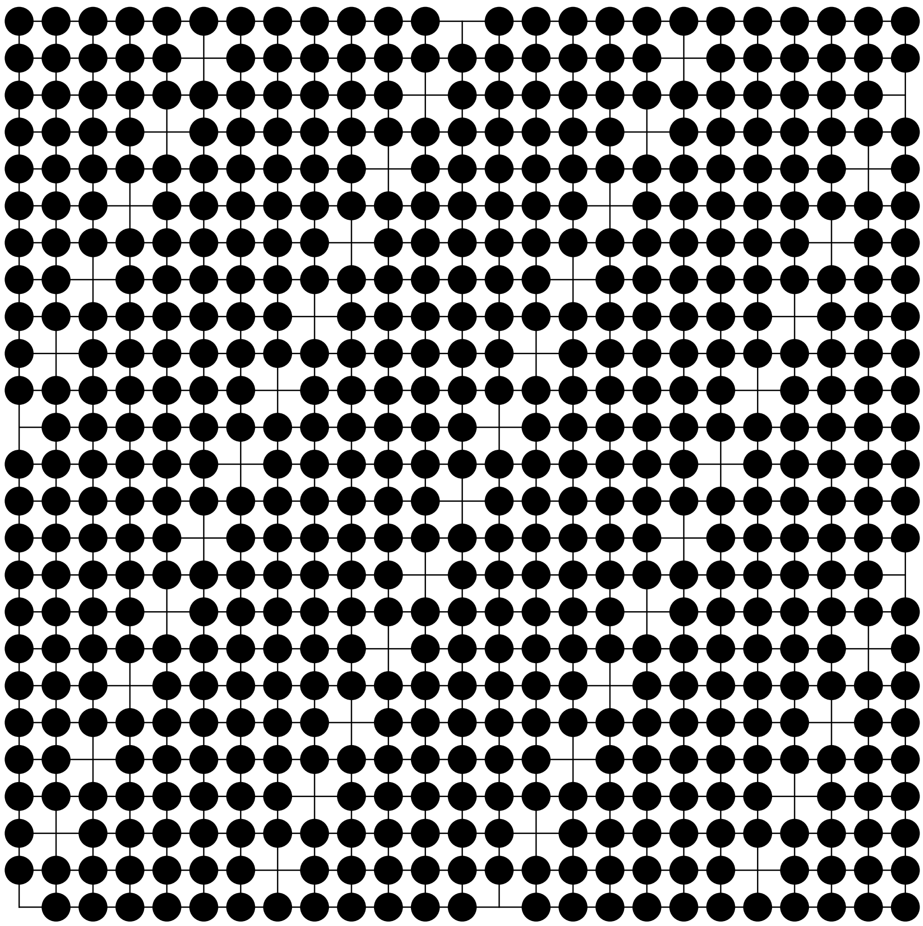


FIGURE 5. A lattice constellation for $k = 12$ generated by vectors $(1, 2)$ and $(0, 13)$.

For instance with the lattice points generated by the vectors $(1, 2)$ and $(2, -1)$, the set of points with $N = 15$ is shown in Fig. 4. In particular, this configuration shows that $f(4) = 1$.

The following result appears to be well-known [1], but we include it here for completeness.

Theorem 4.3. *If there exists $1 < m < k$ such that $m-1$, m and $m+1$ are all coprime with $k+1$, then the lattice construction with $v_1 = (1, m)$ and $v_2 = (0, k+1)$ corresponds to a permutation σ in T_{k+1}.*

Proof. Consider the lattice construction generated by $(1, m)$ and $(0, k+1)$. If m is coprime with $k+1$, then $(m, 2m, \cdots, (k+1)m) \mod (k+1)$ is a permutation σ in S_{k+1} and thus we find in a $k+1$ by $k+1$ subarray empty locations of the form $(i, \sigma(i))$. $i + \sigma(i) \equiv (m+1)i \mod (k+1)$ and $\{i + \sigma(i) \mod (k+1)\}$ is again a permutation since $m+1$ and $k+1$ are coprime. Similarly, $i - \sigma(i) \equiv -(m-1)i \mod (k+1)$ and $\{i - \sigma(i) \mod (k+1)\}$ is a permutation since $m-1$ and $k+1$ are coprime. Thus the conditions of Theorem 3.1 are satisfied and the conclusion follows. □

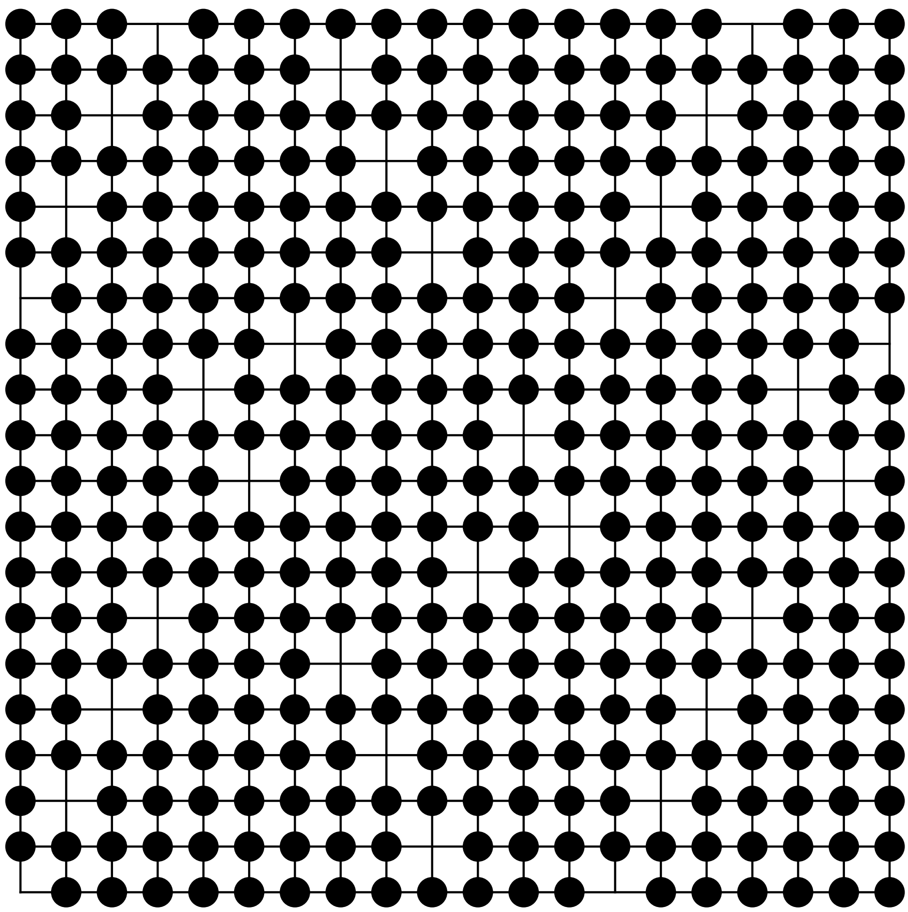


FIGURE 6. A constellation for $k = 12$ corresponding to the permutation $(0, 2, 4, 6, 11, 9, 12, 5, 3, 1, 7, 10, 8)$ and it is not generated by a lattice.

Theorem 4.3 also provides a proof of Corollary 4.1 since if $k+1$ is coprime with 6, then 1, 2 and 3 are all coprime with $k+1$ and we can choose $m = 2$. In particular the lattice construction with $v_1 = (1, 2)$ and $v_2 = (0, k+1)$ generates a permutation σ in T_{k+1}. Fig. 5 shows the construction for $k = 12$.

For $k = 4$, there is only one equivalence class $(0, 2, 4, 1, 3)$ in T_{k+1} that satisfies the conditions of Theorem 3.1. For $k = 6$, there are two equivalent classes $(0, 2, 4, 6, 1, 3, 5)$ and $(0, 3, 6, 2, 5, 1, 4)$. For $k = 10$, there are 4 equivalent classes. In particular, Theorem 4.3 shows that if $k + 1 > 4$ is prime, then there are at least $\frac{k-2}{2}$ equivalent classes in T_{k+1}. This is because each $2 \leq m \leq k - 1$ is coprime with $k + 1$ and the permutation generated by m is the inverse of the permutation generated by $k - 1 - m$ which are equivalent[3]. It is possible to have more than $\frac{k-2}{2}$ equivalent classes as there are permutations in T_{k+1} not generated by a lattice. For $k+1$ coprime with 6, if $k = 4, 6$ and 10, all permutations in T_{k+1} are generated by a lattice. For $k = 12$, there are permutations in T_{k+1} that are not generated by a lattice. One such example is shown in Fig. 6. Such solutions are referred to as *nonlinear* solutions [1].

5. Conclusions

We studied the asymptotic behavior of the minimal number of points needed to generate n line segments of length k using a construction based on permutations of $\{0, 1, \cdots, k\}$ with certain properties. We showed that this construction allows us to create constellations of points where asympotically most points are part of 4 line segments. This construction is equivalent to the modular $(k+1)$-queens problem and thus $f(k) = \frac{k}{4}$ for $k+1$ coprime with 6. If $k + 1$ is even or $k + 1$ is divisible by 3, this construction fails to provide such a constellation. However, results in the modular n-queens problem can still provide an upper bound on $f(k)$ which shows that $\lim_{k\to\infty} \frac{f(k)}{k} = \frac{1}{4}$. Even though these constructions for the modular n-queens problem provide limiting values of $\frac{a_k(n)}{n}$ as $n \to \infty$, for a fixed n the optimal constellation to achieve $a_k(n)$ can be quite different and difficult to compute (see for example `https://oeis.org/A273916/a273916.png`) .

6. Acknowledgements

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