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CIRCLE CHAINS INSIDE THE ARBELOS AND INTEGER SEQUENCES

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Abstract. In this article we study some configurations of chains of mutually tangent circles that can be inscribed inside the arbelos and we find some simple algebraical conditions relating them to certain integer sequences.

1. Introduction

The arbelos (shoemaker's knife in Greek) is a plane figure bounded by three semicircles as shown in Figure 1; Archimedes himself is believed to have been the first mathematicien to study the many properties of such a figure.

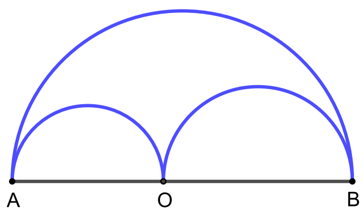

Figure 1. Example of arbelos

Inside the arbelos, it is possible to draw many different kinds of chains of mutually tangent circles; here we take a cue from the paper of C. Bergsten [1] where he built up three different circle chains originating from the so called *twin circles*.

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By drawing the segment OC (see Figure 2), that is perpendicular to the segment AB and divides the arbelos into two smaller regions, the twin circles are two special circles of equal radius so that each of them lies within one of those regions and is tangent to the segment OC and to two of the semi-circular sides. In the next paragraphs we shall present formulas for center

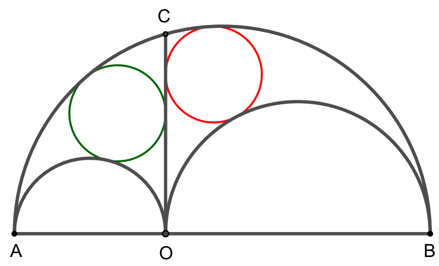

FIGURE 2. Example of arbelos with twin circles

coordinates and radius of the three different kinds of circle chains that can be drawn inside each one of the two parts in which the arbelos is partitioned by the segment OC.

2. First kind of chains

2.1. Center coordinates and radius.

The first kind of chains we are going to consider are the ones that originate from the twin circles and that are directed towards point O which has been chosen as origin of the cartesian axes; as we can notice from Figure 3 we have two chains: one at the left and one at the right of the segment OC respectively.

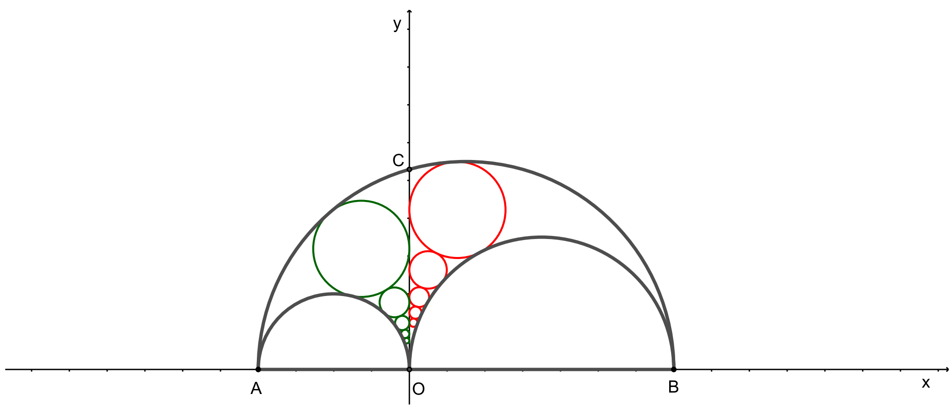

FIGURE 3. First kind of chains

Now, on the basis of the contents of paper [6], it is possible to derive the formulas expressing the center coordinates $\left(x_n^{'}, y_n^{'}\right)$ and radius $R_n^{'}$ of the generic *nth* circle of the chain; notice that here and in the following, we shall use the subscripts r and l to indicate the chains that are at the right and at the left of the segment OC respectively.
If a and b are the radii of the left and rigth semicircles respectively, we have the following expressions for the chains:

$$x_{rn}^{'} = \frac{b}{\left(\sqrt{\frac{a+b}{a}} + n - 1\right)^2} \qquad n \geq 1 \tag{1}$$

$$y_{rn}^{'} = \frac{2b}{\left(\sqrt{\frac{a+b}{a}} + n - 1\right)} \qquad n \geq 1 \tag{2}$$

$$R_{rn}^{'} = \frac{b}{\left(\sqrt{\frac{a+b}{a}} + n - 1\right)^2} \qquad n \geq 1 \tag{3}$$

while, for the left chain, we have:

$$x_{ln}^{'} = -\frac{a}{\left(\sqrt{\frac{a+b}{b}} + n - 1\right)^2} \qquad n \geq 1 \tag{4}$$

$$y_{ln}^{'} = \frac{2a}{\left(\sqrt{\frac{a+b}{b}} + n - 1\right)} \qquad n \geq 1 \tag{5}$$

$$R_{ln}^{'} = \frac{a}{\left(\sqrt{\frac{a+b}{b}} + n - 1\right)^2} \qquad n \geq 1 \tag{6}$$

As put into evidence in [6], it is interesting to notice that, by considering a single chain, the locus of the circle centers is represented by a parabola; therefore, for the right and left chains we respectively have:

$$x = \frac{y^2}{4b} \qquad\qquad x = -\frac{y^2}{4a}$$

Notice also, as one can easily verify, that the focuses of the two parabolas are coincident with the center of the two inner semicircles forming the arbelos i.e.: $(b,0)$ and $(-a,0)$.

2.2. **Integer sequences associated with the chain.**

Let us focus on the right chain by considering the ratios $\frac{b}{R'_{rn}}$, $\frac{2b}{y'_{rn}}$ and $\frac{y'_{rn}}{R'_{rn}}$ that generate the sequences $\left\{\frac{b}{R'_{rn}}\right\}$, $\left\{\frac{2b}{y'_{rn}}\right\}$ and $\left\{\frac{y'_{rn}}{R'_{rn}}\right\}$ respectively. Let us also define the parameter $\lambda = \frac{b}{a}$ with $\lambda \neq 0$; we pose the following question: do exist values of λ so that the sequences $\left\{\frac{b}{R'_{rn}}\right\}$, $\left\{\frac{2b}{y'_{rn}}\right\}$ and $\left\{\frac{y'_{rn}}{R'_{rn}}\right\}$ are integer? The answer is given by the following theorem.

Theorem 2.1. *If $\lambda = k^2 - 1$ where k is an integer and $k \geq 2$, then the sequences $\left\{\frac{b}{R'_{rn}}\right\}$, $\left\{\frac{2b}{y'_{rn}}\right\}$ and $\left\{\frac{y'_{rn}}{R'_{rn}}\right\}$ are integer.*

Proof. *From formulas From formulas (1),(2) and (3), we have:*

$$\frac{b}{R'_{rn}} = \left(\sqrt{1+\lambda} + n - 1\right)^2 \qquad n \geq 1$$

$$\frac{2b}{y'_{rn}} = \left(\sqrt{1+\lambda} + n - 1\right) \qquad n \geq 1$$

and

$$\frac{y'_{rn}}{R'_{rn}} = 2\left(\sqrt{1+\lambda} + n - 1\right) \qquad n \geq 1$$

From the above equations one has that the ratios $\frac{b}{R'_{rn}}$, $\frac{2b}{y'_{rn}}$ and $\frac{y'_{rn}}{R'_{rn}}$ are integer for each $n \geq 1$ only if $\lambda = k^2 - 1$ for any integer $k \geq 2$.

□

Thus, we have that for $k \geq 2$, the following integer sequences are generated:

$$\left\{(k+n-1)^2\right\}_n \tag{7}$$

$$\{(k+n-1)\}_n \tag{8}$$

$$\{2\,(k+n-1)\}_n \tag{9}$$

We immediately see that formula (7) represents, for any k, a sequence composed by only square numbers that is classified in OEIS (The On-Line Encyclopedia of Integer Sequences) [4], by A000290.

As far as formula (8) is concerned, it represents, for any k, the sequence composed by the natural numbers or *counting numbers* (starting from k) that is classified in OEIS by A000027 and finally formula (9) is composed by the even numbers (starting from $2k$) and is classified in OEIS by A005843. If, instead of the right chain, we consider the left chain, the same Theorem 2.1 still holds provided that we define $\lambda = \frac{a}{b}$ instead of $\lambda = \frac{b}{a}$.

2.3. **Relation with harmonic numbers.**

By looking at formula (2), one can notice that the ordinates of the circles center are in harmonic progression being their reciprocals in arithmetic progression; that suggests to see if there may be a connection between them and the harmonic numbers.
For convenience, we remind that the harmonic numbers H_n are defined by means of the following sums:

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \qquad n \geq 1$$

We demonstrate now the following theorem:

Theorem 2.2. *If $\lambda = k^2 - 1$ where k is an integer and $k \geq 2$, then the following equality holds:*

$$\sum_{i=1}^{n} \frac{y'_{ri}}{2b} = H_{n+k-1} - H_{k-1}. \tag{10}$$

Proof. We prove the theorem by induction.
For $n \geq 1$ consider the statement $P(n)$ represented by formula (10).

1. $P(1)$ is true as one can immediately see from formula (2).
2. $P(n) \Rightarrow P(n+1)$ is true because we have:

$$\sum_{i=1}^{n+1} \frac{y'_{ri}}{2b} = \sum_{i=1}^{n} \frac{y'_{ri}}{2b} + \frac{y'_{n+1}}{2b} = H_{n+k-1} - H_{k-1} + \frac{y'_{n+1}}{2b} = H_{n+1+k-1} - H_{k-1}$$

The above chain of equalities proves that $P(n) \Rightarrow P(n+1)$ and, by induction, $P(n)$ is true for every integer $n \geq 1$.

□

3. Second kind of chains

3.1. **Center coordinates and radius.**

The second kind of chains we are going to consider are the ones that originate from the twin circles and that are directed towards point B for the right chain and towards point A for the left chain; in fact, as we can notice from Figure 4 we have two chains: one at the left and one at the right of the segment OC respectively. Now, on the basis of the contents of paper [6], it

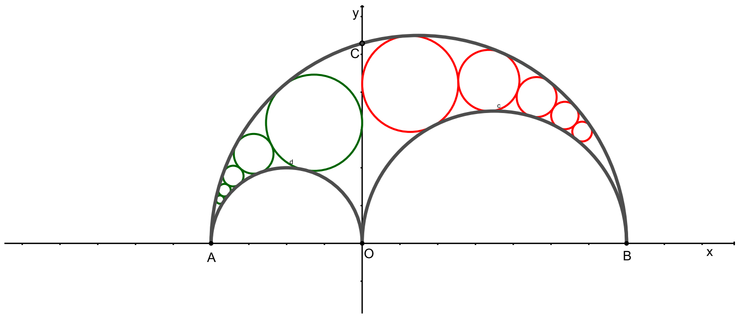

Figure 4. Second kind of chains

is possible to derive the formulas expressing the center coordinates $\left(x_n'', y_n''\right)$ and radius R_n'' of the generic nth circle of the chain.
We have the following expressions for the right chain:

$$x_{rn}'' = 2b - \frac{(a+b)\,b\,(2b+a)}{\left(\sqrt{\frac{a+b}{a}} + n - 1\right)^2 a^2 + (a+b)\,b} \qquad n \geq 1 \tag{11}$$

$$y_{rn}'' = \frac{2\,(a+b)\,ab\left(\sqrt{\frac{a+b}{a}} + n - 1\right)}{\left(\sqrt{\frac{a+b}{a}} + n - 1\right)^2 a^2 + (a+b)\,b} \qquad n \geq 1 \tag{12}$$

$$R_{rn}'' = \frac{(a+b)\,ab}{\left(\sqrt{\frac{a+b}{a}} + n - 1\right)^2 a^2 + (a+b)\,b} \qquad n \geq 1 \tag{13}$$

while for the left chain we have:

$$x_{ln}'' = -2a + \frac{(a+b)\,a\,(2a+b)}{\left(\sqrt{\frac{a+b}{b}} + n - 1\right)^2 b^2 + (a+b)\,a} \qquad n \geq 1 \tag{14}$$

$$(15) \qquad y''_{ln} = \frac{2(a+b)\,ab\left(\sqrt{\frac{a+b}{b}}+n-1\right)}{\left(\sqrt{\frac{a+b}{b}}+n-1\right)^2 b^2+(a+b)\,a} \qquad n \geq 1$$

$$(16) \qquad R''_{ln} = \frac{(a+b)\,ab}{\left(\sqrt{\frac{a+b}{b}}+n-1\right)^2 b^2+(a+b)\,a} \qquad n \geq 1$$

As put into evidence in [6], it is interesting to notice that, by considering a single chain, the locus of the circle centers is represented by an ellipse.
As far as the right chain is concerned, the ellipse has focuses coincident with the center of the outer circle at $(b-a,0)$ and with the center of the right inner circle at $(b,0)$; its equation is given by:

$$\sqrt{(x-b+a)^2+y^2}+\sqrt{(x-b)^2+y^2}=2b+a$$

As far as the left chain is concerned, the ellipse has focuses coincident with the center of the outer circle at $(b-a,0)$ and with the center of the left inner circle at $(-a,0)$; its equation is given by:

$$\sqrt{(x-b+a)^2+y^2}+\sqrt{(x+a)^2+y^2}=2a+b$$

3.2. **Integer sequences associated with the chain.**
Let us focus on the right chain by considering the ratio $\frac{y''_{rn}}{R''_{rn}}$ that generate the sequence $\left\{\frac{y''_{rn}}{R''_{rn}}\right\}$. By considering, as in the previous section, the parameter $\lambda = \frac{b}{a}$, with $\lambda \neq 0$, we pose the following question: do exist values of λ so that the sequence $\left\{\frac{y''_{rn}}{R''_{rn}}\right\}$ is integer? The answer is given by the following theorem.

Theorem 3.1. *If $\lambda = k^2-1$ where k is an integer and $k \geq 2$, then the sequence $\left\{\frac{y''_{rn}}{R''_{rn}}\right\}$ is integer.*

Proof. *From formulas From formulas (12) and (13), we have:*

$$\frac{y''_{rn}}{R''_{rn}} = 2\left(\sqrt{1+\lambda}+n-1\right) \qquad n \geq 1$$

From the above equation one has that the ratio $\frac{y''_{rn}}{R''_{rn}}$ is integer for each $n \geq 1$ only if $\lambda = k^2-1$ for any integer $k \geq 2$.

□

Notice that both the ratios $\frac{y'_{rn}}{R'_{rn}}$ and $\frac{y''_{rn}}{R''_{rn}}$ generate the same sequence composed by the even numbers (starting from $2k$) which is classified in OEIS by A005843.
We just mention (without proofing it for reasons of brevity) that, in this case, the ratios $\frac{b}{R''_{rn}}$ and $\frac{2b}{y''_{rn}}$ do not generate integer sequences but rational sequences.
If, instead of the right chain, we consider the left chain, the same Theorem 3.1 still holds provided that we define $\lambda = \frac{a}{b}$ instead of $\lambda = \frac{b}{a}$.

4. Third kind of chains

4.1. **Center coordinates and radius.**

The third kind of chains we are going to consider are the ones that originate from the twin circles and that are directed towards point C for for both the chains; as we can notice from Figure 5 we have two chains: one at the left and one at the right of the segment OC respectively. We remark that, in practice, this problem represents the study of a chain of mutually tangent circles inscribed inside a circular segment that was the object of two our previous works [2], [3] even if we followed a different approach with respect to the one here presented. Now, on the basis of the contents of paper [1],

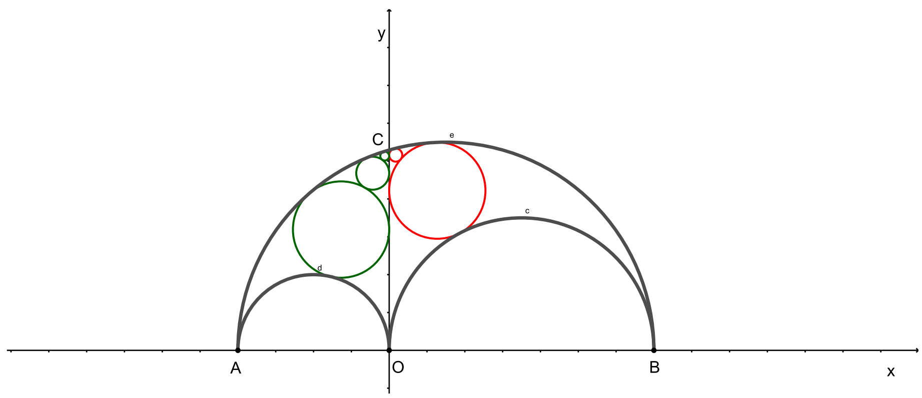

Figure 5. Third kind of chains

[2], [3] it is possible to derive the formulas expressing the center coordinates $\left(x'''_n, y'''_n\right)$ and radius R'''_n of the generic nth circle of the chain.
As far as the right chain is concerned, we have:

$$x'''_{rn} = \frac{4a^n b}{\left[\left(\sqrt{a+b}+\sqrt{b}\right)^n + \left(\sqrt{a+b}-\sqrt{b}\right)^n\right]^2} \qquad n \geq 1 \tag{17}$$

$$y'''_{rn} = \sqrt{4ab - 4a\frac{4a^n b}{\left[\left(\sqrt{a+b}+\sqrt{b}\right)^n + \left(\sqrt{a+b}-\sqrt{b}\right)^n\right]^2}} \qquad n \geq 1 \tag{18}$$

$$(19) \qquad R'''_{rn} = \frac{4a^n b}{\left[\left(\sqrt{a+b}+\sqrt{b}\right)^n + \left(\sqrt{a+b}-\sqrt{b}\right)^n\right]^2} \qquad n \geq 1$$

As far as the left chain is concerned, we have:

$$(20) \qquad x'''_{ln} = -\frac{4ab^n}{\left[\left(\sqrt{a+b}+\sqrt{a}\right)^n + \left(\sqrt{a+b}-\sqrt{a}\right)^n\right]^2} \qquad n \geq 1$$

$$(21) \qquad y'''_{ln} = \sqrt{4ab - 4b\frac{4ab^n}{\left[\left(\sqrt{a+b}+\sqrt{a}\right)^n + \left(\sqrt{a+b}-\sqrt{a}\right)^n\right]^2}} \qquad n \geq 1$$

$$(22) \qquad R'''_{ln} = \frac{4ab^n}{\left[\left(\sqrt{a+b}+\sqrt{a}\right)^n + \left(\sqrt{a+b}-\sqrt{a}\right)^n\right]^2} \qquad n \geq 1$$

As put into evidence in [2], it is interesting to notice that, by considering a single chain, the locus of the circle centers is represented by a parabola; therefore, for the right and left chains we respectively have:

$$x = -\frac{y^2}{4a} + b \qquad\qquad x = \frac{y^2}{4b} - a$$

Notice also, as one can immediately verify, that the focus of both the parabolas is coincident with the center of the outer semicircle forming the arbelos i.e.: $(b-a, 0)$.

4.2. **Integer sequences associated with the chain.**

Let us focus on the right chain by considering the ratios $\frac{b}{R'''_{rn}}$ and $\frac{y'''_{rn}}{R'''_{rn}}$ that generate the sequences $\left\{\frac{b}{R'''_{rn}}\right\}$ and $\left\{\frac{y'''_{rn}}{R'''_{rn}}\right\}$ respectively. By considering, as in the previous sections, the parameter $\lambda = \frac{b}{a}$, with $\lambda \neq 0$, we pose the following question: do exist values of λ so that the sequences $\left\{\frac{b}{R'''_{rn}}\right\}$ and $\left\{\frac{y'''_{rn}}{R'''_{rn}}\right\}$ are integer? The answers is given by the following two theorems.

Theorem 4.1. *If $\lambda = k$ where k is an integer and $k \geq 1$, then the sequence $\left\{\frac{b}{R'''_{rn}}\right\}$ is integer.*

Proof. *First of all it is necessary to remark that the ratio $\frac{b}{R'''_{rn}}$ is related with the Chebyshev polynomial of first kind and order n, indicated by $T(n,x)$, by means of the relation:*

$$(23) \qquad \frac{b}{R'''_{rn}} = T^2\left(n, \sqrt{1+\lambda}\right)$$

Infact, according to [5], the algebraic definition of the Chebyshev polynomial $T(n,x)$ is:

$$T(n,x) = \frac{\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n}{2}$$

By using it, for $x=\sqrt{1+\lambda}$, and by remembering formula (19), one gets formula (23).
For the following, one also need a further representation of $T(n,x)$ as a sum; i.e.:

$$T\left(n,\sqrt{1+\lambda}\right) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \lambda^j \left(\sqrt{1+\lambda}\right)^{n-2j} \tag{24}$$

where $\lfloor \frac{n}{2} \rfloor$ means the nearest integer less than or equal to $\frac{n}{2}$.
Now, we may distinguish two cases:

- *n is an even integer i.e.: $n = 2m$*
  In this case we have:

  $$T\left(n,\sqrt{1+\lambda}\right) = \sum_{j=0}^{m} \binom{2m}{2j} \lambda^j (1+\lambda)^{m-j}$$

  that represents an integer provided that λ is an integer i.e.: $\lambda = k$ and so $T^2\left(n,\sqrt{1+\lambda}\right)$ is an integer too.

- *n is an odd integer i.e.: $n = 2m+1$*
  In this case we have:

  $$T\left(n,\sqrt{1+\lambda}\right) = \sqrt{1+\lambda}\sum_{j=0}^{m} \binom{2m+1}{2j} \lambda^j (1+\lambda)^{m-j}$$

  If we consider the square of the above expression, we have:

  $$T^2\left(n,\sqrt{1+\lambda}\right) = (1+\lambda)\left(\sum_{j=0}^{m} \binom{2m+1}{2j} \lambda^j (1+\lambda)^{m-j}\right)^2$$

  The above expression is always an integer provided that λ is an integer i.e.: $\lambda = k$.

Thus we can conclude that the sequence $\left\{\frac{b}{R'''_{rn}}\right\}$ is integer.

□

Theorem 4.2. *If $\lambda = k^2 - 1$ where k is an integer and $k \geq 2$, then the sequence $\left\{\frac{y'''_{rn}}{R'''_{rn}}\right\}$ is integer.*

Proof. *From formulas appearing in subsection 4.1, one can deduce:*

$$\left(\frac{y'''_{rn}}{R'''_{rn}}\right)^2 = \frac{4a}{R'''_{rn}}\left(\frac{b}{R'''_{rn}} - 1\right) \tag{25}$$

From formula (23) and from the definition of λ, we have:

$$\frac{4a}{R'''_{rn}} = \frac{4T^2\left(n, \sqrt{1+\lambda}\right)}{\lambda} \tag{26}$$

and:

$$\frac{b}{R'''_{rn}} - 1 = T^2\left(n, \sqrt{1+\lambda}\right) - 1 \tag{27}$$

By remembering that the following identity, relating Chebyshev polynomials of first kind $T(n,x)$ and second kind $U(n,x)$, holds:

$$T^2(n,x) - \left(x^2 - 1\right) U^2(n-1,x) = 1$$

one can write, for $x = \sqrt{1+\lambda}$:

$$T^2\left(n, \sqrt{1+\lambda}\right) - 1 = \lambda\, U^2\left(n-1, \sqrt{1+\lambda}\right) \tag{28}$$

Therefore, by means of formulas from (25) to (28), one gets:

$$\frac{y'''_{rn}}{R'''_{rn}} = 2\,T\left(n, \sqrt{1+\lambda}\right) U\left(n-1, \sqrt{1+\lambda}\right) \tag{29}$$

It is now needed the sum representation of $U\left(n, \sqrt{1+\lambda}\right)$ i.e.:

$$U\left(n, \sqrt{1+\lambda}\right) = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2h+1} \lambda^h \left(\sqrt{1+\lambda}\right)^{n-2h}$$

Therefore, by means of it and of the analogous representation for $T\left(n, \sqrt{1+\lambda}\right)$ previously given, one has:

(30)

$$\frac{y'''_{rn}}{R'''_{rn}} = 2 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \lambda^j \left(\sqrt{1+\lambda}\right)^{n-2j} \sum_{h=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2h+1} \lambda^h \left(\sqrt{1+\lambda}\right)^{n-1-2h}$$

By looking at formula (30), we see that it represents the product of two sum of polynomials defined in terms of integer powers of λ and of $\sqrt{1+\lambda}$; in other words, the ratio $\frac{y'''_{rn}}{R'''_{rn}}$ is expressed as a sum of addends of the type:

$$\binom{n}{2j}\binom{n}{2h+1} \lambda^{j+h} \left(\sqrt{1+\lambda}\right)^{2n-1-2j-2h} \tag{31}$$

By remembering that the binomial coefficients are integers and by observing that the exponent of the factor $\sqrt{1+\lambda}$ is always an odd integer number, we have that if $\lambda = k^2 - 1$ with $k \geq 2$ the expression given by (31) is an integer *number and thus also formula given by (30) is integer.*

□

By considering some integer values for k, we obtained some integer sequences that are classified in OEIS:

- ◇ as far as the sequence $\left\{\frac{b}{R'''_{rn}}\right\}$ is concerned, we have found that to $k = 1, 2, 3, 4, 8, 9$ the following sequences correspond respectively: A055997, A171640, A055793, A115032, A055792, A247335. These results are consistent with those that have been published in [3] by following a different approach with respect to the one here presented which is based on the Chebyshev polynomials.
- ◇ as far as the sequence $\left\{\frac{y'''_{rn}}{R'''_{rn}}\right\}$ is concerned to $k = 8$ the sequence A082405 corresponds.

If, instead of the right chain, we consider the left chain, the same Theorems 4.1 and 4.2 still hold provided that we define $\lambda = \frac{a}{b}$ instead of $\lambda = \frac{b}{a}$.

5. Conclusions

We think that this study shows a new nice characteristic of the arbelos that, as far as we know, has never been put into evidence before. In fact, the sequences that we have investigated in the previous sections has a common characteristic: all of them are integer if the geometric parameter that characterizes the arbelos, i.e. the ratio λ between the radii of the two inner circles, assumes a value satisfying the following relation:

$$\lambda = k^2 - 1 \qquad k \in \mathbb{N} \qquad k \geq 2$$

Finally, notice also that the sequence $\{\lambda\}_k$, so generated, is itself classified in OEIS by A005563.

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