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A REFINEMENT OF TÓTH'S INEQUALITY

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Abstract. Let R and r denote the radius of the circumscribed and inscribed circle respectively of a bicentric quadrilateral. In this paper, we establish a refinement of Tóth's inequality which states that $\frac{R}{r} \geq \sqrt{2}$, with equality if and only if the quadrilateral is a square. We make use of analytic methods to prove our result, which allows one to further sharpen the inequality by reducing the constant multiplying the $\frac{R}{r}$ term.

1. Introduction

Let $q := \frac{R}{r}$ denote the ratio of the circumradius to the inradius in either a triangle or bicentric quadrilateral. As a corollary of the classical formula $OI^2 = R^2 - 2rR$ due to Euler (see, e.g., [4, p. 29]) for the distance between the circumcenter and the incenter, one obtains $q \geq 2$ in the case of a triangle, with equality if and only if the triangle is equilateral. One refinement of Euler's inequality, which is due to Jiegen (see [8, p. 63]), states

$$3\sqrt{3}(2-\sqrt{3})r \leq r_1 + r_2 + r_3 \leq \frac{3\sqrt{3}(2-\sqrt{3})}{2}R, \tag{1}$$

where r_1, r_2 and r_3 are the respective radii of the inscribed circles of ΔIAB, ΔIBC and ΔICA and I is the incenter of ΔABC.

Frequently extending comparable results for triangles, a variety of inequalities have been shown for convex quadrilaterals [7], with several of these focusing on the bicentric case (see, e.g., [6, 10, 11, 14]). Perhaps the most well-known of these and simplest states that $q \geq \sqrt{2}$ for a bicentric quadrilateral (see [2, p. 132]), with equality if and only if $ABCD$ is a square. This result is often credited to Tóth from 1948 (see [13]) and it follows as a special case of a more general result later shown by Carlitz [3] concerning the bicentric quadrilateral analogue of Euler's formula for OI.

Tóth's inequality also follows from the inequality

$$4r^2 \leq \Delta \leq 2R^2, \tag{2}$$

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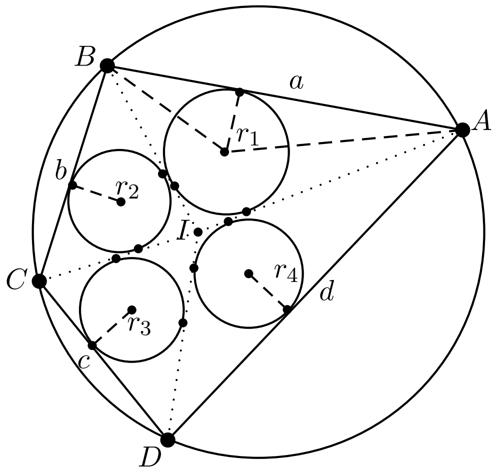

FIGURE 1. Bicentric quadrilateral $ABCD$ with incenter I.

which was shown in [1, p. 64] where Δ denotes the area, and from the equality (9) below as well since the right side of (9) is clearly at least 2. A further inequality interpolating Tóth due to Yun [14] is given by

(3)
$$\frac{r\sqrt{2}}{R} \le \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \le 1,$$

which was subsequently proven in different ways by Josefsson [6] and later Hess [5]. See also the related inequality found in [12] for cyclic quadrilaterals.

Here, we consider a different refinement of Tóth's inequality in analogy to (1) above. Let r_1, r_2, r_3, r_4 be the radii of the inscribed circles in triangles IAB, IBC, ICD, IDA, respectively, where I denotes the incenter of a bicentric quadrilateral $ABCD$ as illustrated in Figure 1. We wish to find an inequality in analogy to (1) which refines Tóth. First consider an inequality of the form

$$a\sqrt{2} \le \frac{r_1 + r_2 + r_3 + r_4}{r} \le a\frac{R}{r}, \tag{4}$$

for a fixed $a > 0$. By Lemma 2.1 below, it is seen that the quantity $\frac{r_1+r_2+r_3+r_4}{r}$ achieves its maximum value when $A = B = \frac{\pi}{2}$ and is minimized as A and B both approach zero. Thus, upon taking $A = B = \frac{\pi}{2}$ in (4) $\big($and hence $\frac{R}{r} = \sqrt{2}\big)$, one must require $a \ge 4 - 2\sqrt{2}$. But then $a\sqrt{2} \ge 4\sqrt{2} - 4$, which is strictly greater than $\frac{3}{2}$, the limiting value of $\frac{r_1+r_2+r_3+r_4}{r}$ as A and B both approach zero, whence the left inequality in (4) cannot hold.

Thus, we seek an inequality of the form

$$a\sqrt{2} \le b - \frac{r_1 + r_2 + r_3 + r_4}{r} \le a\frac{R}{r}, \tag{5}$$

for constants a and b. Note that $b = (a + 4)\sqrt{2} - 4$, which follows from taking $A = B = \frac{\pi}{2}$, with the middle quantity in (5) minimized in this case, as is $\frac{R}{r}$. In the next section, we prove the $a = 1$ case of (5), which we state as the following result.

Theorem 1.1. *We have*

$$\sqrt{2} \leq 5\sqrt{2} - 4 - \frac{r_1 + r_2 + r_3 + r_4}{r} \leq \frac{R}{r}, \tag{6}$$

with equality if and only if $ABCD$ is a square.

We employ analytic methods, including use of trigonometric formulas and certain functional inequalities, to establish Theorem 1.1. By tightening our inequalities, a more technical argument as described in the third section shows that the constant a in (5) may be reduced to approximately $\frac{1}{10}$. Some concluding remarks are made in the fourth section.

2. Proof of the main result

We first establish the following trigonometric formula for $\frac{r_1+r_2+r_3+r_4}{r}$.

Lemma 2.1. *For a bicentric quadrilateral $ABCD$, we have*

$$\begin{aligned} \frac{r_1 + r_2 + r_3 + r_4}{r} = 2 - \frac{1}{2} &\left(\tan \frac{A}{4} + \tan \left(\frac{\pi}{4} - \frac{A}{4} \right) \right) \\ &\times \left(\tan \frac{B}{4} + \tan \left(\frac{\pi}{4} - \frac{B}{4} \right) \right). \end{aligned} \tag{7}$$

Proof. We first show $\frac{r_1}{r} = \frac{1}{2}\left(1 - \tan \frac{A}{4} \tan \frac{B}{4}\right)$. Computing the side length a in ΔIAB in two different ways, we have

$$a = r_1 \left(\cot \frac{A}{4} + \cot \frac{B}{4} \right) = r \left(\cot \frac{A}{2} + \cot \frac{B}{2} \right),$$

and hence

$$\frac{r_1}{r} = \frac{\cot \frac{A}{2} + \cot \frac{B}{2}}{\cot \frac{A}{4} + \cot \frac{B}{4}}.$$

By the identity

$$\cot 2x + \cot 2y = \frac{1}{2}(1 - \tan x \tan y)(\cot x + \cot y),$$

which may be verified using the facts $\cot z = \frac{1}{\tan z}$ and $\tan 2z = \frac{2 \tan z}{1 - \tan^2 z}$, one obtains the formula stated above for $\frac{r_1}{r}$.

By the analogous formulas for r_2, r_3 and r_4, we get

$$\begin{aligned} &\frac{r_1 + r_2 + r_3 + r_4}{r} \\ &= 2 - \frac{1}{2} \left(\tan \frac{A}{4} \tan \frac{B}{4} + \tan \frac{B}{4} \tan \frac{C}{4} + \tan \frac{C}{4} \tan \frac{D}{4} + \tan \frac{D}{4} \tan \frac{A}{4} \right) \\ &= 2 - \frac{1}{2} \Bigg(\tan \frac{A}{4} \tan \frac{B}{4} + \tan \left(\frac{\pi}{4} - \frac{A}{4} \right) \tan \frac{B}{4} \\ &\quad + \tan \left(\frac{\pi}{4} - \frac{A}{4} \right) \tan \left(\frac{\pi}{4} - \frac{B}{4} \right) + \tan \frac{A}{4} \tan \left(\frac{\pi}{4} - \frac{B}{4} \right) \Bigg) \\ &= 2 - \frac{1}{2} \left(\tan \frac{A}{4} + \tan \left(\frac{\pi}{4} - \frac{A}{4} \right) \right) \left(\tan \frac{B}{4} + \tan \left(\frac{\pi}{4} - \frac{B}{4} \right) \right), \end{aligned}$$

as desired. □

Thus, by (7), we have the following equivalent trigonometric version of Theorem 1.1 which we will show below.

Theorem 2.1. *For a bicentric quadrilateral $ABCD$ with circumradius R and inradius r, we have*

$$\sqrt{2} \le 5\sqrt{2}-6+\frac{1}{2}\left(\tan\frac{A}{4}+\tan\left(\frac{\pi}{4}-\frac{A}{4}\right)\right)\left(\tan\frac{B}{4}+\tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\right) \le \frac{R}{r}. \tag{8}$$

Note that the left inequality in (8) follows from the fact that the function $h(x) = \tan(x) + \tan\left(\frac{\pi}{4} - x\right)$ achieves its minimum value of $2\left(\sqrt{2}-1\right)$ on the interval $[0, \frac{\pi}{4}]$ at $x = \frac{\pi}{8}$. The remainder of the section will be devoted to establishing the right inequality in (8).

We will need the following formula for $\frac{R}{r}$.

Lemma 2.2. *Let R and r denote the circumradius and inradius of a bicentric quadrilateral $ABCD$. Then we have*

$$\frac{R^2}{r^2} = \frac{1}{\sin A \sin B} + \frac{1}{\sin^2 A \sin^2 B}. \tag{9}$$

Proof. Let Δ denote the area of quadrilateral $ABCD$. By the well-known formula (see, e.g., [4, p. 60]),

$$4\Delta R = \sqrt{(ab+cd)(ac+bd)(ad+bc)},$$

we have

$$\frac{R^2}{r^2} = \frac{(ab+cd)(ac+bd)(ad+bc)}{16\Delta^2 r^2} = \frac{1}{4\sin A \sin B}\cdot\frac{ac+bd}{r^2}, \tag{10}$$

where the second equality follows from observing

$$\begin{aligned}\frac{1}{4}\sin A \sin B(ab+cd)(ad+bc) &= \frac{1}{2}(ab\sin B + cd\sin D)\cdot\frac{1}{2}(ad\sin A + bc\sin C)\\ &= \Delta\cdot\Delta = \Delta^2.\end{aligned}$$

Upon writing $\frac{a}{r} = \cot\frac{A}{2} + \cot\frac{B}{2}$, $\frac{c}{r} = \cot\frac{C}{2} + \cot\frac{D}{2}$, and so on, we have

$$\begin{aligned}\frac{ac+bd}{r^2} &= \left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(\cot\frac{C}{2}+\cot\frac{D}{2}\right)\\ &\quad+\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\left(\cot\frac{D}{2}+\cot\frac{A}{2}\right)\\ &= \left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\\ &\quad+\left(\tan\frac{A}{2}+\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}+\tan\frac{B}{2}\right)\\ &= \left(2+\cot\frac{A}{2}\tan\frac{B}{2}+\tan\frac{A}{2}\cot\frac{B}{2}\right)\\ &\quad+\left(2+\tan\frac{A}{2}\tan\frac{B}{2}+\cot\frac{A}{2}\cot\frac{B}{2}\right)\\ &= 4+\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)\left(\tan\frac{B}{2}+\cot\frac{B}{2}\right)\\ &= 4+\frac{1}{\left(\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(\sin\frac{B}{2}\cos\frac{B}{2}\right)} = 4+\frac{4}{\sin A\sin B}.\end{aligned}$$

Combining the last equality with (10) yields (9). □

Upon squaring, we seek to show

$$4\left(5\sqrt{2}-6\right)^2 \le 4\frac{R^2}{r^2} - 4\left(5\sqrt{2}-6\right)g(A,B) - g^2(A,B), \tag{11}$$

where

$$g(A,B) := \left(\tan\frac{A}{4} + \tan\left(\frac{\pi}{4}-\frac{A}{4}\right)\right)\left(\tan\frac{B}{4} + \tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\right).$$

Substituting the formula for $\frac{R}{r}$ from Lemma 2.2, we have that the right-hand side of (11) as a function of A and B is given by

$$h(A,B) := 4\csc A\csc B(1+\csc A\csc B) - 4\left(5\sqrt{2}-6\right)g(A,B) - g^2(A,B).$$

Then we seek to minimize $h(A,B)$ subject to $A+B=\ell$, where $0<\ell<2\pi$ is fixed and $0<A,B<\pi$, using the method of Lagrange. Note that upon replacing, if necessary, A by $\pi - A$ or B by $\pi - B$ in h, we may assume $0 < A, B \le \frac{\pi}{2}$ and thus $0 < \ell \le \pi$.

To determine potential local extrema of h, we first seek A and B such that $\frac{\partial h(A,B)}{\partial A} = \frac{\partial h(A,B)}{\partial B}$. This leads to the equality

$$\begin{aligned}
&4\csc A\csc B(1+2\csc A\csc B)(\cot B - \cot A)\\
&= (5\sqrt{2}-6)\left(\sec^2\frac{A}{4} - \sec^2\left(\frac{\pi}{4}-\frac{A}{4}\right)\right)\left(\tan\frac{B}{4}+\tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\right)\\
&\quad - (5\sqrt{2}-6)\left(\tan\frac{A}{4}+\tan\left(\frac{\pi}{4}-\frac{A}{4}\right)\right)\left(\sec^2\frac{B}{4} - \sec^2\left(\frac{\pi}{4}-\frac{B}{4}\right)\right)\\
&\quad + \frac{1}{2}\left(\tan\frac{A}{4}+\tan\left(\frac{\pi}{4}-\frac{A}{4}\right)\right)\left(\sec^2\frac{A}{4} - \sec^2\left(\frac{\pi}{4}-\frac{A}{4}\right)\right)\\
&\qquad \times\left(\tan\frac{B}{4}+\tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\right)^2\\
&\quad - \frac{1}{2}\left(\tan\frac{A}{4}+\tan\left(\frac{\pi}{4}-\frac{A}{4}\right)\right)^2\left(\sec^2\frac{B}{4} - \sec^2\left(\frac{\pi}{4}-\frac{B}{4}\right)\right)\\
&\qquad \times\left(\tan\frac{B}{4}+\tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\right).
\end{aligned}\tag{12}$$

Lemma 2.3. *Let $0 < A, B < \frac{\pi}{2}$. Then equality (12) holds if and only if $A = B$.*

Proof. Clearly (12) holds if $A = B$. Suppose, to the contrary, that (12) holds for some $A \ne B$. Without loss of generality, we may assume $A > B$. Using the fact $\sec^2 x - \sec^2 y = \tan^2 x - \tan^2 y = (\tan x - \tan y)(\tan x + \tan y)$ on the right side of (12), and factoring further the resulting expression, implies

$$\begin{aligned}
&4\csc A\csc B(1+2\csc A\csc B)(\cot B - \cot A)\\
&= k(A,B)\left(\tan\frac{A}{4} - \tan\frac{B}{4} - \tan\left(\frac{\pi}{4}-\frac{A}{4}\right) + \tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\right),
\end{aligned}\tag{13}$$

where

$$k(A,B)=\frac{1}{2}g(A,B)\left(10\sqrt{2}-12+g(A,B)\right).$$

Note that

$$\frac{\cot B-\cot A}{A-B}=\csc^2 u$$

and

$$\frac{\tan\frac{A}{4}-\tan\left(\frac{\pi}{4}-\frac{A}{4}\right)-\left(\tan\frac{B}{4}-\tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\right)}{A-B}=\frac{1}{4}\left(\sec^2\frac{v}{4}+\sec^2\left(\frac{\pi}{4}-\frac{v}{4}\right)\right),$$

for some u and v satisfying $B<u,v<A$, by the mean value theorem.

Then dividing both sides of (13) by $A-B$ implies

(14)

$$4\csc A\csc B(1+2\csc A\csc B)\csc^2 u=\frac{1}{4}k(A,B)\left(\sec^2\frac{v}{4}+\sec^2\left(\frac{\pi}{4}-\frac{v}{4}\right)\right).$$

Since $\csc x>1$ for $0<x<\frac{\pi}{2}$, the left side of (14) is bounded below by 12. On the other hand, note that $0<\tan x+\tan\left(\frac{\pi}{4}-x\right)<1$ and $0<\sec^2 x+\sec^2\left(\frac{\pi}{4}-x\right)<3$ for $0<x<\frac{\pi}{8}$, since the two functions at hand can be shown to be decreasing on the interval $(0,\frac{\pi}{8})$, with the upper bound of the inequality achieved at $x=0$ in both instances. Thus, the right side of (14) is bounded above by $\frac{3}{8}\left(10\sqrt{2}-11\right)<12$. This however contradicts the equality (14) and implies the desired result. □

By the prior lemma, in order to minimize $h(A,B)$ over all possible A and B, we need to consider the function $p(x):=h(4x,4x)$.

Lemma 2.4. *We have* $p(x)\geq 4\left(5\sqrt{2}-6\right)^2$ *for* $0<x\leq\frac{\pi}{8}$, *with equality if and only if* $x=\frac{\pi}{8}$.

Proof. A direct calculation gives $p\left(\frac{\pi}{8}\right)=4\left(5\sqrt{2}-6\right)^2$ (which corresponds to the case when $ABCD$ is a square), so in order to establish the result, it suffices to show $p'(x)<0$ for $0<x<\frac{\pi}{8}$. Then

$$\begin{aligned}p(x)=4\left(\csc^2 4x+\csc^4 4x\right)-4\left(5\sqrt{2}-6\right)\left(\tan x+\tan\left(\frac{\pi}{4}-x\right)\right)^2\\-\left(\tan x+\tan\left(\frac{\pi}{4}-x\right)\right)^4\end{aligned}$$

implies $p'(x)<0$ iff

$$\begin{aligned}&8\csc^2 4x\cot 4x\left(1+2\csc^2 4x\right)\\&\quad>\left(\sec^2\left(\frac{\pi}{4}-x\right)-\sec^2 x\right)\left(\tan\left(\frac{\pi}{4}-x\right)+\tan x\right)\\&\quad\quad\times\left(10\sqrt{2}-12+\left(\tan\left(\frac{\pi}{4}-x\right)+\tan x\right)^2\right).\end{aligned}\tag{15}$$

Since $0<\tan\left(\frac{\pi}{4}-x\right)+\tan x<1<\csc^2 4x$, we have $8\csc^2 4x\left(1+2\csc^2 4x\right)>24>10\sqrt{2}-12+\left(\tan\left(\frac{\pi}{4}-x\right)+\tan x\right)^2$. Thus, to demonstrate (15), it is enough to show

$$\cot 4x>\sec^2\left(\frac{\pi}{4}-x\right)-\sec^2 x,\qquad 0<x<\frac{\pi}{8}.\tag{16}$$

Let $q(x) = \cot 4x - \sec^2\left(\frac{\pi}{4} - x\right) + \sec^2 x$ and we wish to prove $q(x) > 0$. Note that $q\left(\frac{\pi}{8}\right) = 0$ and so to complete the proof, one can show $q'(x) < 0$. Now $0 < \sec^2 x,\ \sec^2\left(\frac{\pi}{4} - x\right) < 2$ for $0 < x < \frac{\pi}{8}$, and thus

$$\sec^2\left(\frac{\pi}{4} - x\right)\tan\left(\frac{\pi}{4} - x\right) + \sec^2 x \tan x < 2\left(\tan\left(\frac{\pi}{4} - x\right) + \tan x\right) < 2 < 2\csc^2 4x.$$

This implies $q'(x) < 0$, and hence (16), as desired. □

We also need to consider the range of values for the boundary function $r(x) := h\left(\frac{\pi}{2}, 4x\right)$.

Lemma 2.5. *We have $r(x) \geq 4\big(5\sqrt{2} - 6\big)^2$ for $0 < x \leq \frac{\pi}{8}$, with equality if and only if $x = \frac{\pi}{8}$.*

Proof. Since $r\left(\frac{\pi}{8}\right) = 4\big(5\sqrt{2} - 6\big)^2$, it suffices to show $r'(x) < 0$ for $0 < x < \frac{\pi}{8}$. Note that

$$r(x) = 4\big(\csc 4x + \csc^2 4x\big) - 8\big(16 - 11\sqrt{2}\big)\left(\tan x + \tan\left(\frac{\pi}{4} - x\right)\right) - 4\big(3 - 2\sqrt{2}\big)\left(\tan x + \tan\left(\frac{\pi}{4} - x\right)\right)^2,$$

and hence $r'(x) < 0$ iff

$$2\csc 4x \cot 4x(1 + 2\csc 4x) > \left(\sec^2\left(\frac{\pi}{4} - x\right) - \sec^2 x\right) \times \left(16 - 11\sqrt{2} + \big(3 - 2\sqrt{2}\big)\left(\tan\left(\frac{\pi}{4} - x\right) + \tan x\right)\right). \tag{17}$$

Since $0 < \tan\left(\frac{\pi}{4} - x\right) + \tan x < 1 < \csc^2 4x$ for $0 < x < \frac{\pi}{8}$ and $16 - 11\sqrt{2} + \big(3 - 2\sqrt{2}\big) < 6$, inequality (17) follows from (16), as desired. □

Proof of Theorem 2.1:

We complete the proof of (11), which establishes the right-hand inequality in (8). Given $0 < \ell \leq \pi$, let S_ℓ denote the subset of $\mathbb{R}^2$ comprising those ordered pairs (A, B) such that $A + B = \ell$ with $0 < A, B \leq \frac{\pi}{2}$. We wish to minimize $h(A, B)$ over all points (A, B) in S_ℓ for each fixed ℓ. If $\ell = \pi$, then only $(A, B) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is possible, which corresponds to $ABCD$ being a square and there is equality in (11). So assume $0 < \ell < \pi$, and we show that (11) holds strictly in this case. We consider S_ℓ as a subset of all points in $\mathbb{R}^2$ such that $A + B = \ell$ when discussing its boundary and interior points. For each ℓ, the point $(A, B) = \left(\frac{\ell}{2}, \frac{\ell}{2}\right)$ corresponds to an interior critical point of h on the set S_ℓ, by (12), and is the only such point, by Lemma 2.3. Furthermore, by Lemma 2.4, we have $h\left(\frac{\ell}{2}, \frac{\ell}{2}\right) > 4\big(5\sqrt{2} - 6\big)^2$ for $0 < \ell < \pi$.

In order to minimize h on S_ℓ, we must also consider the values of h at possible boundary points of S_ℓ. Suppose now $0 < \ell \leq \frac{\pi}{2}$. Though there are no boundary points of S_ℓ in this case, we still need to consider the behavior of h as A or B approaches zero. By symmetry of A and B, we need only consider the case when A approaches zero. From the formula for h, it is seen that $\lim_{h\to 0^+}(h(A,\ell-A))$ is infinite and hence $h(A,B) > 4\big(5\sqrt{2}-6\big)^2$ for all (A,B) in S_ℓ when $0<\ell\le\frac{\pi}{2}$.

Now assume $\frac{\pi}{2}<\ell<\pi$. Then S_ℓ has boundary points when A or B equals $\frac{\pi}{2}$. By Lemma 2.5, we have $h\left(\frac{\pi}{2},\ell-\frac{\pi}{2}\right) > 4\big(5\sqrt{2}-6\big)^2$ as $\frac{\pi}{2}<\ell<\pi$, which implies inequality (11) holds strictly for all points in S_ℓ in this case as well. Combining the previous cases implies (11) for all possible (A,B), with equality if and only if $A=B=\frac{\pi}{2}$. □

3. A sharpened version of the result

In this section, we consider a sharpened version of the inequality of the form

$$0 \le 4\big(\sqrt{2}-1\big) - \frac{r_1+r_2+r_3+r_4}{r} \le d\left(\frac{R}{r}-\sqrt{2}\right), \qquad d>0. \tag{18}$$

Note that inequality (6) corresponds to the $d=1$ case and we seek to find d as small as possible in (18).

Proceeding as in the proof of (8) above, we attempt to show for $d>0$ fixed and all relevant A and B,

$$h_d(A,B) \ge 4\left((d+4)\sqrt{2}-6\right)^2,$$

where

$$h_d(A,B) = 4d^2\csc A\csc B(1+\csc A\csc B) - 4\left((d+4)\sqrt{2}-6\right)g(A,B) - g^2(A,B)$$

and $g(A,B)$ is as before.

We will make use of the functions

$$\alpha(x) := \frac{\tan\left(\frac{\pi}{4}-x\right)+\tan x}{\csc 4x}, \qquad \beta(x) := \frac{\tan\left(\frac{\pi}{4}-x\right)-\tan x}{\cot 4x}$$

for $0<x<\frac{\pi}{8}$ and

$$\gamma(x,y) := \frac{\sec^2 y+\sec^2\left(\frac{\pi}{4}-y\right)}{1+2\csc 4x\csc 4y}$$

for $0<y<x<\frac{\pi}{8}$. One can show $0<\alpha(x)<2\big(\sqrt{2}-1\big)$, $0<\beta(x)<2-\sqrt{2}$ and $0<\gamma(x,y)<\frac{4}{3}\big(2-\sqrt{2}\big)$ for x and y in the given ranges.

We generalize Lemma 2.3 as follows.

Lemma 3.1. *Let $0<A,B<\frac{\pi}{2}$. Then we have $\frac{\partial h_d(A,B)}{\partial A}=\frac{\partial h_d(A,B)}{\partial B}$ if and only if $A=B$ when $d\ge c$, where*

$$c=\frac{\sqrt{2}+\sqrt{8+9\sqrt{2}}}{3\big(10+7\sqrt{2}\big)}\approx 0.09995.$$

Proof. In analogy to (14) above, the equality $\frac{\partial h_d(A,B)}{\partial A}=\frac{\partial h_d(A,B)}{\partial B}$ when $A>B$ implies

$$\begin{aligned}&4d^2\csc A\csc B(1+2\csc A\csc B)\csc^2 u\\ &=\frac{1}{4}k_d(A,B)\left(\sec^2\frac{v}{4}+\sec^2\left(\frac{\pi}{4}-\frac{v}{4}\right)\right)\end{aligned} \tag{19}$$

for some $A > u, v > B$, where

$$k_d(A,B) = \frac{1}{2}g(A,B)\left(2(d+4)\sqrt{2} - 12 + g(A,B)\right)$$

and $g(A,B)$ is as before. We show that (19) is impossible when $A > B$ for d as given by demonstrating that the left side of (19) is strictly greater than the right. In order to do so, it is enough to show

$$\begin{aligned} &4d^2 \csc A \csc B(1 + 2\csc A \csc B)\csc^2 A \\ &> \frac{1}{4}k_d(A,B)\left(\sec^2 \frac{B}{4} + \sec^2\left(\frac{\pi}{4} - \frac{B}{4}\right)\right), \end{aligned} \tag{20}$$

since the functions $\csc^2 x$ and $\sec^2 x + \sec^2\left(\frac{\pi}{4} - x\right)$ are decreasing for x in the relevant ranges. Upon dividing both sides by $\csc A \csc B(1 + 2\csc A \csc B)$, we have that inequality (20) may be rewritten as

$$4d^2 \csc^2 A > \frac{1}{8}\alpha(A/4)\alpha(B/4)\gamma(A/4, B/4)\left(2(d+4)\sqrt{2} - 12 + g(A,B)\right). \tag{21}$$

Based on the upper bounds for the functions α, γ and g, inequality (21) is ensured of holding if d satisfies

$$32d^2 \geq 4\left(\sqrt{2} - 1\right)^2 \cdot \frac{4}{3}\left(2 - \sqrt{2}\right) \cdot \left(2(d+4)\sqrt{2} - 11\right).$$

The last inequality is equivalent to $d \geq c$ as d is positive, which implies the result. □

Let $p_d(x) := h_d(4x, 4x)$. We extend the proof of Lemma 2.4, which involved minimizing h in the case when $A = B$, to general d. To do so, note that $p_d\left(\frac{\pi}{8}\right) = 4\left((d+4)\sqrt{2} - 6\right)^2$ with $p_d'(x) < 0$ for $0 < x < \frac{\pi}{8}$ iff

$$8d^2\left(1 + 2\csc^2 4x\right) > \alpha^2(x)\beta(x)\left(2(d+4)\sqrt{2} - 12 + g(4x,4x)\right). \tag{22}$$

Then (22) is guaranteed to hold if it is required

$$6d^2 \geq \left(\sqrt{2} - 1\right)^2\left(2 - \sqrt{2}\right)\left(2(d+4)\sqrt{2} - 11\right),$$

which once again implies $d \geq c$, where c is as above.

Let $r_d(x) := h_d\left(\frac{\pi}{2}, 4x\right)$. In analogy to the proof of Lemma 2.5, note that

$$r_d(\pi/8) = 4\left((d+4)\sqrt{2} - 6\right)^2,$$

with $r_d'(x) < 0$ for $0 < x < \frac{\pi}{8}$ iff

$$\begin{aligned} &2d^2(1 + 2\csc 4x) > \alpha(x)\beta(x) \\ &\times\left(\left(\sqrt{2} - 1\right)\left((d+4)\sqrt{2} - 6\right) + \left(3 - 2\sqrt{2}\right)\sqrt{g(4x,4x)}\right). \end{aligned} \tag{23}$$

Inequality (23) is guaranteed to hold if d satisfies

$$3d^2 \geq \left(\sqrt{2} - 1\right)\left(2 - \sqrt{2}\right)\left(\left(\sqrt{2} - 1\right)\left((d+4)\sqrt{2} - 6\right) + 3 - 2\sqrt{2}\right),$$

which implies

$$d \geq \frac{2 - \sqrt{2} + \sqrt{14\sqrt{2} - 18}}{3\left(3\sqrt{2} + 4\right)} \approx 0.07793.$$

Thus, when $d \geq c$, we have that Lemmas 2.3–2.5 from the previous section may be extended to h_d. The rest of the proof of (8) is then seen to apply, upon making the suitable adjustments, which leads to the following result.

Theorem 3.1. *We have*

$$0 \leq 4\big(\sqrt{2}-1\big) - \frac{r_1+r_2+r_3+r_4}{r} \leq c\left(\frac{R}{r}-\sqrt{2}\right), \tag{24}$$

where $c = \frac{\sqrt{2}+\sqrt{8+9\sqrt{2}}}{3\left(10+7\sqrt{2}\right)} \approx 0.09995$.

Upon considering numerical values in the range of the ratio

$$\frac{4\big(\sqrt{2}-1\big) - \frac{r_1+r_2+r_3+r_4}{r}}{\frac{R}{r}-\sqrt{2}}$$

over all possible non-square bicentric quadrilaterals, the smallest value of d that could work in (18) is at least 0.04737 approximately. We leave it as an open question to determine the best possible value of d in (18).

4. Concluding remarks

In this paper, we have obtained a refinement of Tóth's inequality for bicentric quadrilaterals, establishing an equivalent trigonometric version of the result by analytic methods. We then sharpened our result reducing the constant multiplying the $\frac{R}{r}$ term by an order of magnitude.

One might also consider analogues of (1) and (24) for bicentric polygons. Let R_n and r_n denote respectively the circumradius and inradius of a bicentric n-gon. Then we have (see [13, Eqn. (2)])

$$\frac{R_n}{r_n} \geq \sec(\pi/n), \qquad n \geq 3, \tag{25}$$

which reduces to the inequalities of Euler and Tóth when $n = 3$ and $n = 4$, respectively. More recently, a number of inequalities for bicentric n-gons have been given in [11], particularly in the cases when $n = 6$ and $n = 8$; see also [9], where a general inequality is proven for cyclic n-gons.

In light of (1) and (24), one could attempt to prove a refinement of (25) of the form

$$0 \leq \frac{r_1+\cdots+r_n}{r} - \alpha \leq \beta\left(\frac{R_n}{r_n} - \sec(\pi/n)\right) \tag{26}$$

or

$$0 \leq \alpha - \frac{r_1+\cdots+r_n}{r} \leq \beta\left(\frac{R_n}{r_n} - \sec(\pi/n)\right), \tag{27}$$

where α and β are constants that depend upon n. The exact form of the inequality that should be sought appears it might depend upon the parity of n. Note that (1) corresponds to the $n = 3$ case of (26) with $\alpha = 3\sqrt{3}\big(2-\sqrt{3}\big)$ and $\beta = \frac{3\sqrt{3}\left(2-\sqrt{3}\right)}{2}$, whereas (24) would be the $n = 4$ case of (27) with $\alpha = 4\big(\sqrt{2}-1\big)$ and $\beta = c$.

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