INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 12 (2023), No. 1, 35 - 45

TWO NON-CONGRUENT REGULAR POLYGONS HAVING VERTICES AT THE SAME DISTANCES FROM THE POINT

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Abstract. For the given regular plane polygon and an arbitrary point in the plane of the polygon, the distances from the point to the vertices of the polygon are defined. We proved that there is one more non-congruent regular polygon having the vertices at the same distances from the point. The sizes of both regular polygons are uniquely determined by these distances. In general case, geometrical construction of the second regular polygon is given. It is proved that there are two points in the plane, which separately have the same set of the distances to the vertices of two non-congruent regular polygons with a shared vertex.

1. Introduction

The concept of the cyclic averages are introduced in [5], [6]. For a regular polygon with n vertices P_n, there are an $n-1$ number of the cyclic averages:

$$S_n^{(2)}, S_n^{(4)}, \dots, S_n^{(2n-2)}.$$

For an arbitrary point M in the plane of the regular polygon P_n, we use the notation $M(d_1, d_2, \dots, d_n, L)$ where d_i are the distances from this point to the vertices A_i of the regular polygon P_n and L is the distance between the point and the center O of the polygon. If the radius of the circumcircle Ω of the regular polygon P_n is R, we denote such polygon by $P_n(R)$.

Keywords and phrases: Cyclic averages, Regular polygon, Pompeiu theorem, Equilateral triangle.
(2020)Mathematics Subject Classification: 51M15, 51N20, 51N35.
Received: 17.01.2022. In revised form: 12.09.2022. Accepted: 07.08.2022

The cyclic averages are defined as sums of the like even powers of distances d_i to the vertices of $P_n(R)$:

$$S_n^{(2)} = \frac{1}{n}\sum_{i=1}^{n} d_i^2,\ S_n^{(4)} = \frac{1}{n}\sum_{i=1}^{n} d_i^4,\ \dots,\ S_n^{(2m)} = \frac{1}{n}\sum_{i=1}^{n} d_i^{2m},$$
$$\text{where } m = 1, 2, \dots, n-1.$$

The cyclic averages can be expressed only in terms of R and L:

$$S_n^{(2m)} = (R^2+L^2)^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{2k}{k} R^{2k}L^{2k}(R^2+L^2)^{m-2k}. \tag{1.1}$$

If we are given no more than the distances $d_1, d_2, \dots, d_n$ from a point to the vertices of an n-gon, there obviously is an infinity of n-gons determined by the n distances. If, however, the polygon is required to be regular, can the n distances uniquely determine the sizes of the polygon? In the present article we investigate this problem.

2. General case. Existence

Let us fix the n distances

$$d_1, d_2, \dots, d_n$$

and consider the L and R as unknowns.

The number of the cyclic averages is characteristic of the regular polygon but each regular polygon has at least two cyclic averages – $S_n^{(2)}$ and $S_n^{(4)}$. From (1.1) they equal:

$$S_n^{(2)} = R^2 + L^2, \tag{2.1}$$

$$S_n^{(4)} = (R^2+L^2)^2 + 2R^2L^2. \tag{2.2}$$

Substituting (2.1) into (2.2):

$$2R^2L^2 = S_n^{(4)} - (S_n^{(2)})^2. \tag{2.3}$$

The relations (1.1), (2.1) and (2.3) give us the conditions, which must be satisfied by the $d_1, d_2, \dots, d_n$ if they serve as the distances from the point to the vertices of the regular polygon

$$S_n^{(2m)} = (S_n^{(2)})^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2^k}\binom{m}{2k}\binom{2k}{k}\big(S_n^{(4)} - (S_n^{(2)})^2\big)^k (S_n^{(2)})^{m-2k},$$

where $m = 3, \dots, n-1$. But we initially assumed they are such distances, so we consider (2.1) and (2.2) only.

From (2.1) and (2.3) R^2 and L^2 are the solutions of the equation

$$X^2 - S_n^{(2)}X + \frac{1}{2}\big(S_n^{(4)} - (S_n^{(2)})^2\big) = 0,$$

so we get two pairs of the solutions:

$$\text{I.}\qquad R_1^2 = \frac{1}{2}\left(S_n^{(2)} + \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right),$$
$$L_1^2 = \frac{1}{2}\left(S_n^{(2)} - \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right);$$
$$\text{II.}\qquad R_2^2 = \frac{1}{2}\left(S_n^{(2)} - \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right),$$
$$L_2^2 = \frac{1}{2}\left(S_n^{(2)} + \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right).$$

If one of them exists, automatically exists another one. Algebraically, it means the following inequalities must be held:

$$(*)\qquad 3(S_n^{(2)})^2 - 2S_n^{(4)} \geq 0,$$

$$(**)\qquad S_n^{(2)} - \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}} \geq 0.$$

Indeed,

$$3(S_n^{(2)})^2 - 2S_n^{(4)} = 3(R^2+L^2)^2 - 2\big((R^2+L^2)^2 + 2R^2L^2\big) = (R^2 - L^2)^2,$$

and from (2.3) follows

$$S_n^{(4)} \geq (S_n^{(2)})^2,$$

which proves $(**)$.

Denote by Ω_1 and Ω_2 the circumcircles of the regular polygons $P_n(R_1)$ and $P_n(R_2)$, by O_1 and O_2 their centers, respectively. Therefore

$$L_1 = MO_1 \text{ and } L_2 = MO_2.$$

From the solutions I and II follows:

$$R_1 > R_2 \text{ and } L_1 < L_2,$$

so the first solution corresponds to the larger regular polygon, while the second solution corresponds to the smaller one. The distances from the M point to the centers is longer for the smaller polygon.

If

$$3(S_n^{(2)})^2 = 2S_n^{(4)},$$

the point M lies on the circumcircle,

$$R_1 = L_1 = R_2 = L_2.$$

This is degenerate case – both regular polygons are congruent. We obtain:

Theorem 2.1. *If the point of the distances $d_1, d_2, \ldots, d_n$ to the vertices of the regular polygon $P_n(R_1)$, does not lie on the circumcircle Ω_1 of $P_n(R_1)$, there is one more non-congruent regular polygon $P_n(R_2)$ having the vertices at the same $d_1, d_2, \ldots, d_n$ distances from the point. If the point lies on the circumcircle Ω_1 of $P_n(R_1)$ there is no more non-congruent regular polygon having the vertices at the same distances from the point.*

For the first solution (larger polygon) $L_1 < R_1$ i.e. the point M lies inside the circumcircle Ω_1, while for the second solution (smaller polygon) $L_2 > R_2$ i.e point M lies outside the circumcircle Ω_2.

Let us summarize the obtained results.

Theorem 2.2. *If the arbitrary point $M(d_1, d_2, \ldots, d_n, L_1)$ lies inside the circumcircle Ω_1 of the regular polygon $P_n(R_1)$, there is one more regular polygon $P_n(R_2)$ having the vertices at the same distances from the point $M(d_1, d_2, \ldots, d_n, L_2)$ and holds:*

$$\sqrt{\frac{1}{2}\left(S_n^{(2)} + \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right)} = R_1 = L_2$$
$$> R_2 = L_1 = \sqrt{\frac{1}{2}\left(S_n^{(2)} - \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right)},$$

i.e. the point M lies outside the circumcircle Ω_2 of the regular polygon $P_n(R_2)$.

Theorem 2.3. *If the arbitrary point $M(d_1, d_2, \ldots, d_n, L_1)$ lies outside the circumcircle Ω_1 of the regular polygon $P_n(R_1)$, there is one more regular polygon $P_n(R_2)$ having the vertices at the same distances from the point $M(d_1, d_2, \ldots, d_n, L_2)$ and holds:*

$$\sqrt{\frac{1}{2}\left(S_n^{(2)} - \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right)} = R_1 = L_2$$
$$< R_2 = L_1 = \sqrt{\frac{1}{2}\left(S_n^{(2)} + \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right)},$$

i.e. the point M lies inside the circumcircle Ω_2 of the regular polygon $P_n(R_2)$.

3. Equilateral triangle

Algebraic Backround

The well-known the Pompeiu theorem states [8]:

let given an equilateral triangle and any point in its plane. Then the distances from the point to the vertices d_1, d_2, d_3 are lengths of the sides of a triangle.

We call a triangle with sides d_1, d_2, d_3 a Pompeiu triangle [9]. The Pompeiu triangle is degenerate if the point lies on the circumcircle of the equilateral triangle, because by Van Schooten's theorem the largest distance equals to the sum of the others.

According to Theorem 2.1, for the given Pompeiu triangle there are two equilateral triangles – the larger and the smaller. In [1], [3], [4], [9] are investigated case of the larger equilateral triangle i.e. the point lies inside of the circumcircle, both equilateral triangles are considered by H. Eves [7].

For the equilateral triangle $P_3(R)$ and the point $M(d_1, d_2, d_3, L)$ the $S_3^{(2)}$ and $S_3^{(4)}$ cyclic averages equal:

$$S_3^{(2)} = \frac{1}{3}\,(d_1^2 + d_2^2 + d_3^2), \quad S_3^{(4)} = \frac{1}{3}\,(d_1^4 + d_2^4 + d_3^4).$$

The inequality $(*)$ gives

$$3(S_n^{(2)})^2 - 2S_n^{(4)} = \frac{1}{3}\left((d_1^2 + d_2^2 + d_3^2)^2 - 2(d_1^4 + d_2^4 + d_3^4)\right) = \frac{16}{3}\,\Delta^2(d_1, d_2, d_3),$$

the symbol – $\Delta_{(d_1,d_2,d_3)}$ denotes the area of the triangle whose sides have lengths d_1, d_2, d_3. Then the inequality $(**)$ turns into will-known Weitzenböck's inequality [2]

$$d_1^2 + d_2^2 + d_3^2 \geq 4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}.$$

Two equilateral triangles are:

I. The larger triangle, M lies inside Ω_1;

$$R_1^2 = \frac{1}{6}\left(d_1^2 + d_2^2 + d_3^2 + 4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}\right),$$
$$L_1^2 = \frac{1}{6}\left(d_1^2 + d_2^2 + d_3^2 - 4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}\right).$$

II. The smaller triangle, M lies outside Ω_2;

$$R_2^2 = \frac{1}{6}\left(d_1^2 + d_2^2 + d_3^2 - 4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}\right),$$
$$L_2^2 = \frac{1}{6}\left(d_1^2 + d_2^2 + d_3^2 + 4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}\right).$$

Geometrical Construction

We perform the constructions in two ways:

A. Given the Pompeiu triangle and construct both equilateral triangles.

B. Given one of the equilateral triangle and the point and construct the second one.

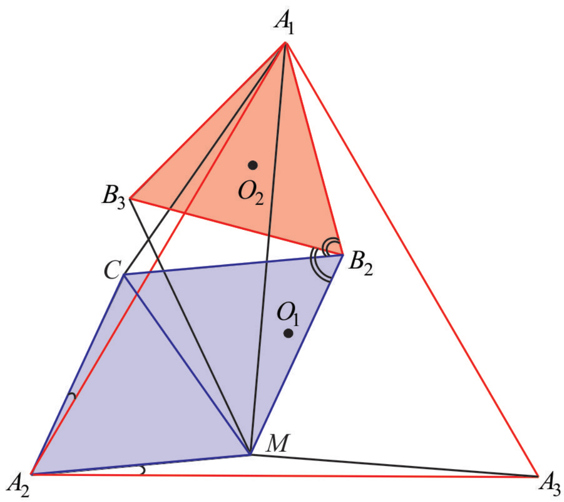

FIGURE 1

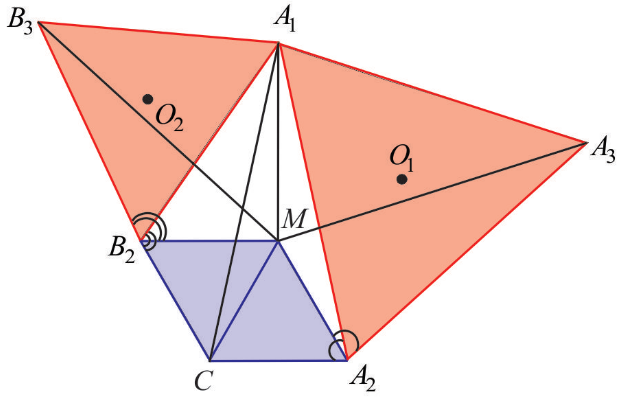

FIGURE 2

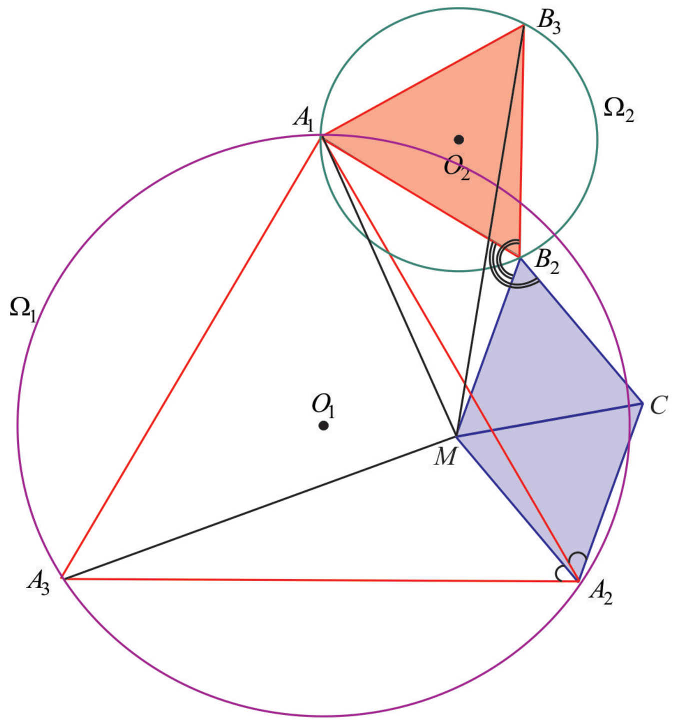

FIGURE 3

A. If given the Pompeiu triangle we take one vertex as the point M, see Fig. 1 and Fig. 2. If MCA_1 is the Pompeiu triangle we construct around one side, for example MC, two auxiliary equilateral triangles MCA_2 and MCB_2. Obtained the auxiliary points A_2 and B_2 connect to the third vertex A_1. Two line segments A_2A_1 and B_2A_1 serve as the sides of the desired two equilateral triangles – $A_1A_2A_3$ and $A_1B_2B_3$. Indeed,

$$MA_3 = CA_1 = MB_3;$$

because of the congruency of the triangles:

$$MA_2A_3 = CA_2A_1 \text{ and } CB_2A_1 = MB_2B_3.$$

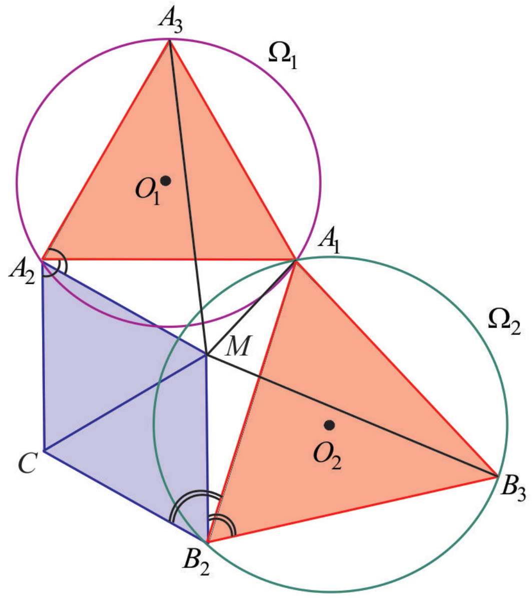

FIGURE 4

B. If the point M inside of the circumcircle Ω_1, it means the triangle $A_1A_2A_3$ is the larger, see Fig. 3. We construct around one distance, for example MA_2, one auxiliary equilateral triangle MA_2C obtained the auxiliary point C. Again around line segment MC construct the second auxiliary equilateral triangle – MCB_2. Connect the obtained point B_2 to the vertex A_1. The line segment B_2A_1 serve as the side of the desired smaller equilateral triangle $A_1B_2B_3$.

If the point M outside of the circumcircle Ω_1, it means the triangle $A_1A_2A_3$ is the smaller, see Fig. 4. Repeat above-mentioned steps, we obtain the larger equilateral triangle – $A_1B_2B_3$.

4. General case. Construction

For the regular polygon P_n, when $n > 3$ constructions by using the auxiliary triangles are impossible. In general case Theorem 2.2 and Theorem 2.3 give us the method of construction for the second regular polygon from the given one and the point. From these theorems:

$$R_2 = L_1 \text{ and } L_2 = R_1. \tag{$***$}$$

Conditions $(***)$ are necessary but not sufficient for the construction.

Let us consider the distances from the given point M to the vertices of the second regular polygon as unknowns:

$$x_1, x_2, \ldots, x_n.$$

From (1.1) and $(***)$ they satisfy

$$\sum_{i=1}^{n} x_i^{2m} = \sum_{i=1}^{n} d_i^{2m}, \quad \text{where} \quad m = 1, 2, \ldots, n-1. \tag{4.1}$$

We have the $n-1$ equations, so to determine the unknowns uniquely, let us consider one unknown as one of the distances – d_i. Without loss of generality take

$$x_1 = d_1. \tag{4.2}$$

Then, from the elementary properties of the symmetric functions, from (4.1) it follows that

$$d_2^2, \ldots, d_n^2 \;\text{ and }\; x_2^2, \ldots, x_n^2$$

are roots of the same equation of degree $n-1$. Consequently $x_2, \ldots, x_n$ are a permutation of the $d_2, \ldots, d_n$; therefore both of them are the same set of the distances:

$$\{x_2, \ldots, x_n\} = \{d_2, \ldots, d_n\}.$$

So, the conditions $(***)$ and (4.2) are the sufficient conditions to identify the second regular polygon having the vertices at the same distances from the point.

Constructions for a square and a regular pentagon are given in Fig. 5 and Fig. 6, respectively. We describe the method of construction, which is true for any regular polygon.

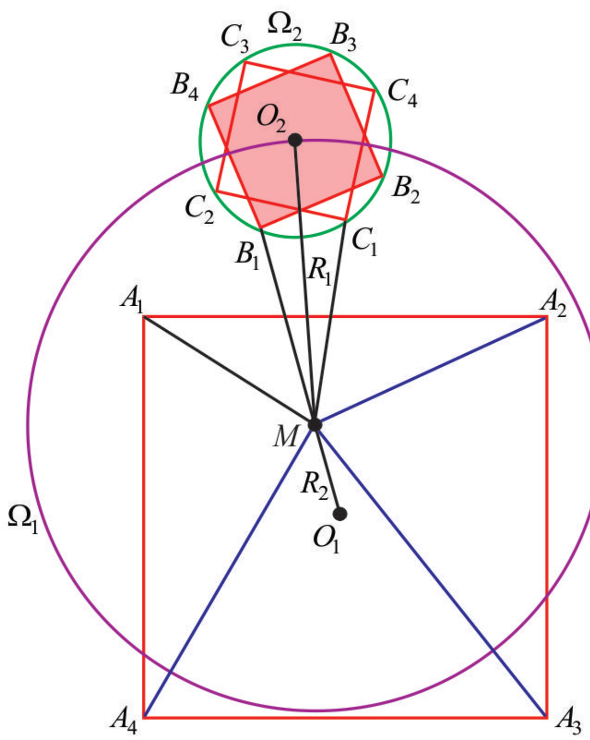

Figure 5

If given regular polygon $A_1A_2\cdots A_n$ and the point M in its plane, draw the circle Ω_1 with center M and the radius R_1. Choose the point O_2 on the Ω_1, which is the center of the second desired regular polygon – $B_1B_2\cdots B_n$. Construct the circle Ω_2, whose center is O_2 and the radius equals $R_2 = O_1M$. From the point M as the center draw the auxiliary circle with radius MA_1. The intersection points of the auxiliary and Ω_2 circles – B_1 and C_1 are the vertices of the desired polygon separately. In fact, we construct two regular polygons – $B_1B_2\cdots B_n$ and $C_1C_2\cdots C_n$ which have the same set of the distances from the point M to the vertices of original $A_1A_2\cdots A_n$ polygon:

$$\{MA_i\} = \{MB_i\} = \{MC_i\},$$
$$\text{where } \ i = 1, 2, \ldots, n.$$

For the square case, see Fig. 5, by corresponding enumeration of the vertices:

$$MA_i = MB_i = MC_i, \ \text{ where } \ i = 1, \ldots, 4.$$

For the regular pentagon case, see Fig. 6:

$$MA_i = MB_i = MC_i, \ \text{ where } \ i = 1, \ldots, 5.$$

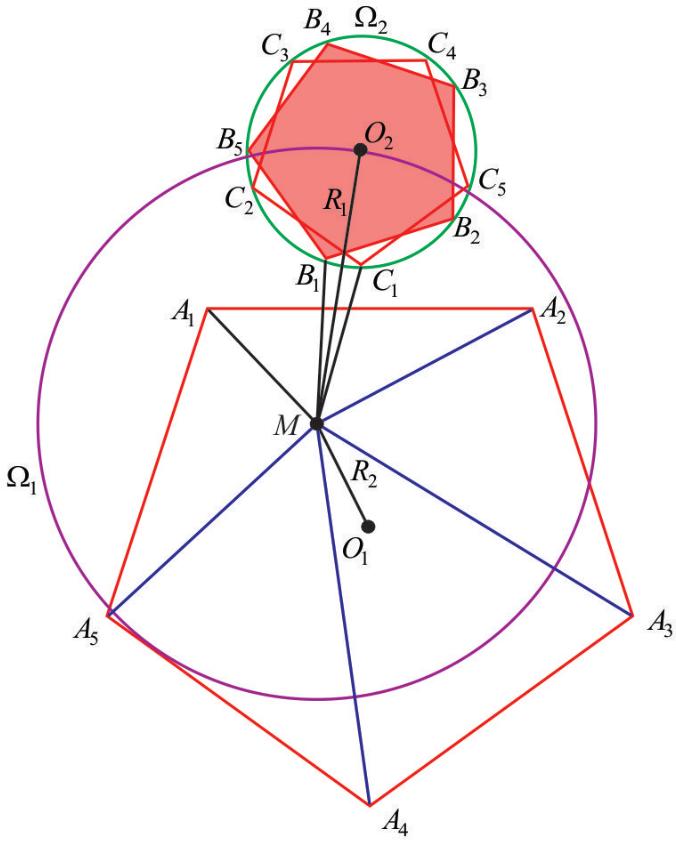

Figure 6

5. Two regular polygons and two points theorem

Until now we consider the case when the regular polygon and the point were given. Let us consider the case when two non-congruent regular polygons (with the same number of vertices) are given initially. Is there a point in the plane of two non-congruent regular polygons, from where the distances to the vertices of these polygons are the same? From the construction method it is clear – such point exists, if

$$|R_1 - R_2| \le O_1O_2 \le R_1 + R_2,$$

and one of the distances MA_i and MB_i must be equal to each other (4.2). If two polygons have one shared vertex this equality automatically holds. From the construction method the point M must be at distance R_1 from the O_2, and at distance R_2 from the O_1, i.e. intersection of two circles – $\Omega_1(O_2, R_1)$ and $\Omega_2(O_1, R_2)$. But in general case, there are two points of such properties. So we obtain – two regular polygons and two points theorem.

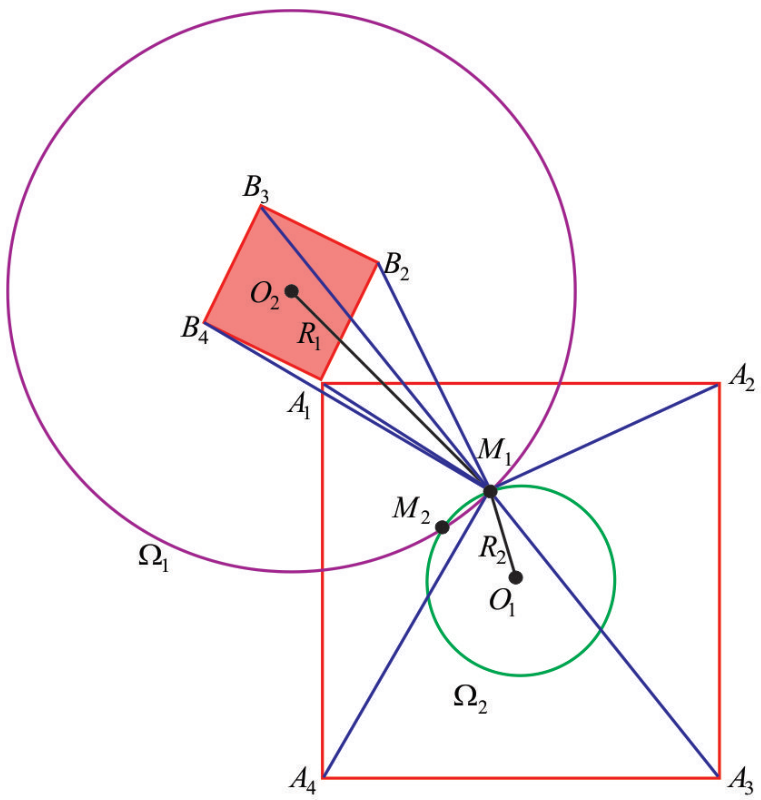

Figure 7

Theorem 5.1. *If two non-congruent regular polygons* $A_1A_2\cdots A_n$ *and* $B_1B_2\cdots B_n$ *have one shared vertex, there are two points* M_1 *and* M_2 *in the plane, separately having the same set of the distances to the vertices of the polygons:*

$$\{M_1A_i\} = \{M_1B_i\}, \quad \{M_2A_i\} = \{M_2B_i\}, \quad \text{where } i = 1, \ldots, n.$$

The M_1 *and* M_2 *are intersection points of two circles –* $\Omega_1(O_2, R_1)$ *and* $\Omega_2(O_1; R_2)$*, where* O_1 *and* O_2 *are the centers of the circumcircles of* $A_1A_2\cdots A_n$ *and* $B_1B_2\cdots B_n$*,* R_1 *and* R_2 *their radii, respectively.*

The construction of M_1 and M_2 is given for the squares in Fig. 7, the shared vertex is A_1.

The same distances are:

$$M_1A_2 = M_1B_2, \quad M_1A_3 = M_1B_3, \quad M_1A_4 = M_1B_4$$

and

$$M_2A_2 = M_2B_4, \quad M_2A_3 = M_2B_3, \quad M_2A_4 = M_2B_2.$$

If the shared vertex and the centers of the circumcircles are collinear, there is only one point having the same set of the distances to the vertices of two non-congruent regular polygons.

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