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# THEORETICAL CONSIDERATIONS ON AN ISSUE FROM ROMANIAN NATIONAL MATHEMATICAL OLYMPIAD 2019

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ABSTRACT. At the National Mathematical Olympiad in 2019, to the $7^{th}$ grade, the following problem was given, author Traian Preda.

Let $ABCD$ be a square and $E$ be a point on the side $(CD)$. The squares $AENM$ and $EBQP$ are constructed outside the triangle $ABE$ (Fig. 1). Prove that:

$a)$ $ND = PC$;

$b)$ $ND \perp PC$.

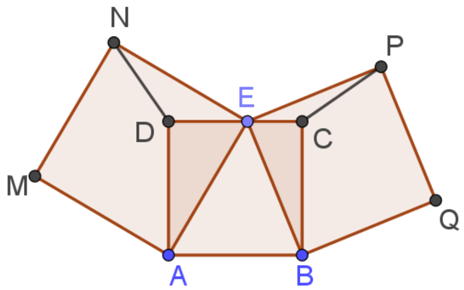


FIGURE 1

We turn our attension to point $a)$. The question is: why was the construction of some squares outside chosen? And automatically the next question: can we replace the squares with other regular polygons and keep the relation $a)$?

In this note we will demonstrate an important result and based on it, we will give some applications.

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**Lemma 1.** *Let $ABCD$ be a square and $E$ be a point on the side $(CD)$. Outside the triangle $ABE$ are constructed the regular polygons $AEM_1 \dots M_{n-2}$ in the trigonometric direction and $BEN_1 \dots N_{N-2}$ in the opposite trigonometric direction, where $n \in \mathbb{N}$, $n \geq 3$ (Fig. 2).*
*Show that*

$$(1) \qquad DM_m^2 = x^2\left(x^2+(x+y)^2\right)\frac{\sin^2\frac{m\pi}{n}}{\sin^2\frac{\pi}{n}} + 2x^2\frac{\sin\frac{m\pi}{n}\cos\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}} - 2x(x+y)\frac{\sin\frac{m\pi}{n}\sin\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}} \qquad \textit{and}$$

$$(2) \qquad CN_m^2 = y^2 + \left(y^2+(x+y)^2\right)\frac{\sin^2\frac{m\pi}{n}}{\sin^2\frac{\pi}{n}} + 2y^2\frac{\sin\frac{m\pi}{n}\cos\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}} - 2y(x+y)\frac{\sin\frac{m\pi}{n}\sin\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}},$$

*where $DE = x$, $CE = y$ and $m \in \{1, 2, \dots, n-2\}$*

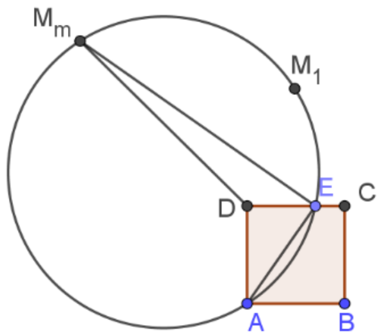


FIGURE 2

**Proof.** We note $m(\measuredangle AED) = \alpha$.
In triangle $ADE$ we have $AE^2 = x^2 + (x+y)^2 = EM_1^2$,

$$\sin(\measuredangle DEA) = \frac{DA}{AE} = \frac{x+y}{\sqrt{x^2+(x+y)^2}} \qquad \text{and}$$

$$\cos(\measuredangle DEA) = \frac{DE}{AE} = \frac{x}{\sqrt{x^2+(x+y)^2}}.$$

For calculation $\cos(\measuredangle DEM_m)$ keep in mind that in a regular polygon with $n$ sides the measure of an angle has the value $\frac{n-2}{n}\pi$ and
$m(\widehat{AE}) = m(\widehat{EM_1}) = m(\widehat{AM}_{n-2}) = \frac{2\pi}{n}$, so $m(\measuredangle M_1M_mE) = \frac{\pi}{n}$.
Then $m(\measuredangle EM_1M_m) = \frac{(n-m)\pi}{n}$ and $m(\measuredangle AEM_m) = \frac{(n-m-1)\pi}{n}$.
In the triangle $DEM_m$, we have $m(\measuredangle DEM_m) = |m(\measuredangle AEM_m) - m(\measuredangle AED)|$ and taking the above observations into account, we obtain that $m(\measuredangle DEM_m) =$

$\left|\dfrac{(n-m-1)\pi}{n}-\alpha\right|$. Since the sosine function is an even function, we have

$$\begin{aligned}\cos(\measuredangle DEM_m) &= \cos\left(\frac{(n-m-1)\pi}{n}-\alpha\right) = \cos\left(\pi-\frac{(m+1)\pi}{n}-\alpha\right) = \\ &= -\cos\left(\frac{(m+1)\pi}{n}+\alpha\right) = \\ &= -\cos\frac{(m+1)\pi}{n}\cdot\cos\alpha+\sin\frac{(m+1)\pi}{n}\cdot\sin\alpha,\end{aligned}$$

from where

(3)

$$\cos(\measuredangle DEM_m)=\frac{1}{\sqrt{x^2+(x+y)^2}}\left[-x\cdot\cos\frac{(m+1)\pi}{n}+(x+y)\sin\frac{(m+1)\pi}{n}\right].$$

In the triangle $EM_1M_m$, applying the sine theorem we have $\dfrac{EM_1}{\sin(\measuredangle EM_mM_1)}=\dfrac{EM_m}{\sin(\measuredangle EM_1M_m)}$, from where $EM_m=\sqrt{x^2+(x+y)^2}\dfrac{\sin\frac{(n-m)\pi}{n}}{\sin\frac{\pi}{n}}$.

Because $\sin\dfrac{(n-m)\pi}{n}=\sin\left(\pi-\dfrac{m\pi}{n}\right)=\sin\dfrac{m\pi}{n}$, it results that

$$EM_m=\sqrt{x^2+(x+y)^2}\,\frac{\sin\frac{m\pi}{n}}{\sin\frac{\pi}{n}}. \tag{4}$$

From the cosine theorem applied to the triangle $M_nDE$, we have

$$DM_m^2=DE^2+EM_m^2-2DE\cdot EM_m\cdot\cos(\measuredangle DEM_m)$$

and taking (3) and (4) into account, relation (1) follows.The relation (2) is obtained from relation (1) replacing $x$ with $y$ and $y$ with $x$.

**Theorem 1.** *Let $ABCD$ be a square and $E$ be a point on the side $(CD)$. Outside the triangle $ABE$ are constructed the regular polygons $AEM_1...M_{n-2}$ in the trigonometric direction and $BEN_1\dots N_{n-2}$ in the opposite trigonometric direction, where $n\in N$, $n\geq 3$, $DE=x$, $CE=y$ and $m\in\{1,2,\dots,n-2\}$.*

*Show that*

*a) if $x=y$, it result that $DM_m=CN_m$;*

*b) if $x\neq y$, then $DM_m=CN_m$ if and only if*

$$1+\frac{\sin^2\frac{m\pi}{n}}{\sin^2\frac{\pi}{n}}+2\frac{\sin\frac{m\pi}{n}\cos\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}}-2\frac{\sin\frac{m\pi}{n}\sin\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}}=0.$$

**Proof.** Taking (1) and (2) into account, point *a*) follows.

If $x\neq y$ the relation $DM_m=CN_m$ is equivalent after calculus by

$$\begin{aligned}(x^2-y^2)+(x^2-y^2)\frac{\sin^2\frac{m\pi}{n}}{\sin^2\frac{\pi}{n}}+2(x^2-y^2)\frac{\sin\frac{m\pi}{n}\cos\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}}-\\ -2(x^2-y^2)\frac{\sin\frac{m\pi}{n}\sin\frac{(m+1)\pi}{n}}{\sin\frac{\pi}{n}}=0,\end{aligned}$$

from where point *b*) follows.

**Theorem 2.** *In the conditions of Theorem* 1*, if $E$ is not middle of $DC$, we have that $DM_1 = CM_1$ if and only if $n = 4$ (Fig.* 3*).*

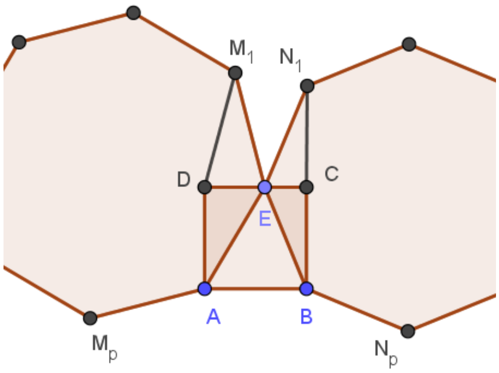


FIGURE 3

**Proof.** We consider $m = 1$ in Theorem 1 and we obtain the equation $1 + \cos\frac{2\pi}{n} - \sin\frac{2\pi}{n} = 0$.

This is equivalent to $\cos\frac{2\pi}{n} - \sin\frac{2\pi}{n} = -1$, equivalent to $\sin\left(\frac{\pi}{4} - \frac{2\pi}{n}\right) = -\frac{\sqrt{2}}{2}$, equivalent to $\sin\left(\frac{(n-8)\pi}{4n}\right) = -\frac{\sqrt{2}}{2}$. The solutions of this trigonometric equation are $\frac{n-8}{4n}\pi \in \left\{(-1)^k \cdot \arcsin\left(-\frac{\sqrt{2}}{2}\right) + k\pi, k \in \mathbb{Z}\right\}$, or $\frac{n-8}{4n}\pi \in \left\{(-1)^{k+1} \cdot \frac{\pi}{4} + k\pi, k \in \mathbb{Z}\right\}$, from where $n = \frac{8}{1 - 4k + (-1)^{k+2}}$, $k \in \mathbb{Z}$.

For $k$ =even number $=2q$, $q \in \mathbb{Z}$, we have $n = \frac{8}{1 - 8q + 1} = \frac{8}{2 - 8q} = \frac{4}{1 - 4q} \in \mathbb{N}$, from where $1 - 4q \in D_4 = \{1, 2, 4\}$, it result $q = 0$, so $n = 4$.

For $k$ = odd number $= 2q + 1$, $q \in \mathbb{Z}$, we have $n = \frac{8}{1 - 4(1q + 1) - 1} = \frac{8}{-4(2q + 1)} = \frac{-2}{2q + 1} \in N$, from where $2q + 1 \in \{-1, -2\}$. It result $q = -1$, so $n = -1$, value that does not agrees.

**Remark 1.** For $n = 4$ just point *a*) is obtained from the initial problem.

**Theorem 3.** *In the conditions of Theorem* 1*, if $E$ it is not middle of $DC$, we have that $DM_{n-2} = CN_{n-2}$ f and only if $n = 12$.*

**Proof.** We consider $m = n - 2$ in Theorem 1 and taking into account that

$$\sin\frac{m\pi}{n} = \sin\frac{(n-2)\pi}{n} = \sin\left(\pi - \frac{2\pi}{n}\right) = \sin\frac{2\pi}{n},$$

$$\sin\frac{(m+1)\pi}{n} = \sin\frac{(n-1)\pi}{n} = \sin\left(\pi - \frac{\pi}{n}\right) = \sin\frac{\pi}{n},$$

$$\cos\frac{(m+1)\pi}{n} = \cos\frac{(n-1)\pi}{n} = \cos\left(\pi - \frac{\pi}{n}\right) = -\cos\frac{\pi}{n}, \quad \text{and}$$
$$\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n},$$

we have that $DM_{n-2} = CN_{n-2}$ is equivalent to

$$1 + \frac{\sin^2\frac{2\pi}{n}}{\sin^2\frac{\pi}{n}} + 2\frac{\sin\frac{2\pi}{n}\left(-\cos\frac{\pi}{n}\right)}{\sin\frac{\pi}{n}} - 2\frac{\sin\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}} = 0,$$

equivalent to $\sin\frac{2\pi}{n} = \frac{1}{2}$.

We get the solutions $\frac{2\pi}{n} \in \left\{(-1)^k \cdot \arcsin\left(\frac{1}{2}\right) + k\pi,\ k \in \mathbb{Z}\right\}$, equivalent to $\frac{2\pi}{n} \in \left\{(-1)^{k+1} \cdot \frac{\pi}{6} + k\pi,\ k \in \mathbb{Z}\right\}$, from where $\frac{2\pi}{n} = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$, or $n = \frac{12}{12k+1}$, $k \in \mathbb{Z}$, with the solution $k = 0$, so $n = 12$.

The other option $\frac{2\pi}{n} = \frac{5\pi}{6} + 2k\pi$, leads to $n = \frac{12}{12k+5} \notin \mathbb{N}$, $\forall\, k \in \mathbb{Z}$.

**Remark 2.** Figure 4 shows the conclusion of Theorem 3.

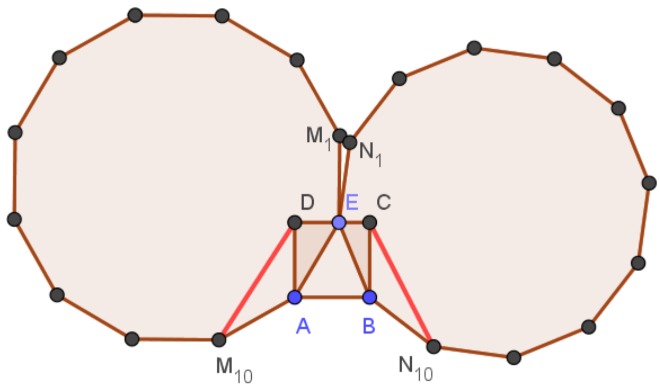


FIGURE 4

**Theorem 4.** *In the conditions of Theorem* 1*, if $E$ it is not the middle of $DC$, we have that $DM_{n-3} = CN_{n-3}$ if and only if $n = 8$ or $n = 4$.*

**Proof.** We keep that in mind

$$\sin\frac{m\pi}{n} = \sin\frac{(n-3)\pi}{n} = \sin\left(\pi - \frac{3\pi}{n}\right) = \sin\frac{3\pi}{n} = 3\sin\frac{\pi}{n} - 4\sin^3\frac{\pi}{n},$$
$$\sin\frac{(m+1)\pi}{n} = \sin\frac{(n-2)\pi}{n} = \sin\left(\pi - \frac{2\pi}{n}\right) = \sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n} \quad \text{and}$$
$$\cos\frac{(m+1)\pi}{n} = \cos\frac{(n-2)\pi}{n} = \cos\left(\pi - \frac{2\pi}{n}\right) = -\cos\frac{2\pi}{n} = 2\sin^2\frac{\pi}{n} - 1.$$

We note $\sin\frac{\pi}{n} = t \in (0,1]$ and then $\cos\frac{\pi}{n} = \sqrt{1-t^2}$. We consider $m = n-3$ in Theorem 1 and we obtain the equations, respectively equivalent

$$1 + \frac{(3t-4t^3)^2}{t^2} + 2\frac{(3t-4t^3)(2t^2-1)}{t} - 2\frac{(2t-4t^3)2t\sqrt{1-t^2}}{t} = 0,$$
$$1 + \frac{t^2(3-4t^2)^2}{t^2} + 2\frac{t(3-4t^2)(2t^2-1)}{t} - 4(3t-4t^3)\sqrt{1-t^2} = 0,$$
$$1 + (3-4t^2)^2 + 2(3-4t^2)(2t^2-1) - 4(3t-4t^3)\sqrt{1-t^2} = 0,$$
$$1 - t^2 = (3t-4t^3)\sqrt{1-t^2},\ 1-t^2 = (3t-4t^3)^2,$$
$$1 - t^2 = t^2(3-4t^2)^2.$$

The last equation is of degree III in $t^2$ and has as solutions on $t^2 = \frac{2-\sqrt{2}}{4}$, $t^2 = \frac{2+\sqrt{2}}{4}$ and $t^2 = \frac{1}{2}$. The following values are convenient $t = \frac{\sqrt{2-\sqrt{2}}}{2}$, $t = \frac{\sqrt{2+\sqrt{2}}}{2}$ and $t = \frac{\sqrt{2}}{2}$.

For $t = \frac{\sqrt{2-\sqrt{2}}}{2} = \sin\frac{\pi}{8}$, we have $\sin\frac{\pi}{n} = \sin\frac{\pi}{8}$, equivalent to $\frac{\pi}{n} = \frac{\pi}{8} + 2k\pi$, $k \in \mathbb{Z}$ or $\frac{\pi}{n} = \frac{7\pi}{8} + 2k\pi$, $k \in \mathbb{Z}$. It is obtained from the first relation $n = \frac{8}{16k+1}$, $k \in \mathbb{Z}$, which is natural number for $k = 0$, from where $n = 8$. The second relationship does not lead to natural solutions.

For $t = \frac{\sqrt{2+\sqrt{2}}}{2}$, because $\frac{\sqrt{2+\sqrt{2}}}{2} = \sin\frac{3\pi}{8}$, we have $\sin\frac{\pi}{n} = \sin\frac{3\pi}{8}$, equivalent to $\frac{\pi}{n} = \frac{3\pi}{8} + 2k\pi$, $k \in \mathbb{Z}$ or $\frac{\pi}{n} = \frac{5\pi}{8} + 2k\pi$, $k \in \mathbb{Z}$. None of the relationships lead to natural solutions for $n$.

For $t = \frac{\sqrt{2}}{2}$, we have $\sin\frac{\pi}{n} = \sin\frac{\pi}{4}$, equivalent to $\frac{\pi}{n} = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$ or $\frac{\pi}{n} = \frac{3\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$. It is obtained from the first relation $n = \frac{4}{8k+1}$, $k \in \mathbb{Z}$, which is natural number for $k = 0$, from where $n = 4$, that is point $a)$ of the original issue. The second relation does not lead to natural solutions.

**Remark 3.** Figure 5 shows the conclusion of Theorem 4.

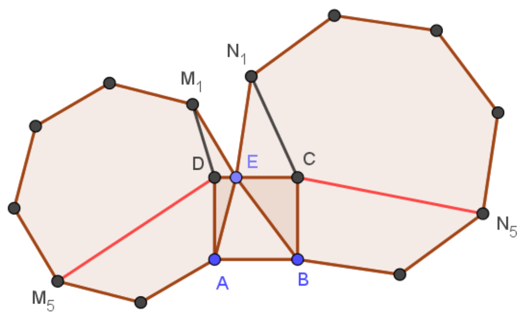


FIGURE 5

**Theorem 5.** *In the conditions of Theorem* 1*, if* $E$ *it is not the middle of* $DC$ *and* $n = 2p$*,* $p \geq 2$*, we have that* $DM_{p-1} = CN_{p-1}$ *if and only if* $n = 4$.

**Proof.** We consider $m = p - 1$ in Theorem 2 and taking that

$$\sin\frac{m\pi}{n} = \sin\frac{(p-1)\pi}{2p} = \sin\left(\frac{\pi}{2} - \frac{\pi}{2p}\right) = \cos\frac{\pi}{2p}\,,$$

$$\sin\frac{(m+1)\pi}{n} = \sin\frac{p\pi}{2p} = \sin\frac{\pi}{2} = 1 \quad \text{and}$$

$$\cos\frac{(m+1)\pi}{n} = \cos\frac{p\pi}{2p} = \cos\frac{\pi}{2} = 0.$$

Equivalent equations are obtained $1 + \dfrac{\cos^2\frac{\pi}{2p}}{\sin^2\frac{\pi}{2p}} - 2\dfrac{\cos\frac{\pi}{2p}}{\sin\frac{\pi}{2p}} = 0$, $\left(\cot\dfrac{\pi}{2p} - 1\right)^2 = 0$, $\cot\dfrac{\pi}{2p} = 1$, from where $\dfrac{\pi}{2p} = \dfrac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$. We get $p = \dfrac{2}{4k+1}$ $k \in \mathbb{Z}$, which is natural number for $k = 0$, from where $p = 2$, so $n = 4$.

## References

[1] Preda, T., *Subject* 2*,* $7^{th}$ *grade*, Mathematical Gazette, Supplement to the $70^{th}$ National Mathematical Olympiad (in Romanian), Romania, 22-26 april (2019), 6.

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