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MORE NEW CHARACTERIZATIONS OF EXTANGENTIAL QUADRILATERALS

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Abstract. We prove an additional 16 new necessary and sufficient conditions for when a convex quadrilateral can have an excircle.

1. Introduction

This paper is the second part in our study of new characterizations of extangential quadrilaterals. The first part was [4], and we recommend the reader to study that paper before continuing with this one. Earlier studies were conducted by the first author in [1, 2].

Before we start proving more new necessary and sufficient conditions for when a convex quadrilateral can have an *excircle* (a circle tangent to the extensions of all four sides), we remind the reader of the following three very useful characterizations that will be applied in several of the proofs: A convex quadrilateral $ABCD$ can have an excircle outside the biggest of the two vertex angles at A or C if and only if either

$$AB + BC = CD + DA \tag{1}$$

or

$$AJ + JC = CK + KA, \tag{2}$$

where $J = AB \cap CD$ and $K = BC \cap DA$ (here only non-trapezoids are considered so that these points exist), or

$$BJ + JD = DK + KB. \tag{3}$$

There are similar conditions for an excircle outside of B or D, but like in the previous paper [4], we almost exclusively study the case with an excircle outside of A or C.

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2. Tangent circles

We begin by reviewing a few triangle formulas that will be used in several of the proofs. They are well-known, so their derivations are left as an exercise for readers to whom whey are unfamiliar. The proofs are based on the *two tangent theorem*, which states that the two tangents to a circle from an external point have equal lengths ($AE = AF$ in Figure 1).

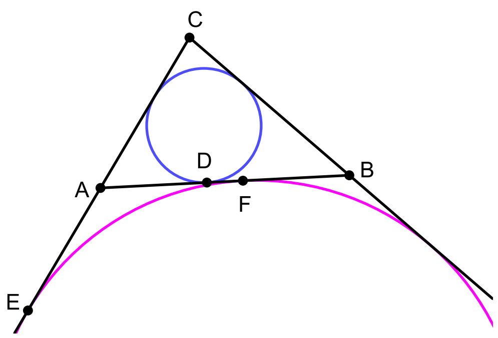

Figure 1. Tangent points for the incircle and one excircle to a triangle

Lemma 2.1. *In triangle ABC, suppose the incircle is tangent to AB at D, and that the excircle tangent to AB is tangent to the extension of CA at E. Then*

$$AD = \tfrac{1}{2}(AB - BC + CA), \qquad AE = \tfrac{1}{2}(AB + BC - CA),$$
$$CE = \tfrac{1}{2}(AB + BC + CA).$$

The first characterization has somewhat different formulations for an excircle outside of A and C, so we only state it in the latter case.

Theorem 2.1. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Then the excircle to triangles ADJ and ABK that are tangent to DJ and BK respectively are tangent to the extension of AB at the same point if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of C.*

Proof. Suppose the considered excircles to triangles ADJ and ABK are tangent to the extension of AB at L_1 and L_2 respectively (see Figure 2). Applying the third formula in the lemma yields

$$AL_1 = \tfrac{1}{2}(AJ + JD + DA), \qquad AL_2 = \tfrac{1}{2}(AK + KB + BA)$$

so

$$2(AL_1 - AL_2) = AJ + JD + DA - AK - KB - BA = BJ + JD - DK - KB.$$

The two triangle excircles are tangent on the extension of AB if and only if

$$L_1L_2 = 0 \quad \Leftrightarrow \quad AL_1 = AL_2 \quad \Leftrightarrow \quad BJ + JD = DK + KB$$

which proves the theorem according to (3). $\square$

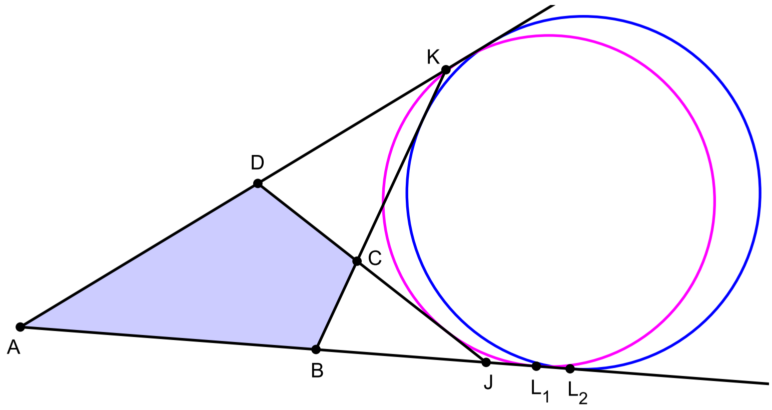

FIGURE 2. $ABCD$ is extangential $\Leftrightarrow$ $L_1L_2 = 0$

Thus the quadrilateral is extangential if and only if the two circles in Figure 2 coincide, in which case that is the excircle to the quadrilateral.

There were several potential tangent circles in Section 3 of [4]. Now we prove that these are also characterizations of extangential quadrilaterals.

Theorem 2.2. *Two excircles belonging to each of the two triangles created by a diagonal in a convex quadrilateral, which are tangent to two adjacent sides of the quadrilateral, are tangent to each other on the extension of that diagonal if and only if it is an extangential quadrilateral.*

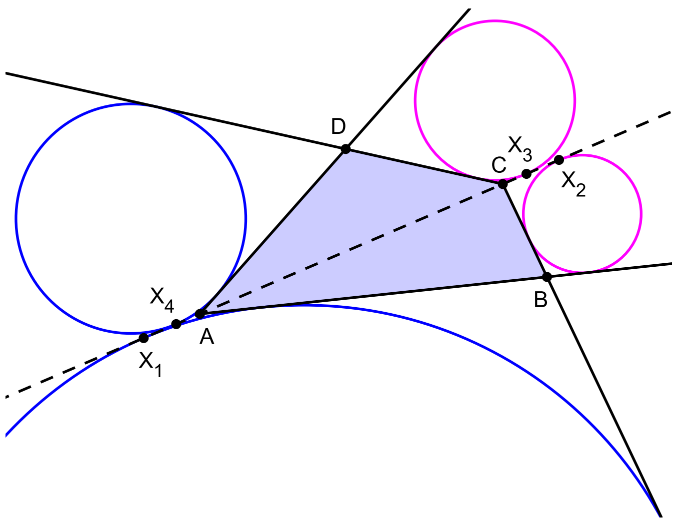

FIGURE 3. $X_1X_4 = 0$ $\Leftrightarrow$ $ABCD$ is extangential $\Leftrightarrow$ $X_2X_3 = 0$

Proof. There are two possible pairs of excircles that fit the description in the theorem. The proof in each case is the same, so we only study one of them.

Suppose the two excircles are tangent to the extension of diagonal AC at X_1 and X_4 (see Figure 3). From the second formula in the lemma, we get

$$AX_1 = \tfrac{1}{2}(AB + BC - AC), \qquad AX_4 = \tfrac{1}{2}(CD + DA - AC)$$

so

$$2(AX_1 - AX_4) = AB + BC - CD - DA.$$

Hence

$$AX_1 = AX_4 \quad \Leftrightarrow \quad AB + BC = CD + DA$$

which proves that the two circles are tangent to each other at $X_1 = X_4$ if and only if the quadrilateral can have an excircle outside A or C according to (1). □

Another pair of tangent excircles appears in the following theorem.

Theorem 2.3. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Then there is one excircle from each of triangles ACJ and ACK that are tangent to each other on the extension of diagonal AC if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

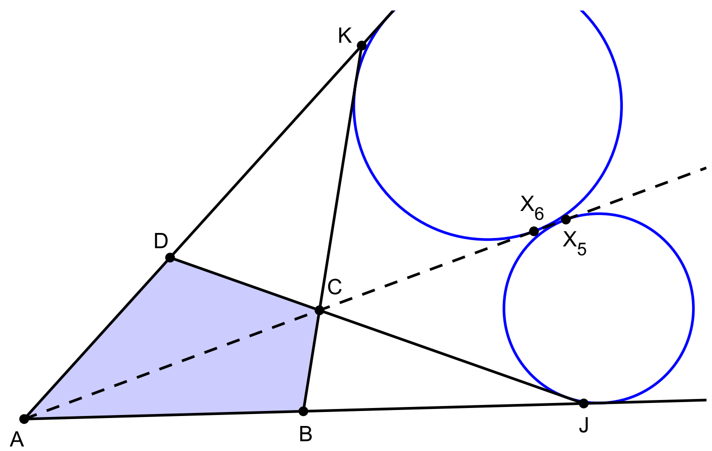

Figure 4. $ABCD$ is extangential $\quad \Leftrightarrow \quad X_5X_6 = 0$

Proof. Suppose the two excircles are tangent to the extension of diagonal AC at X_5 and X_6 (see Figure 4). We have

$$CX_5 = \tfrac{1}{2}(-AC + AJ + CJ), \qquad CX_6 = \tfrac{1}{2}(-AC + AK + CK)$$

so

$$2(CX_5 - CX_6) = AJ + CJ - AK - CK$$

and we get that

$$CX_5 = CX_6 \quad \Leftrightarrow \quad AJ + CJ = AK + CK$$

which proves that the two circles are tangent to each other at $X_5 = X_6$ if and only if the quadrilateral can have an excircle outside A or C according to (2).

We note that there is another pair of excircles to the same triangles that are tangent at the same point on the extension of AC outside of A if and only if $ABCD$ is an extangential quadrilateral. The proof in that case is the same except that $A \leftrightarrow C$. □

Next we have a pair of circles that are tangent on the other diagonal.

Theorem 2.4. *Consider the two triangles created by diagonal BD in a convex quadrilateral $ABCD$. The incircle in one of the triangles and the excircle to the other triangle that is tangent to BD are tangent to each other on BD if and only if $ABCD$ is extangential with an excircle outside of A or C.*

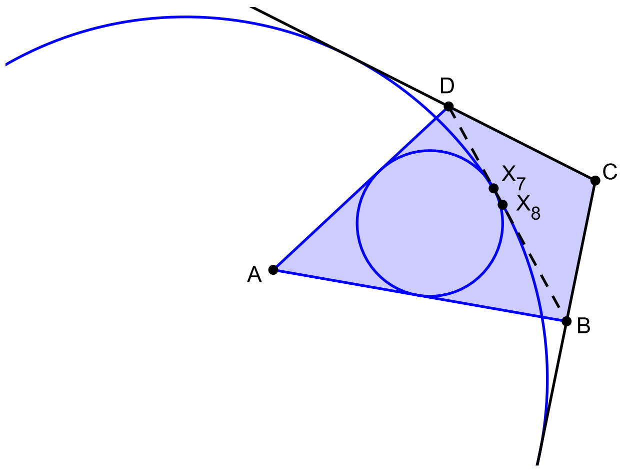

FIGURE 5. $ABCD$ is extangential $\quad\Leftrightarrow\quad X_7X_8 = 0$

Proof. There are two possible cases with similar proofs, so we only study one of them.

Suppose the incircle in ABD and the excircle to BCD are tangent to BD at X_7 and X_8 respectively (see Figure 5). It holds that

$$BX_7 = \tfrac{1}{2}(AB + BD - DA), \qquad BX_8 = \tfrac{1}{2}(CD + BD - BC).$$

Then

$$2(BX_7 - BX_8) = AB - DA + BC - CD$$

and so

$$BX_7 = BX_8 \quad \Leftrightarrow \quad AB + BC = CD + DA$$

which proves that the two circles are tangent to each other at $X_7 = X_8$ if and only if the quadrilateral is extangential according to (1). □

3. Concurrent lines

In this section we prove five characterizations regarding concurrent lines. The first is about the same configuration as in the previous theorem.

Theorem 3.1. *In a convex quadrilateral $ABCD$, suppose the incircle in triangle ABC is tangent to AB and DA at G and F respectively, and the excircle to triangle BCD that is tangent to BD is tangent to the extensions of BC and CD at H and I respectively. Then GH and FI intersect on AC if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

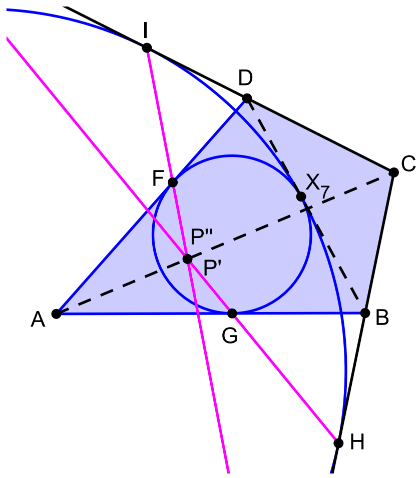

FIGURE 6. $ABCD$ is extangential $\quad\Leftrightarrow\quad$ GH, FI, AC are concurrent

Proof. ($\Rightarrow$) In an extangential quadrilateral, applying Menelaus' theorem (with non-directed distances, see Figure 6) in triangles ABC and ADC, where transversals GH and FI intersect AC at points P' and P'' respectively, we get

$$\frac{AG}{GB}\cdot\frac{BH}{HC}\cdot\frac{CP'}{P'A}=1=\frac{AF}{FD}\cdot\frac{DI}{IC}\cdot\frac{CP''}{P''A} \tag{4}$$

where $GB = BX_7 = BH$ and $FD = DX_7 = DI$ according to the two tangent theorem and X_7 denote the point where the incircle and excircle are tangent to BD according to Theorem 2.4. Thus (4) is simplified into

$$\frac{CP'}{P'A}=\frac{CP''}{P''A}$$

which implies that $P' = P''$, since these two points divide AC in the same ratio.

($\Leftarrow$) We do a contrapositive proof of the converse. If $ABCD$ is not extangential, assume without loss of generality that $BX_7 > BX_8$ where X_7 and X_8 are the points where the incircle and excircle are tangent to BD respectively (see Figure 5). Then

$$GB = BX_7 > BX_8 = BH, \qquad FD = DX_7 < DX_8 = DI.$$

Applying Menelaus' theorem twice as in the proof of the direct theorem and combining those equalities, we get by using $AG = AF$ and $HC = IC$ and the inequalities just observed that

$$\frac{GB}{GB}\cdot\frac{CP'}{P'A}>\frac{BH}{GB}\cdot\frac{CP'}{P'A}=\frac{DI}{FD}\cdot\frac{CP''}{P''A}>\frac{FD}{FD}\cdot\frac{CP''}{P''A}.$$

Hence

$$\frac{CP'}{P'A}>\frac{CP''}{P''A}$$

so $P' \neq P''$, completing the proof. □

Next we have two excircles and two concurrent lines on diagonal AC.

Theorem 3.2. *In a convex quadrilateral $ABCD$, suppose the excircles to triangles ABC and ADC outside of BC and CD are tangent to the sides AB, BC, CD, DA or their extensions at L, M, N, O respectively. Then LM and ON intersect on AC if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

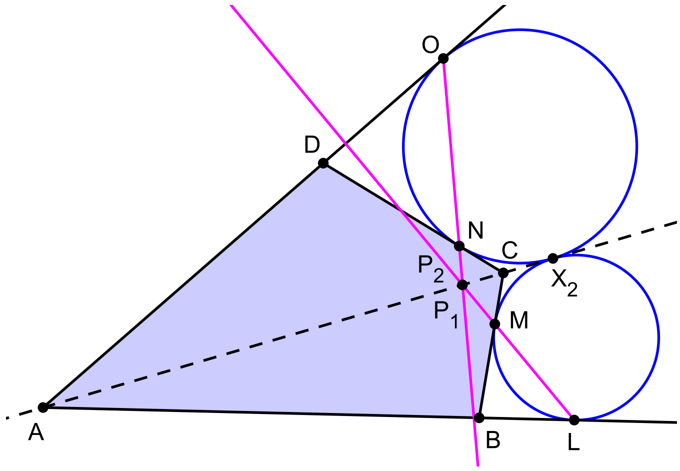

FIGURE 7. If $ABCD$ is extangential $\Rightarrow$ LM, ON, AC are concurrent

Proof. ($\Rightarrow$) Applying Menelaus' theorem (see Figure 7) in triangles ABC and ADC, where transversals LM and ON intersect AC at points P_1 and P_2 respectively, we get

$$\frac{AL}{LB}\cdot\frac{BM}{MC}\cdot\frac{CP_1}{P_1A}=1=\frac{AO}{OD}\cdot\frac{DN}{NC}\cdot\frac{CP_2}{P_2A}. \tag{5}$$

The two tangent theorem states that $OD = DN$ and $LB = BM$. When $ABCD$ is extangential, we also have $AL = AX_2 = AO$ and $MC = CX_2 = NC$ according to Theorem 2.2, where X_2 is the point on the extension of AC where the two excircles are tangent to each other. Then (5) is simplified as

$$\frac{CP_1}{P_1A}=\frac{CP_2}{P_2A}$$

which means that $P_1 = P_2$ since these two points divide AC in the same ratio.

($\Leftarrow$) We do a contrapositive proof of the converse. Suppose without loss of generality that the excircles to triangles ABC and ADC are tangent to the extension of AC at points X_2 and X_3 respectively such that $CX_2 > CX_3$ (see Figure 8). Applying Menelaus' theorem twice as in the proof of the direct theorem and simplifying each equality using $LB = BM$ and $OD = DN$, we get by combining the remaining equalities that

$$\frac{AL}{MC}\cdot\frac{CP_1}{P_1A}=\frac{AO}{NC}\cdot\frac{CP_2}{P_2A},$$

which we rewrite as

$$\frac{AL}{AO}\cdot\frac{NC}{MC}=\frac{P_1A}{CP_1}\cdot\frac{CP_2}{P_2A}. \tag{6}$$

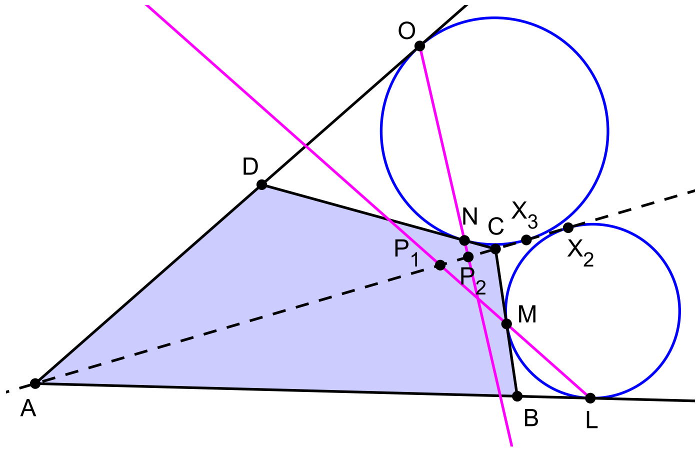

FIGURE 8. Here LM, ON, AC are not concurrent

According to the two tangent theorem, the left hand side can be expressed as

$$\frac{AL}{AO}\cdot\frac{NC}{MC}=\frac{AX_2}{AX_3}\cdot\frac{CX_3}{CX_2}=\frac{AC+CX_2}{AC+CX_3}\cdot\frac{CX_3}{CX_2}=\frac{\frac{AC}{CX_2}+1}{\frac{AC}{CX_3}+1}. \tag{7}$$

Since $CX_2>CX_3$, it follows that $\frac{1}{CX_2}<\frac{1}{CX_3}$, and thus

$$\frac{AC}{CX_2}<\frac{AC}{CX_3} \quad\Rightarrow\quad \frac{AC}{CX_2}+1<\frac{AC}{CX_3}+1 \quad\Rightarrow\quad \frac{\frac{AC}{CX_2}+1}{\frac{AC}{CX_3}+1}<1. \tag{8}$$

Combining (6), (7) and (8) yields

$$\frac{P_1A}{CP_1}\cdot\frac{CP_2}{P_2A}=\frac{AL}{AO}\cdot\frac{NC}{MC}=\frac{\frac{AC}{CX_2}+1}{\frac{AC}{CX_3}+1}<1$$

so

$$\frac{CP_2}{P_2A}<\frac{CP_1}{P_1A}$$

and it follows that $P_1\neq P_2$. This completes the proof of the converse. □

In the next theorem there are two excircles tangent to the extensions of two adjacent sides.

Theorem 3.3. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J=AB\cap CD$ and $K=BC\cap DA$. Suppose the excircles to triangles ACJ and ACK outside of CJ and CK are tangent to the extensions of the sides AB, CD, BC, DA at P, Q, R, S respectively. Then PQ and RS intersect on AC if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

Proof. Applying Menelaus' theorem in triangles AJC and AKC with transversals PQ and SR respectively (see Figure 9) yields

$$\frac{AP}{PJ}\cdot\frac{JQ}{QC}\cdot\frac{CP_3}{P_3A}=\frac{AS}{SK}\cdot\frac{KR}{RC}\cdot\frac{CP_4}{P_4A}$$

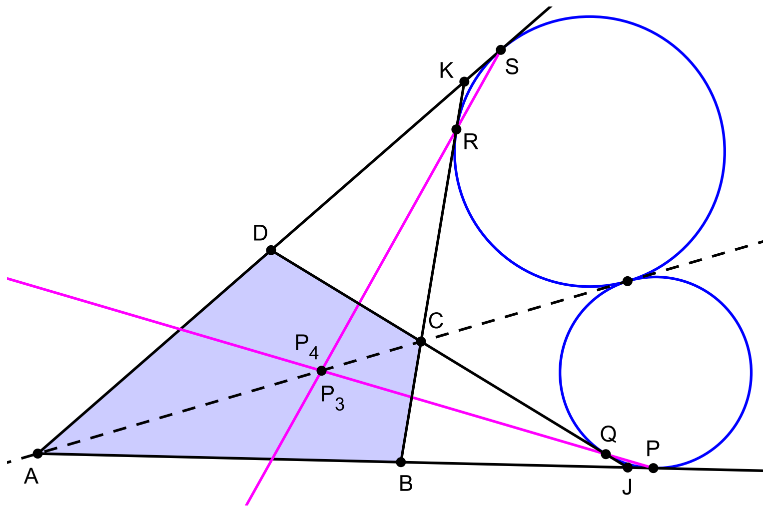

FIGURE 9. $ABCD$ is extangential $\Leftrightarrow$ PQ, RS, AC are concurrent

where P_3 and P_4 are the points where PQ and SR intersect AC respectively. By the two tangent theorem and Theorem 2.3, this is simplified into

$$\frac{CP_3}{P_3A} = \frac{CP_4}{P_4A}$$

which means that $P_3 = P_4$.

The converse can be proved with a contrapositive proof in the same way as in the previous theorem. We let the reader write down the proof as an exercise. □

There is also the following concurrency related to the configuration in the previous theorem:

Theorem 3.4. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose the excircles to triangles ACJ and ACK outside of CJ and CK are tangent to the extensions of the sides AB, CD, BC, DA at P, Q, R, S respectively. Then SP and RQ intersect on the extension of JK if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

Proof. ($\Rightarrow$) Applying Menelaus' theorem in triangles CJK and AJK with transversals RQ and SP respectively (see Figure 10), we get

$$\frac{CQ}{QJ} \cdot \frac{JP_5}{P_5K} \cdot \frac{KR}{RC} = \frac{AP}{PJ} \cdot \frac{JP_6}{P_6K} \cdot \frac{KS}{SA}$$

where P_5 and P_6 are the points where RQ and SP intersect JK respectively. Here $QJ = PJ$ and $KR = KS$ according to the two tangent theorem. When $ABCD$ is extangential, we also have $CQ = CX_5 = RC$ and $AP = AX_5 = SA$ according to Theorem 2.3, where X_5 is the point on the extension of AC where the two excircles are tangent to each other. Then we get

$$\frac{JP_5}{P_5K} = \frac{JP_6}{P_6K}$$

which means that $P_5 = P_6$.

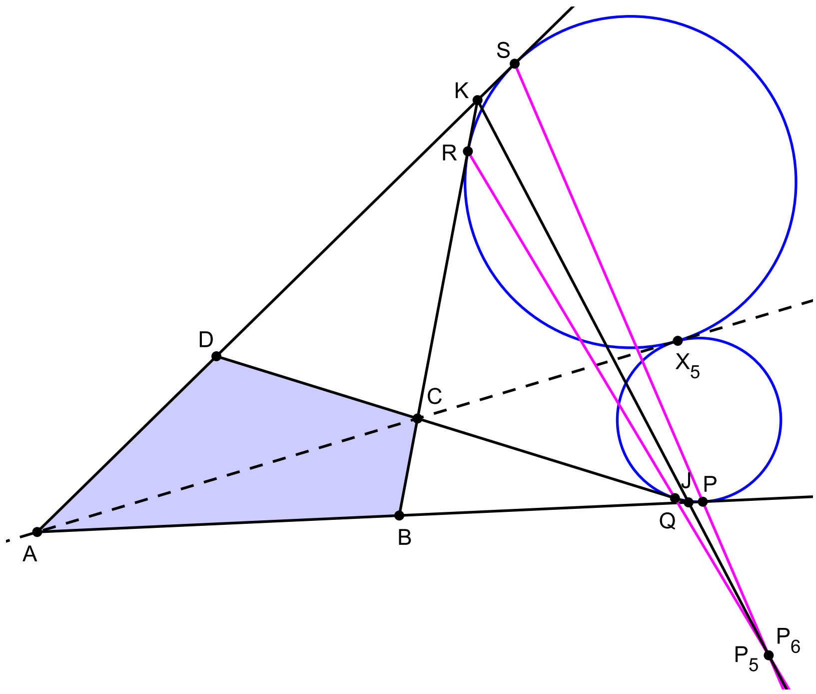

FIGURE 10. $ABCD$ is extangential $\Leftrightarrow$ SP, RQ, JK are concurrent

($\Leftarrow$) If $ABCD$ is not extangential, suppose without loss of generality that the excircles to triangles AJC and AKC are tangent to the extension of AC at points X_5 and X_6 respectively such that $CX_5 > CX_6$. Menelaus' theorem used twice yields after simplification

$$\frac{CQ}{RC} \cdot \frac{JP_5}{P_5K} = \frac{AP}{SA} \cdot \frac{JP_6}{P_6K}$$

which via the two tangent theorem can be rewritten as

$$\frac{CX_5}{CX_6} \cdot \frac{AX_6}{AX_5} = \frac{P_5K}{JP_5} \cdot \frac{JP_6}{P_6K}$$

since $CQ = CX_5$, $RC = CX_6$, $AP = AX_5$ and $SA = AX_6$. Now using $AX_6 = AC + CX_6$ and $AX_5 = AC + CX_5$ and also $CX_5 > CX_6$, we get

$$\frac{P_5K}{JP_5} \cdot \frac{JP_6}{P_6K} = \frac{\frac{AC}{CX_6} + 1}{\frac{AC}{CX_5} + 1} > 1.$$

Hence

$$\frac{JP_6}{P_6K} > \frac{JP_5}{P_5K}$$

which proves that $P_6 \neq P_5$. □

Finally there is the following variant of the previous theorem, illustrated in Figure 11. The proof is very similar, so it is omitted.

Theorem 3.5. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose the excircles to triangles ACJ and ACK outside of CJ and CK are tangent to the extensions of the sides*

AB, CD, BC, DA at P, Q, R, S respectively. Then PR and QS intersect on JK if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.

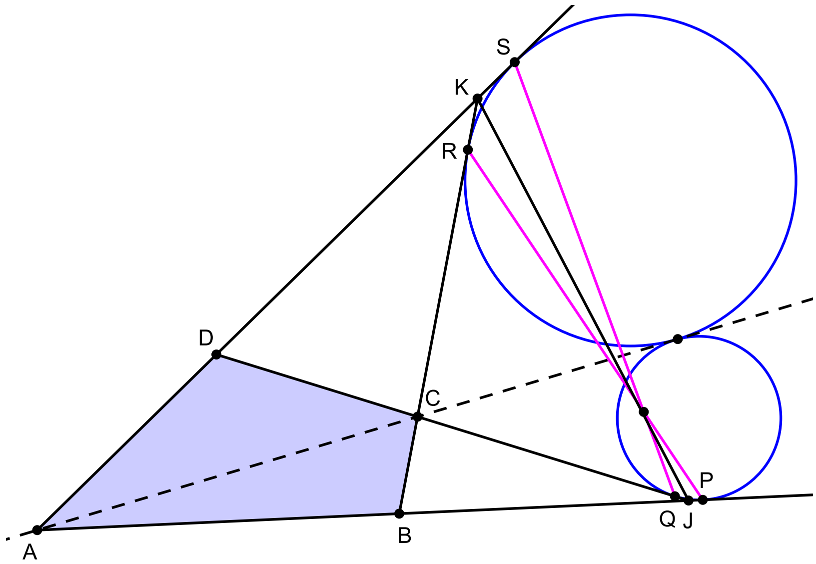

FIGURE 11. $ABCD$ is extangential $\Leftrightarrow$ PR, QS, JK are concurrent

4. Cyclic quadrilaterals

In this section we shall prove three necessary and sufficient conditions for when a convex quadrilateral can have an excircle that concerns cyclic quadrilaterals, that is, quadrilaterals whose vertices all lie on a circle. We start with one characterization featuring an isosceles trapezoid (which is a special case of a cyclic quadrilateral).

Theorem 4.1. *In a convex quadrilateral $ABCD$, let the incircles in triangles ABD and BCD be tangent to AB and BC at G and G' respectively, and the excircles to the same triangles that are tangent to BD be tangent to the extensions of AB and BC at H' and H respectively. Then $GHH'G'$ is an isosceles trapezoid if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

Proof. ($\Rightarrow$) In an extangential quadrilateral $ABCD$, one of the incircles and one of the excircles are tangent to each other on BD in pairs according to Theorem 2.4. Let those tangent points be X_8 and X_9, see Figure 12. Then $BH' = BX_9 = BG'$ and $BG = BX_8 = BH$, so triangles $BH'G'$ and BGH are isosceles. With the vertical angles at B, this proves that GH and $G'H'$ are parallel and the diagonals $GH' = HG'$, so $GHH'G'$ is an isosceles trapezoid.

($\Leftarrow$) When $GHH'G'$ is an isosceles trapezoid, $BH' = BG'$. Suppose the incircle in BCD and the excircle to ABD are tangent to BD at X_9 and X_{10} respectively. Then $BG' = BX_9$ and $BH' = BX_{10}$. Hence $BX_9 = BX_{10}$ and the converse is true according to Theorem 2.4. □

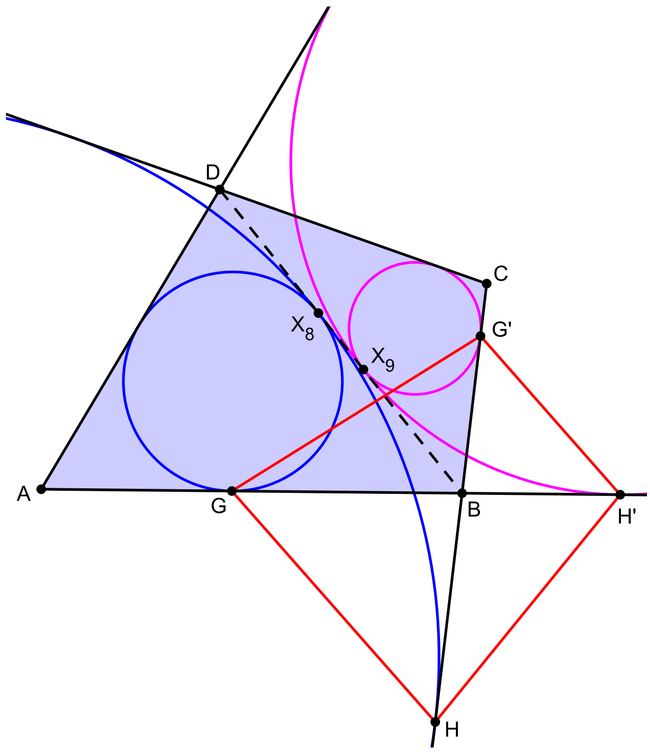

FIGURE 12. $ABCD$ is extangential $\Leftrightarrow$ $GHH'G'$ is an isosceles trapezoid

There is of course a second isosceles trapezoid associated with the four tangent points belonging to the other two extended sides of the quadrilateral.

The next characterization is about the same configuration that appeared in Theorem 2.4.

Theorem 4.2. *Suppose the incircle in triangle ABD and the excircle to triangle BCD that are tangent to BD in a convex quadrilateral $ABCD$ are also tangent to the sides DA, AB, BC, CD or their extensions at F, G, H, I respectively. Then $FGHI$ is a cyclic quadrilateral if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

Proof. ($\Rightarrow$) In all convex quadrilaterals $ABCD$, we have $\angle AGF = \frac{1}{2}(\pi - \angle A)$ and $\angle CIH = \frac{1}{2}(\pi - \angle C)$ since triangles AGF and CIH are isosceles according to the two tangent theorem (see Figure 13). When $ABCD$ is extangential, then the incircle and the considered excircle to triangles ABD and BCD are tangent to BD at the same point X_7 (by Theorem 2.4). Thus $BH = BX_7 = BG$ and $DI = DX_7 = DF$ according to the two tangent theorem. We get $\angle BGH = \frac{1}{2}\angle B$ and $\angle FID = \frac{1}{2}\angle D$ by applying the isosceles triangle theorem and the exterior angle theorem. Then

$$\angle FGH = \pi - \angle AGF + \angle BGH = \frac{\pi}{2} + \frac{\angle A + \angle B}{2}$$

and

$$\angle FIH = \angle FID - \angle HIC = \frac{\angle C + \angle D}{2} - \frac{\pi}{2}.$$

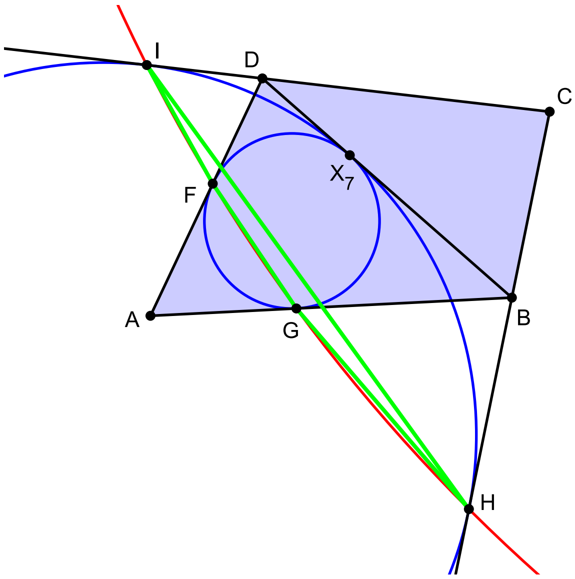

FIGURE 13. $ABCD$ is extangential $\Leftrightarrow$ $FGHI$ is cyclic

Hence

$$\angle FGH + \angle FIH = \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

which proves that $FGHI$ is a cyclic quadrilateral.

($\Leftarrow$) We do a contrapositive proof of the converse. In a convex quadrilaterals $ABCD$ that is not extangential, suppose without loss of generality that the incircle and the considered excircle to triangle ABD are tangent to BD at points X_7 and X_8 respectively such that $BX_7 > BX_8$. Then we have

$$BH = BX_8 < BX_7 = BG, \qquad DF = DX_7 < DX_8 = DI.$$

Thus $\angle BGH < \frac{1}{2}\angle B$ and $\angle FID < \frac{1}{2}\angle D$ since a shorter side is opposite a smaller angle in a triangle. We get

$$\angle FGH = \pi - \angle AGF + \angle BGH < \frac{\pi}{2} + \frac{\angle A + \angle B}{2}$$

and

$$\angle FIH = \angle FID - \angle HIC < \frac{\angle C + \angle D}{2} - \frac{\pi}{2}.$$

Hence

$$\angle FGH + \angle FIH < \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

which proves that $FGHI$ is not a cyclic quadrilateral. $\square$

The following theorem is about the same configuration as in Theorem 2.2.

Theorem 4.3. *Suppose the two excircles belonging to the triangles created by a diagonal that are tangent to two adjacent sides of a convex quadrilateral $ABCD$, the extension of that diagonal, and the extensions of the other two sides of the quadrilateral, are tangent to the sides AB, BC, CD, DA or their extensions at L, M, N, O respectively. Then $LMNO$ is a cyclic* *quadrilateral if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

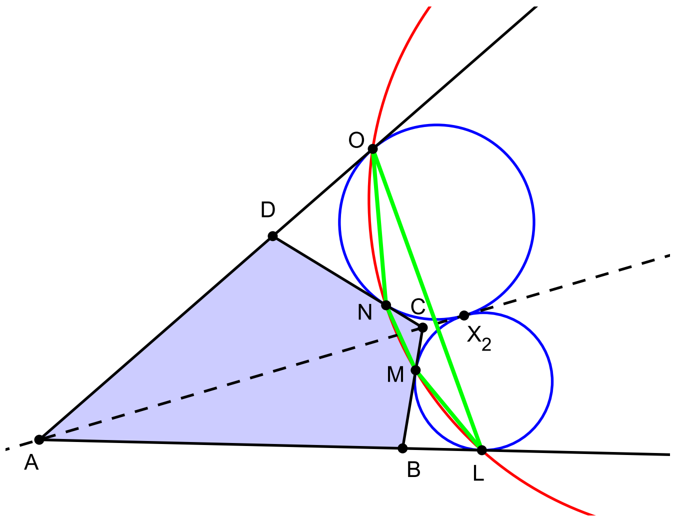

FIGURE 14. $ABCD$ is extangential $\Leftrightarrow$ $LMNO$ is cyclic

Proof. ($\Rightarrow$) In an extangential quadrilateral, we have

$$\angle MLO = \frac{\pi - \angle A}{2} - \frac{\pi - (\pi - \angle B)}{2} = \frac{\pi}{2} - \frac{\angle A + \angle B}{2}$$

(see Figure 14) and

$$\angle MNO = \frac{\pi - \angle C}{2} + \pi - \frac{\pi - (\pi - \angle D)}{2} = \frac{3\pi}{2} - \frac{\angle C + \angle D}{2}.$$

Hence

$$\angle MLO + \angle MNO = \frac{4\pi}{2} - \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

which proves that $LMNO$ is a cyclic quadrilateral.

($\Leftarrow$) When $ABCD$ is not extangential, then using the same method as in the proof of the converse of Theorem 4.2, we either have

$$\angle MLO + \angle MNO > \frac{4\pi}{2} - \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

or

$$\angle MLO + \angle MNO < \frac{4\pi}{2} - \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

depending on the order of the tangency points for the triangle excircles on the extension of AC (which are different points by Theorem 2.2). In either case, $LMNO$ is not a cyclic quadrilateral. $\square$

Next we return to the configuration of Theorem 2.3.

Theorem 4.4. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose an excircle from each of the triangles ACJ and ACK are tangent to the extensions of AB, CD, BC, DA at P, Q, R, S respectively. Then $PQRS$ is a cyclic quadrilateral if and* *only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.*

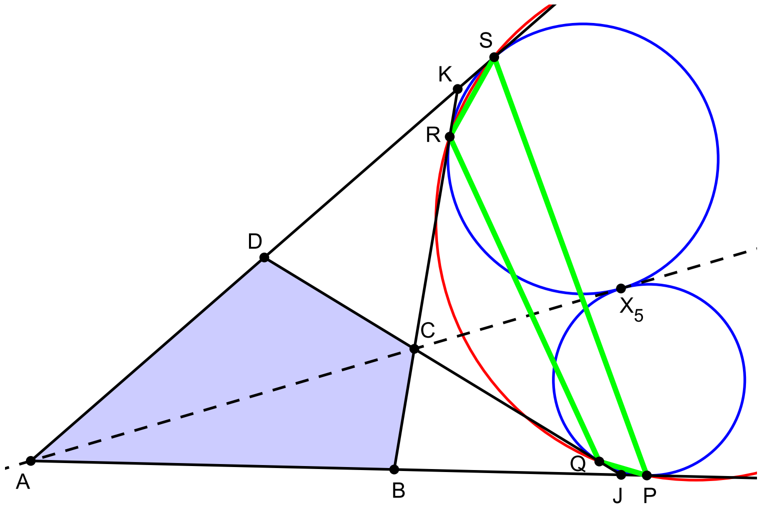

FIGURE 15. $ABCD$ is extangential $\Leftrightarrow$ $PQRS$ is cyclic

Proof. ($\Rightarrow$) In an extangential quadrilateral, we have according to Theorem 2.3 that the two triangle excircles are tangent to AC at the same point, say X_5. Then $AP = AX_5 = AS$, so $\angle APS = \frac{1}{2}(\pi - \angle A)$ since triangle APS is isosceles (see Figure 15). Triangle PJQ is also isosceles according to the two tangent theorem, so by the exterior angle theorem we have $\angle PJQ = \angle A + \angle D$. Then $\angle JPQ = \frac{1}{2}(\pi - \angle A - \angle D)$ and we get

$$\angle QPS = \frac{\pi - \angle A}{2} - \frac{\pi - \angle A - \angle D}{2} = \frac{\angle D}{2}.$$

By similar arguments, we now have $\angle CRQ = \frac{1}{2}(\pi - \angle C)$ and $\angle KRS = \frac{1}{2}(\pi - \angle A - \angle B)$. Thus

$$\angle QRS = \pi - \frac{\pi - \angle A - \angle B}{2} - \frac{\pi - \angle C}{2} = \frac{\angle A + \angle B + \angle C}{2}.$$

Finally

$$\angle QPS + \angle QRS = \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

confirming that $PQRS$ is a cyclic quadrilateral.

($\Leftarrow$) When $ABCD$ is not extangential, assume without loss of generality that $AS = AX_5 < AX_6 = AP$ where X_5 and X_6 are the points where the excircles to triangles ACK and ACJ are tangent to AC. Then $\angle APS < \frac{1}{2}(\pi - \angle A)$ since a shorter side in a triangle is opposite a smaller angle. We still have $\angle PJQ = \angle A + \angle D$ and $\angle JPQ = \frac{1}{2}(\pi - \angle A - \angle D)$, so

$$\angle QPS = \angle APS - \angle JPQ < \tfrac{\angle D}{2}.$$

Since $\angle CRQ > \frac{1}{2}(\pi - \angle C)$ and $\angle KRS = \frac{1}{2}(\pi - \angle A - \angle B)$, we get

$$\angle QRS = \pi - \angle KRS - \angle CRQ < \frac{\angle A + \angle B + \angle C}{2}.$$

Hence

$$\angle QPS + \angle QRS < \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

which proves that $PQRS$ is not cyclic. □

We note that there is another case in Theorem 4.2 when the triangle incircle and excircle change roles, and there is also another case in Theorem 4.3 when the two triangle excircles are tangent to sides DA and AB instead of BC and CD. Even Theorem 4.4 has a second case with excircles tangent to AJ and AK instead of CJ and CK. The six cases in these three theorems yield a total of six circles, and outside an extangential quadrilateral there is its excircle. We leave it as an exercise for the reader to prove that all these seven circles are in fact concentric with center E when the quadrilateral is extangential (and otherwise non of those circles exist). This is illustrated in Figure 16.

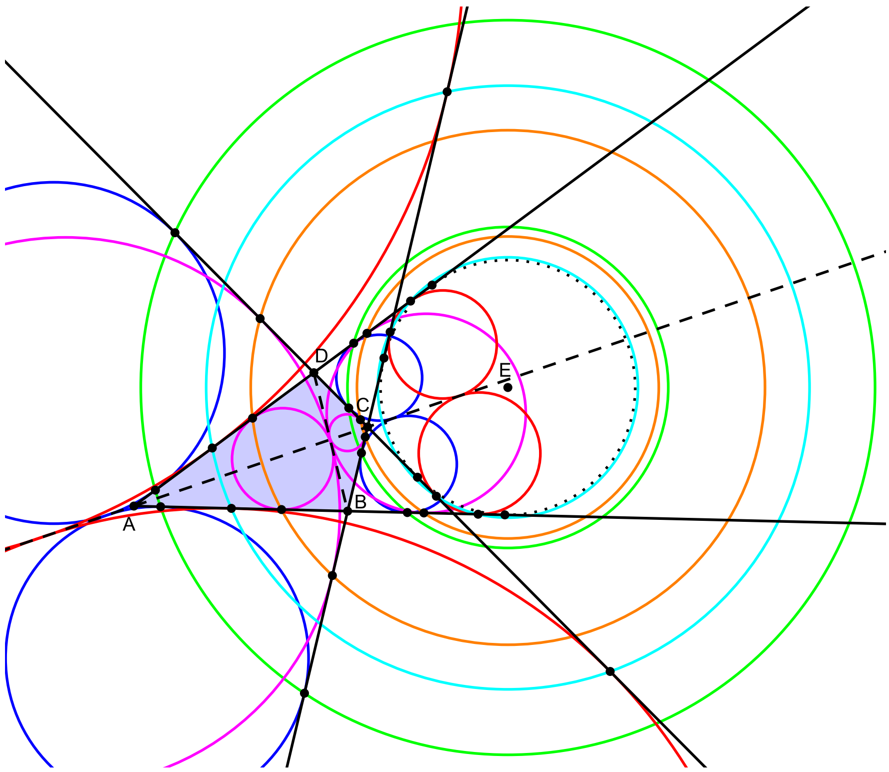

Figure 16. Six circles concentric with the excircle (dotted)

5. Miscellaneous characterizations

Here we prove three characterizations that did not fit into any of the previous sections.

Theorem 5.1. *In a convex quadrilateral $ABCD$, suppose the excircles to triangles ABC and ACD that are tangent to AB, BC, CD, DA are tangent to AB and AD or their extensions at T_1, T_2, T_3, T_4 respectively. Then T_1T_4*

is parallel to T_2T_3 if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.

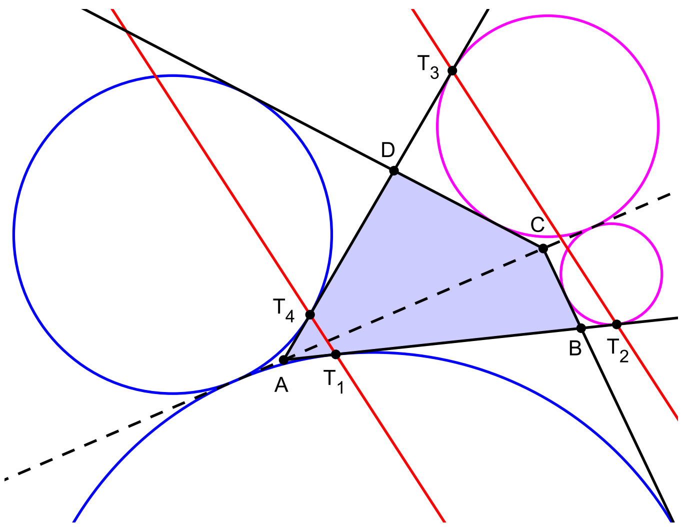

FIGURE 17. $ABCD$ is extangential $\Leftrightarrow$ $T_1T_4 \parallel T_2T_3$

Proof. We have (see Figure 17)

$$AT_1 = \tfrac{1}{2}(AB + BC - AC), \qquad AT_4 = \tfrac{1}{2}(CD + DA - AC)$$

and

$$AT_2 = \tfrac{1}{2}(AB + BC + AC).$$

Then

$$T_1T_2 = AT_2 - AT_1 = \tfrac{1}{2}(AB + BC + AC) - \tfrac{1}{2}(AB + BC - AC) = AC$$

and in the same way $T_4T_3 = AC$. The lines T_1T_4 and T_2T_3 are parallel if and only if

$$\frac{AT_1}{T_1T_2} = \frac{AT_4}{T_4T_3}$$

according to the intercept theorem, which is equivalent to

$$\frac{\frac{1}{2}(AB + BC - AC)}{AC} = \frac{\frac{1}{2}(CD + DA - AC)}{AC}.$$

This in turn is equivalent to $AB + BC = CD + DA$, which is the necessary and sufficient condition (1) for an excircle outside of A or C. □

In [5, p. 133] it was proved that a convex quadrilateral with consecutive sides a, b, c, d and diagonals p, q is tangential (has an incircle) if and only if $ac-bd = pq\cos\theta$ where θ is the angle between the diagonals that is opposite side a. This theorem is attributed to Simionescu. We have the following related condition for the existence of an excircle.

Theorem 5.2. *Let a convex quadrilateral have consecutive sides a, b, c, d and diagonals p, q. Then*

$$bd - ac = pq\cos\theta$$

if and only if it is an extangential quadrilateral, where θ is the angle between the diagonals that is opposite side a.

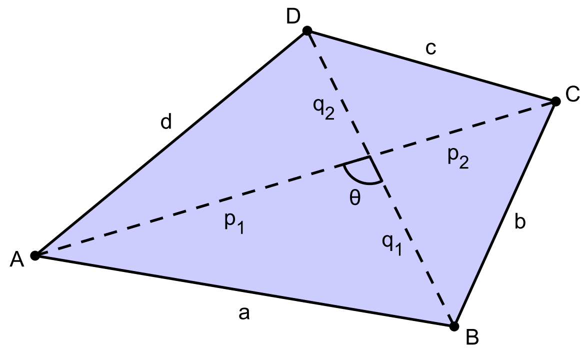

FIGURE 18. $ABCD$ is extangential $\Leftrightarrow$ $bd - ac = pq \cos\theta$

Proof. Suppose the diagonals divide each other in parts with lengths p_1, p_2 and q_1, q_2 (see Figure 18). Applying the law of cosines, we get

$$a^2 = p_1^2 + q_1^2 - 2p_1q_1 \cos\theta, \qquad b^2 = p_2^2 + q_1^2 + 2p_2q_1 \cos\theta,$$
$$c^2 = p_2^2 + q_2^2 - 2p_2q_2 \cos\theta, \qquad d^2 = p_1^2 + q_2^2 + 2p_1q_2 \cos\theta$$

since $\cos(\pi - \theta) = -\cos\theta$. Then

$$\begin{aligned} b^2 + d^2 - a^2 - c^2 &= 2\cos\theta(p_2q_1 + p_1q_2 + p_1q_1 + p_2q_2) \\ &= 2\cos\theta(p_2(q_1 + q_2) + p_1(q_2 + q_1)) \\ &= 2\cos\theta(p_2 + p_1)(q_1 + q_2) \\ &= 2pq\cos\theta \end{aligned}$$

where $p = p_1 + p_2$ and $q = q_1 + q_2$. We can rewrite this equality as

$$pq\cos\theta = \frac{(b-d)^2 - (a-c)^2}{2} + bd - ac.$$

Hence

$$bd - ac = pq\cos\theta \quad \Leftrightarrow \quad (b-d)^2 = (a-c)^2.$$

This equation has the two solutions $b - d = a - c$ and $b - d = -(a - c)$, which are equivalent to $a + d = b + c$ and $a + b = c + d$. These are the characterizations for an excircle outside the biggest of the angles $\{B, D\}$ and $\{A, C\}$ respectively. $\square$

In the last characterization we have two similar triangles.

Theorem 5.3. *In a convex quadrilateral $ABCD$ where the external angle bisectors at B and D intersect at E, let T, U, V, W be the centers of the excircles to triangles ABC and ACD that are tangent to AB, BC, CD, DA respectively. Then*

$$ET \cdot EV = EU \cdot EW$$

if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of A or C.

Proof. ($\Rightarrow$) In an extangential quadrilateral, the excircles tangent to BC, CD and DA, AB are tangent to the extension of AC in pairs according to Theorem 2.2. Thus UV is parallel to TW, and since ET and EW are the external angle bisectors at B and D, we have that triangles EUV and ETW are similar. Then

$$\frac{EU}{ET} = \frac{EV}{EW}$$

and the equality in the theorem follows.

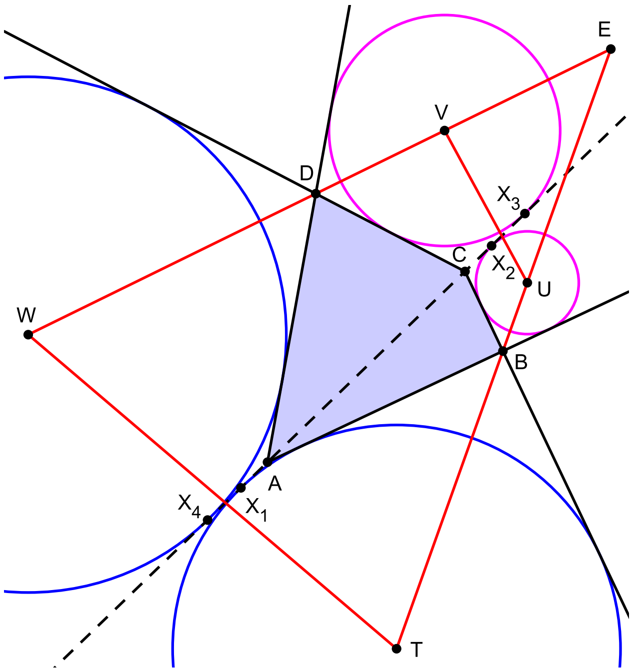

FIGURE 19. $ABCD$ is extangential $\Leftrightarrow$ $ET \cdot EV = EU \cdot EW$

($\Leftarrow$) When $ABCD$ is not extangential, assume without loss of generality that $CD + DA > AB + BC$. If the excircles tangent to DA, AB, BC, CD are tangent to the extension of AC at X_4, X_1, X_2, X_3 respectively (see Figure 19), then

$$AX_4 = \tfrac{1}{2}(DA - AC + CD) = CX_3,$$
$$AX_1 = \tfrac{1}{2}(BC - AC + AB) = CX_2$$

and it follows directly that $AX_4 > AX_1$. This means that angle EVU now is greater than in the proof of the direct theorem, while angle EWT is smaller, so triangles EUV and ETW are no longer directly similar. Then

$$\frac{EU}{ET} \neq \frac{EV}{EW}$$

which completes the proof. $\square$

From the equality in the previous theorem we can derive a nice formula for the exradius ρ of an extangential quadrilateral expressed in terms of the four exradii of the two subtriangles created by a diagonal that are tangent to the four sides of the quadrilateral.

Theorem 5.4. *In an extangential quadrilateral $ABCD$ with an excircle outside of A or C, suppose the excircles to triangles ABC and ACD that are tangent to AB, BC, CD, DA have radii R_a, R_b, R_c, R_d respectively. Then the radius of the excircle to the quadrilateral is given by*

$$\rho = \left| \frac{R_a R_c - R_b R_d}{R_a + R_b - R_c - R_d} \right|.$$

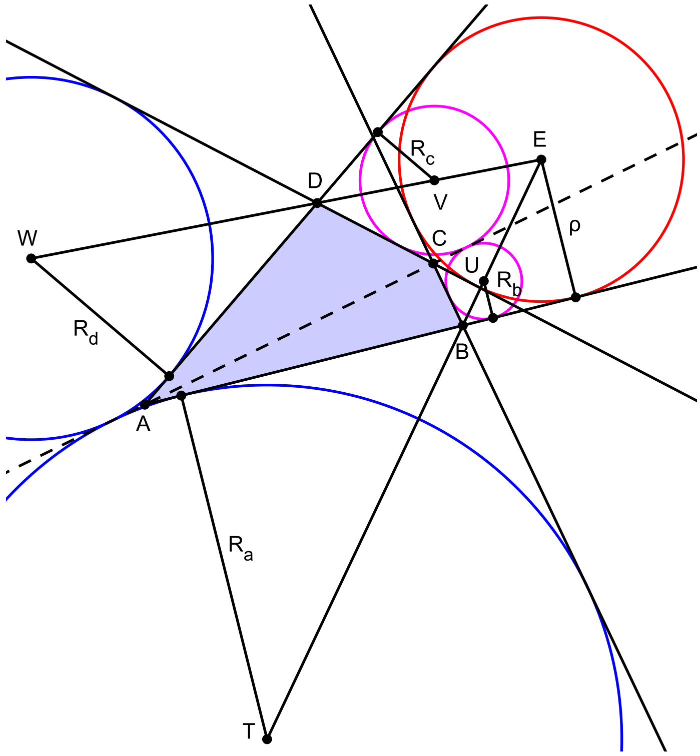

FIGURE 20. The five exradii

Proof. First we consider the case with an excircle outside of C. We have (see Figure 20) that

$$\sin \frac{\pi - B}{2} = \frac{\rho}{EB} \quad \Rightarrow \quad EB = \frac{\rho}{\cos \frac{B}{2}}.$$

Then

$$ET = EB + BT = \frac{\rho}{\cos \frac{B}{2}} + \frac{R_a}{\cos \frac{B}{2}} = \frac{\rho + R_a}{\cos \frac{B}{2}}$$

and

$$EU = \frac{\rho - R_b}{\cos \frac{B}{2}}.$$

In the same way

$$EW = \frac{\rho + R_d}{\cos \frac{D}{2}}, \qquad EV = \frac{\rho - R_c}{\cos \frac{D}{2}}.$$

Inserting these four expressions into $ET \cdot EV = EU \cdot EW$ yields

$$\frac{(\rho + R_a)}{\cos\frac{B}{2}} \cdot \frac{(\rho - R_c)}{\cos\frac{D}{2}} = \frac{(\rho - R_b)}{\cos\frac{B}{2}} \cdot \frac{(\rho + R_d)}{\cos\frac{D}{2}}$$

which we simplify into

$$\rho(R_a - R_c - R_d + R_b) = R_aR_c - R_bR_d.$$

Hence

$$\rho = \frac{R_aR_c - R_bR_d}{R_a + R_b - R_c - R_d}.$$

There are situations when both the numerator and the denominator are negative (when R_d is big in comparison to R_a), but the quotient still gives a positive radius.

In the case with an excircle outside of A, the only difference is that all the terms in the denominator change signs. In order to get a formula that always work for both cases with an excircle outside A or C, we put an absolute value around the entire quotient. □

The formula does not work in all extangential quadrilaterals. For example in a kite, it gives the undefined expression 0/0 for the exradius.

6. Concluding remarks

This paper together with [1, 2, 4] review or prove a total of 59 characterizations of extangential quadrilaterals. When having such a large collection, the reader might get the impression that all properties (necessary conditions) of these quadrilaterals are also sufficient conditions for their existence. But that is not true. Let us conclude by studying one example where the converse is not true since it also includes the possibility of another class of quadrilaterals.

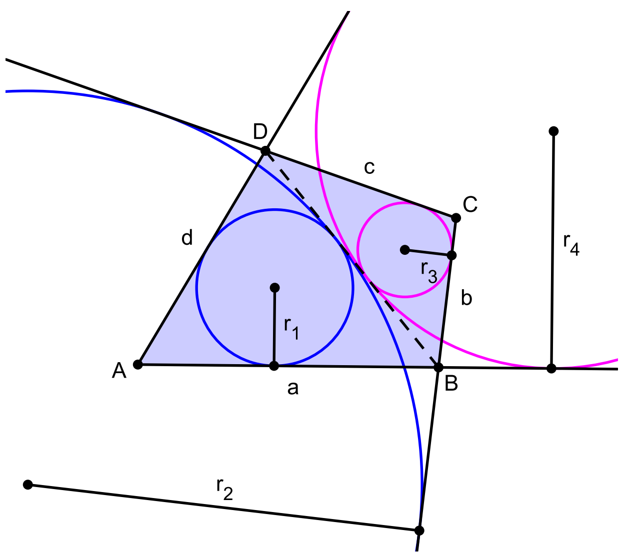

Figure 21. The four subtriangle radii

In a convex quadrilateral $ABCD$ with sides $AB = a$, $BC = b$, $CD = c$, and $DA = d$, we denote the inradii of triangles ABD and BCD by r_1 and r_3, and the radii of their excircles that are tangent to BD by r_2 and r_4 respectively (see Figure 21). It is well-known that the radius of the incircle r and the radius of the excircle r_z that is tangent to side z of a triangle with sides x, y, z are given by the formulas

$$r = \frac{1}{2}\sqrt{\frac{(-x+y+z)(x-y+z)(x+y-z)}{x+y+z}}$$

and

$$r_z = \frac{1}{2}\sqrt{\frac{(x+y+z)(-x+y+z)(x-y+z)}{x+y-z}}.$$

Then $4r_1r_2$ is equal to

$$\sqrt{\frac{(-a+q+d)(a-q+d)(a+q-d)}{(a+q+d)} \cdot \frac{(a+q+d)(-a+q+d)(a+q-d)}{(a-q+d)}}$$

where $BD = q$, which is simplified into

$$4r_1r_2 = (-a+d+q)(a-d+q).$$

In the same way it holds that

$$4r_3r_4 = (-b+c+q)(b-c+q).$$

Using basic algebra, we get

$$4(r_1r_2 - r_3r_4) = (b-c+a-d)(b-c-a+d).$$

Hence $r_1r_2 = r_3r_4$ is equivalent to either $a + b = c + d$ (an extangential quadrilateral) or $a + c = b + d$ (a tangential quadrilateral), so this radii equality is a necessary but *not* a sufficient condition for the existence of an excircle to $ABCD$.

There are at least 100 published characterizations of tangential quadrilaterals, as listed at the end of our recent paper [3], the third part of an extensive study of new characterizations of quadrilaterals that can have an incircle. Since tangential and extangential quadrilaterals have very similar properties, we make the prediction that there are dozens of yet unpublished characterizations of the latter of these classes of quadrilaterals awaiting to be discovered and proved.

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