PENCILS OF LINES

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Abstract. All the points, lines and curves encountered here are presumed to lie in a plane. A set of lines is called a pencil if any member of the set is a linear combination of two other members of the set. The lines are necessarily concurrent. Likewise, a set of conics (or cubics) form a pencil if each is expressible as a linear combination of two other members of the set. Each member of a pencil of conics will pass through four fixed points and for cubics there will be nine such points. These are referred to as the base points of the pencil.

Here is presented a test for a general curve to be a pencil of lines and it is used to provide simplified proofs of standard results and also some new ones, particularly for the harmonic envelope of two conics and the associated problem for cubic curves.

1. Introduction.

Let $S$ be a homogeneous polynomial of degree $n$ in $\mathbf{r} = (r_1, r_2, r_3)$: The plane curve represented by $S(\mathbf{r}) = 0$, where now $(r_1, r_2, r_3)$ are taken to be homogeneous coordinates, is called a curve of the $n$th degree or $n$-curve. The symbol $S$ will be used for both the curve and its polynomial representation. Various names are used for homogeneous coordinates, e.g. areal, barycentric and trilinear, but they all depend upon choosing three points as the triangle of reference which have coordinates $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. As with Cartesian axes, these points may be chosen in the most convenient way for the problem at hand. Fuller explanations and motivation for the use of such coordinates (rather than Cartesian coordinates) are to be found in [8] and the classic texts of [4], [5] and [7]. The website [3] provides a wealth of information on many specific cubic curves.

If $X(\alpha)$ is any point, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then the polar curve of $X$ in the $n$-curve $S$ is the $(n - 1)$-curve $S_X$ defined by

\[
S_X(\mathbf{r}) = \sum_{i=1}^{n} \alpha_i S(\mathbf{r})
\]

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(1) \[ nS_X = \alpha_1 \frac{\partial S}{\partial r_1} + \alpha_2 \frac{\partial S}{\partial r_2} + \alpha_3 \frac{\partial S}{\partial r_3} \equiv \alpha \cdot \nabla S = 0. \]

If \( Y(\beta) \) is any other point then it is readily seen that
\[ (S_X)_Y \equiv (S_Y)_X \]
and so either may be written as \( S_{XY} \) or \( S_{YX} \). In particular, if \( S \) is a conic (i.e. \( n = 2 \)) then the polar curve of \( X \) (simply referred to as the polar of \( X \)) in \( S \) is a line and (2) gives the classical result that if the polar of \( X \) passes through \( Y \) then the polar of \( Y \) passes through \( X \).

A simple result which will be much used in what follows is that when \( U \) is an \( m \)-curve and \( V \) an \( n \)-curve then
\[ (m + n)(UV)_X \equiv mU_XV + nV_XU \Rightarrow (U^k)_X \equiv U^{k-1}U_X \]
for any positive integer \( k \). The next result is also quite elementary but is nevertheless crucial to this investigation.

**Theorem 1.1.** If there is a point \( X(\alpha) \) for which
\[ S_X \equiv 0 \]
for the \( n \)-curve \( S \) then \( S \) consists of \( n \) lines with common point \( X \).

**Proof.** Choose the triangle of reference so that the coordinates of \( X \) are \((1, 0, 0)\). Then (4) implies that \( \partial S/\partial r_1 \) is zero for all \( r \) and so \( S \) is a homogeneous polynomial of degree \( n \) in \( r_2 \) and \( r_3 \). Thus \( S \) may be factorised into \( n \) linear terms and so consists of \( n \) lines through \((1, 0, 0)\) which is \( X \).

This article demonstrates how this theorem may be used in the context of conics \((n = 2)\) and cubics \((n = 3)\) to deduce a variety of results. Some of these are well-established (even elementary) but others are believed to be novel.

2. Preliminary Examples.

2.1. Conics and Cubics as Sets of Lines. As an illustration of how Theorem 1 may be used, consider the (familiar) problem of determining when a conic is a pair of lines. When \( n \) has the value 2, \( S \) can be written as
\[ S \equiv r^T Sr = 0 \Rightarrow S_X \equiv r^T S\alpha = 0 \]
where \( S \) is a symmetric 3×3 matrix. If this last equation is in fact an identity (i.e. valid for all \( r \)) then the equation \( S\alpha = 0 \) has to have a non-trivial solution for \( \alpha \) (the coordinates of \( X \)) and so the determinant of \( S \) is zero.

When \( n = 3 \), \( S \) can be written as
\[ S \equiv \sum_{i,j,k} s_{ijk} r_i r_j r_k = 0 \Rightarrow S_X \equiv \sum_{i,j,k} s_{ijk} \alpha_i r_j r_k = 0 \]
where the summations are all from 1 to 3 and the value of \( s_{ijk} \) is unaltered by any permutation of the suffices. If this last equation is an identity then the equations
\[ \sum_i s_{ijk} \alpha_i = 0 \quad (1 \leq j \leq 3, 1 \leq k \leq 3) \]
all have to possess the same solution for \( \alpha \). This condition may be written as
\[
|s_{1ij} + \mu s_{2ij} + \nu s_{3ij}| = 0 \quad \forall \mu, \nu
\]
but this is not especially easy to check for any given \( S \).

### 2.2. Simple Cases.
- If \( L \) and \( M \) are fixed lines and \( Q \) is defined by
  \[
  Q \equiv LM_X - ML_X = 0
  \]
  then of course \( Q_X \) is identically zero and indeed \( Q \) is the line joining \( X \) to the common point of \( L \) and \( M \). If \( X \) is varied then the envelope of the lines (5) is just this common point. This observation is a special case of a general result encountered later.
- Let \( S \) be a 2-curve (i.e. conic), \( X(\alpha) \) any point and define \( Q \) by
  \[
  Q \equiv SS_{XX} - (S_X)^2 = 0.
  \]
  It follows from (3) that \( Q_X \) is identically zero and it is well known that \( Q \) gives the pair of tangents from \( X \) to \( S \).
  If \( Q \) is considered to be a function of \( r \) and \( \alpha \) then
  \[
  Q(\alpha, r) \equiv Q(r, \alpha)
  \]
  and
  \[
  Q(r, \alpha) \equiv q(R) \quad \text{where} \quad R = r \times \alpha.
  \]
  It is convenient to use the vector product notation but of course \( r \) and \( \alpha \) are the the coordinates of points and not vectors. If \( S \) is the symmetric matrix associated with the conic \( S \) then
  \[
  S = r^T S r \quad \text{and} \quad q(R) = R^T \text{adj}(S) R.
  \]
- If \( X \) is a common point of the conics \( U \) and \( V \) then
  \[
  UV_X = VU_X
  \]
gives the three lines joining \( X \) to the other common points of the conics.

### 3. Lines from Conics.

#### 3.1. Two Lines from Two Conics.
With \( U \) and \( V \) any conics, the conic
\[
(7) \quad Q \equiv UXV - 2UXV_X + UVX_X = 0
\]
is a line-pair for any point \( X(\alpha) \). Let \( Y(\beta) \) be any point (different from \( X \)) on one of these lines. Then the coordinates of points on this line may be parametrized by \( t \) as
\[
r = \alpha + t\beta.
\]
Let the points where this line meets \( U \) be \( A_1 \) and \( A_2 \) where \( t = a_i \) at \( A_i \) so that
\[
(8) \quad UX + 2tUXY + t^2UYY = 0
\]
has roots \( a_1, a_2 \). Likewise, this same line meets \( V \) at \( B_1, B_2 \) when \( t = b_1, b_2 \), the roots of
\[
(9) \quad VX + 2tVXY + t^2VYY = 0.
\]
Since $Y$ lies on (7) we also have
\begin{equation}
U_{XX}V_{YY} - 2U_{XY}V_{XY} + U_{YY}V_{XX} = 0.
\end{equation}

The equations (8), (9) and (10) imply that
\begin{equation}
2a_1a_2 - (a_1 + a_2)(b_1 + b_2) + 2b_1b_2 = 0
\end{equation}
which can be written as
\begin{equation}
(a_1 - b_1)(a_2 - b_2) + (a_1 - b_2)(a_2 - b_1) = 0.
\end{equation}

This shows that the pair of points $B_1, B_2$ divide the pair $A_1, A_2$ harmonically.

If the four points $A_1, A_2, B_1, B_2$ are considered as an unordered set of points
and their $t$ values are $t_1, t_2, t_3, t_4$ then the value of
\begin{equation}
\frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_4)(t_2 - t_3)}
\end{equation}
is $-1, 1/2$ or $2$.

It is now seen that (7) gives the family of lines which meet $U$ and $V$ at
points which form an harmonic range. This family has an envelope, the harmonic envelope of $U$ and $V$. The equation of this envelope and related
information may be found in [9], [4] or [7]. The approaches of these authors
are surprisingly different and none commence with (7). In Appendix B.1 a
proof that an envelope exists for a more general problem is given. This is
appropriate because the technique used also applies to cubic curves which
are considered in Section 4.

There is one case which is relatively simple and has echoes later. When
the conics $U$ and $V$ have four distinct, common points and the diagonal-
point triangle of the quadrangle formed by these common points is used as
the triangle of reference, their equations take the form
\begin{align}
U & \equiv a_1r_1^2 + b_1r_2^2 + c_1r_3^2 = 0, \\
v & \equiv a_2r_1^2 + b_2r_2^2 + c_2r_3^2 = 0
\end{align}
and their harmonic envelope is
\begin{align}
\frac{r_1^2}{b_1c_2 + b_2c_1} + \frac{r_2^2}{c_1a_2 + c_2a_1} + \frac{r_3^2}{a_1b_2 + a_2b_1} = 0.
\end{align}

A different way of expressing the result concerning the lines (7) is as
follows;
the pair of lines given by (7) are the tangents from $X$ to a
conic, called the harmonic envelope of $U$ and $V$. Furthermore, these tangents meet $U$ and $V$ in a harmonic range.

The dual of this result is
the locus of the point $X$ for which the four tangents from $X$
to two conics $U$ and $V$ form a harmonic pencil is a conic,
called the harmonic locus.

This latter conic is ascribed to Salmon in [11]. When $U$ and $V$ are as given
in (14) and (15) Salmon’s conic is
\begin{align}
a_1a_2(b_1c_2 + b_2c_1)r_1^2 + b_1b_2(c_1a_2 + c_2a_1)r_2^2 + c_1c_2(a_1b_2 + a_2b_1)r_3^2 = 0.
\end{align}

It is worthy of note that if $X$ is chosen to be a common point of $U$ and $V$ then the following result is obtained:
The tangents to two conics are drawn at their four common points. These eight lines are tangents to a third conic.

The dual to this is

the four common tangents to two conics define eight points of contact. There exists a conic which passes through these eight points.

Comment. It is suggested that these results deserve to be better known. In [4] this entire topic is relegated to a chapter entitled ‘Miscellaneous Properties’. [9] dismisses them as ‘simple properties’ and [7] is much more concerned with invariants. Perhaps unsurprisingly, [6] does give a more considered account but this classic text (probably the most comprehensive one on conics) is perhaps unlikely to appeal to current students.

3.2. The Conic \( V \) is a Repeated Line. When \( L \) is a line and the conic \( V \) is \( L^2 \), the line-pair given by (7) becomes

\[
U(L_X)^2 + L(U_{XX}L - 2U_XL_X) = 0.
\]

Common points of the line \( L \) and conic \( U \) certainly satisfy this equation. Thus if the line \( L \) meets the conic \( U \) at the points \( A \) and \( B \) then (16) gives the lines \(XA \) and \( XB \). Furthermore, let the line \(XA \) meet \( U \) again at \( C \) and the line \( XB \) meet \( U \) again at \( D \). Then at \( C \) and \( D \), \( U \) is zero but \( L \) is not zero and so, since \( C \) and \( D \) lie on (16),

\[
U_{XX}L - 2U_XL_X = 0
\]

and this is the equation of the line \( CD \). This shows that the lines \( AB, CD \) and \( U_X \) are concurrent. The point \( X \) is a diagonal point of the quadrangle \( ABCD \) and its polar curve \( U_X \) passes through the common point of \( AB \) and \( CD \). Since this common point is also a diagonal point of \( ABCD \) the following result has been established:

The diagonal-point triangle of a quadrangle is a self-polar triangle for any conic circumscribing the quadrangle.

This is a standard result in elementary projective geometry but it is hoped that the method of proof exhibited here is of interest.

3.3. Four Lines from Two Conics. It appears that constructing three lines from two conics only gives degenerate solutions and so the next interesting possibility is to have four lines. For any conics \( U \) and \( V \),

\[
Q \equiv (UV_{XX} - VU_{XX})^2 - 4(UV_X - VU_X)(U_XV_{XX} - V_XU_{XX}) = 0
\]

gives the four lines joining \( X \) to the common points of \( U \) and \( V \). Setting \( V = L^2 \) gives the lines of (16) twice.


4.1. Pencils of Cubics. With \( U \) and \( V \) any cubic curves and \( X(\alpha) \) any point, the cubic \( Q \) is now defined by

\[
Q(\alpha, r) \equiv U_{XXX}V - 3U_{XX}V_X + 3U_XV_{XX} - UV_{XXX} = 0.
\]

It is easily confirmed that

\[
Q_X \equiv 0 \quad \text{and} \quad Q(\alpha, r) \equiv -Q(r, \alpha)
\]
and so \( Q \) consists of a set of three lines through \( X \).

If \( U \) and \( V \) are members of a pencil of cubics (so that they can be expressed as a linear combination of two pre-selected cubics) then clearly \( Q \) depends only upon \( X \) and the nine common points (the base points) of the pencil, which is quite unlike the situation for (7). Since knowledge of any eight of these common points determines the ninth, \( Q \) in fact only depends upon eight points (and \( X \)). However, this set of points defines a plethora of other points.

For any position of \( X \), let \( L \) be one of the lines of \( Q \) and choose any point \( Y(\beta) \) different from \( X \) on \( L \). The coordinates of a point on \( L \) may be expressed as \( (\alpha + t\beta) \) and the value of \( U \) at this point is

\[
U(\alpha + t\beta) = U_{XXX} + 3tU_{XXY} + 3t^2U_{XYY} + t^3U_{YYY} = f(t). \tag{19}
\]

If the roots of \( f(t) \) are \( a_1, a_2, a_3 \) then the points \( (\alpha + a_i\beta) \) are where \( L \) meets \( U \). These points are given the labels \( A_1, A_2, A_3 \). Similarly, the points \( B_1, B_2, B_3 \) are where \( L \) meets \( V \) and they will have coordinates \( (\alpha + b_i\beta) \) where the \( b_i \) are the roots of

\[
V(\alpha + t\beta) = V_{XXX} + 3tV_{XXY} + 3t^2V_{XYY} + t^3V_{YYY} = 0. \tag{20}
\]

Since \( Y \) lies on the line \( L \) which is a solution to \( Q \), it follows that

\[
U_{XXX}V_{YYY} - 3U_{XXY}V_{XYY} + 3U_{XYY}V_{XXX} - U_{YYY}V_{XXX} = 0 \tag{21}
\]

and now equations (19) and (20) imply that

\[
3a_1a_2a_3 - (a_2a_3 + a_3a_1 + a_1a_2)(b_1 + b_2 + b_3)
+ (a_1 + a_2 + a_3)(b_2b_3 + b_3b_1 + b_1b_2) - 3b_1b_2b_3 = 0. \tag{22}
\]

Perhaps triharmonic is a reasonable name for two sets of three points on a line satisfying this relationship. If each of the \( a_i \) and \( b_i \) is replaced by \((a_i + d)\) and \((b_i + d)\) then this last equation is unchanged. In fact (22) may be written as

\[
\sum (a_1 - b_{\pi(1)})(a_2 - b_{\pi(2)})(a_3 - b_{\pi(3)}) = 0 \tag{23}
\]

where the summation is over all permutations of \( \{1, 2, 3\} \). The results (22) and (23) may be compared with (11) and (12).

Now let \( M \) be a fixed line and suppose that the points at which \( M \) meets \( U \) and \( V \) satisfy this triharmonic relationship. Then this relationship will remain true for any two members of the pencil.

For any pencil of cubics and point \( X(\alpha) \), equation (17) gives three lines. In the case of two conics, the analogous procedure produces a pair of lines which have an envelope (the harmonic envelope). The proof that the three lines do indeed have an envelope is given in Appendix B.2 for a slightly more general situation. The envelope \( E \) has degree six in \( r \). This envelope will depend only upon the base points of the pencil defined by \( U \) and \( V \) (that is, their nine common points). By contrast, in the case of two conics the envelope depended very much upon the precise choice of the two conics. In general the formula for the sextic curve \( E \) is very lengthy. However, two cases were identified in which the algebra is reasonable.
Figure 1. The three lines of $Q$ for a typical point $X$ and the envelope $E$ when $U$ and $V$ are given by (24).

- Let $U$ and $V$ have a common self-polar triangle so that it is possible to write
  
  $\begin{align*}
  U &\equiv f_1 r_1^3 + g_1 r_2^3 + h_1 r_3^3 + 6k_1 r_1 r_2 r_3 = 0, \\
  V &\equiv f_2 r_1^3 + g_2 r_2^3 + h_2 r_3^3 + 6k_2 r_1 r_2 r_3 = 0.
  \end{align*}$

  In this case,$\,$

  $Q \equiv FR_1^3 + GR_2^3 + HR_3^3 = 0 \text{ where } (R_1, R_2, R_3) = \mathbf{R} = \mathbf{r} \times \alpha$

  and the envelope is

  $E \equiv \pm \sqrt{\frac{r_1^3}{F}} \pm \sqrt{\frac{r_2^3}{G}} \pm \sqrt{\frac{r_3^3}{H}} = 0$

  where

  $F = g_1 h_2 - g_2 h_1, \quad G = h_1 f_2 - h_2 f_1 \text{ and } H = f_1 g_2 - f_2 g_1.$

- Each of $U$ and $V$ consists of three lines; specifically let

  \begin{align*}
  (24) \quad U &\equiv (r_2 - r_3)(r_3 - r_1)(r_1 - r_2) = 0, \\
  V &\equiv 2r_1 r_2 r_3 = 0.
  \end{align*}

Then

$Q \equiv R_1^2 (R_2 + R_3) + R_2^2 (R_3 + R_1) + R_3^2 (R_1 + R_2) = 0 \quad (\mathbf{R} = \mathbf{r} \times \alpha)$

and

$E \equiv 36s_3^2 + 20s_1^3 s_3 - 72s_1 s_2 s_3 - 6s_2^3 + 42s_1^2 s_2^2 - 22s_1^4 s_2 + 3s_1^6 = 0, \quad s_k = r_1^k + r_2^k + r_3^k.$

Figure 1 shows a typical configuration.
There is a particular kind of cubic pencil which needs separate mention because the set of three lines under consideration evaporate and this case is discussed in the next section. This will presume knowledge of the Hessian of a curve and a short account of its principal properties is included in Appendix A. This also shows how Theorem 1 may be used to prove one of these properties for a cubic curve.

4.2. Hesse Pencils. Let $U$ be a non-degenerate cubic and $H$ its Hessian (which is also a cubic curve) and consider the pencil formed by $U$ and $H$. Such a pencil is termed a Hesse pencil, see [2] for a full account. Direct computation reveals that (17) now gives an identity, i.e. $Q$ is zero for any choice of $X$. Thus

\[ \text{if } V \text{ is a linear combination of } U \text{ and the Hessian of } U \text{ then} \]
\[ Q \equiv 0 \text{ for all } X. \]

The following is a partial converse. Let
\[ U \equiv f r_1^3 + g r_2^3 + h r_3^3 + 6 k r_1 r_2 r_3 = 0 \quad (k \neq 0) \]
and let $V$ have its full form (i.e with ten terms) then it is only necessary to require $Q \equiv 0$ when $X$ is $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ to deduce that
\[ V \equiv \lambda U + \mu r_1 r_2 r_3 \]
for some constants $\lambda$ and $\mu$, so that $V$ is a linear combination of $U$ and its Hessian. Even with $k = 0$ the conclusion remains valid provided that $fgh \neq 0$. However, there are some degenerate situations in which $Q \equiv 0$ even when no Hessians are involved; for example
\[ U = r_1^3, \quad V = r_1^2 r_2 \Rightarrow Q \equiv 0 \forall \alpha. \]

Clearly if $V$ is any member of a Hesse pencil then the Hessian of $V$ also belongs to this pencil. The analysis of the previous section now establishes the following result.

\[ \text{A line meets two members of a Hesse pencil at } A_1, A_2, A_3 \text{ and } B_1, B_2, B_3. \text{ Then these six points will have a triharmonic relationship.} \]

As a special case, consider a line through one of the base points $O$ of the Hesse pencil. Choose $A_3$ and $B_3$ to be $O$ and measure the distances $a_i$ and $b_i$ from $O$. Then (22) gives
\[ \frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{b_1} + \frac{1}{b_2}. \]

Expressed slightly differently, this gives the following.

\[ \text{A fixed line through a base point } O \text{ of a Hesse pencil meets a member of the pencil at two other points } A_1, A_2. \text{ If the point } Y \text{ on this line is such that } O, Y \text{ divide } A_1, A_2 \text{ harmonically then } Y \text{ is independent of the cubic chosen.} \]

As a further development, it is next shown that if the line is varied in direction (it must still pass through $O$) then the locus of $Y$ is a straight line.

It is convenient to change the notation for $U$ slightly so that
\[ U \equiv (f r_1)^3 + (g r_2)^3 + (h r_3)^3 + 6 \lambda r_1 r_2 r_3 = 0, \]
(with parameter $\lambda$) is a Hesse pencil. The base point may be chosen to be $O(0, 1/g, -1/h)$ so that points on any line through $O$ can be written as
\[ r = (0, 1/g, -1/h) + t(1, l, -1 - l) \]
(the direction of the line is specified by $l$ and $t$ determines the position). This line meets $U$ at $t = 0, t_1, t_2$ and $Y$ is where $t = 2t_1t_2/(t_1 + t_2)$. This gives the relationship $gy = hz$ for the coordinates of $Y$. Thus the locus of $Y$ is the straight line through the points 
\[ (1, 0, 0), (1/f, 1/g, 1/h), (w/f, 1/g, 1/h), (w^2/f, 1/g, 1/h) \]
which, with $w$ a complex cube root of unity, are a set of four points, one on each of the four self-polar triangles of any member of the pencil.

4.3. The General Case. With $U$ and $V$ quartics, it is easy to verify that
\[ Q \equiv U_{XXXX}V - 4U_{XXXY}V_X + 6U_{XXV}V_{XX} - 4U_XV_{XXX} + UV_{XXX} = 0 \]
gives a set of four lines through $X(\alpha)$. This notation is clearly unsuited to high-order curves and so let
\[ U^{(0)} \equiv U, \ U^{(1)} \equiv U_X, \ldots, U^{(n)} \equiv U_{XX...X} \]
with similar notation for $V$. Then, for curves of degree $n$,
\[ Q \equiv \sum_{i=0}^{n} (-1)^i \binom{n}{i} U^{(n-i)}V^{(i)} = 0 \]
will be a set of $n$ lines. As before, let $Y(\beta)$ be a fixed point on one of the lines and then any point on it will have coordinates $(\alpha + t\beta)$. If this line meets $U$ ($V$) at the points $A_i$ ($B_i$) and the value of $t$ is $a_i$ ($b_i$) where $1 \leq i \leq n$ then the equations for $a_i$ and $b_i$ are
\[ \sum_{i=0}^{n} \binom{n}{i} t^i U^{(n-i,i)} = 0 \text{ and } \sum_{i=0}^{n} \binom{n}{i} t^i V^{(n-i,i)} = 0 \]
where the suffix $(n-i, i)$ indicates a string of $(n-i)$ $X$’s followed by a string of $i$ $Y$’s.

Introducing $S_j$ for the sums of the products of the $a_i$ taken $j$ at a time (and likewise $T_j$ for the $b_i$) it is seen that
\[ S_j = (-1)^j \binom{n}{j} \frac{U^{(j,n-j)}}{U^{(0,n)}} \text{ and } T_j = (-1)^j \binom{n}{j} \frac{V^{(j,n-j)}}{V^{(0,n)}}. \]
Finally, substituting into (25) gives
\[ \sum_{i=0}^{n} (-1)^i S^{n-i}T_i \binom{n}{i} = 0 \]
as the general form of equation (22) and constitutes a generalization of the notion of harmonic range. Likewise (23) becomes
\[ \sum_{\pi} \prod_{i=1}^{n} (a_i - b_{\pi(i)}) = 0 \Rightarrow \sum_{\pi} \prod_{i=1}^{n} A_iB_{\pi(i)} = 0 \]
where the summations are now over all permutations of $\{1, 2, \ldots, n\}$. If $n$ is even then $U$ and $V$ may be set equal so as to obtain $n$ lines through every point for a given $n$-curve. When $n = 2$ these are just the pair of tangents to
the conic. But when \( n = 4 \) the four values for \( a_i \) are real only in degenerate cases.

5. Tangents to a Cubic Curve.

With \( U \) any cubic curve and \( X(\alpha) \) any point, the results (3) and (4) may be used to show that the 6-curve

\[
Q \equiv U^2(U_{XXX})^2 - 6UU_XU_{XX}U_{XXX} + 4(U_X)^3U_{XX} + 4U(U_{XX})^3 - 3(U_XU_{XX})^2 = 0
\]

represents six straight line through \( X \). Clearly points at which \( U = 0 = U_X \) lie on \( Q \) and it follows that \( Q \) is the set of six tangents from \( X \) to \( U \).

The tangent at any point \( A \) on a cubic curve will meet the cubic at one other point, the tangent point of \( A \), denoted by \( A^* \). With \( A \) as the point of contact of one of the tangents from \( X \) to \( U \), then at \( A^*, U = 0 \neq U_X \) and (27) shows that \( A^* \) lies on the conic

\[
4U_XU_{XXX} - 3(U_{XX})^2 = 0.
\]

The form of this equation implies that this conic makes double contact with \( U_X \), the common chord being \( U_{XXX} \). This conic is mentioned in Section 6. The fact that these six tangent points lie on a conic is a particular case of a result in [10] viz. if six points on a cubic also lie on a conic then their tangent points lie on a conic.

When \( X \) lies on \( U \), there are only four tangents from \( X \) to \( U \) (disregarding the tangent at \( X \)). Setting \( U_{XXX} = 0 \) in (27) gives

\[
4UU_X - 3(U_X)^2 = 0
\]

as the 4-curve representing these four tangents. From this, an algebraic proof of Salmon’s theorem (that the cross-ratio of these four tangents is independent of \( X \)) may be obtained. Only a brief outline is given here, more details are given in [5] (the first edition is a little more explicit) or [9].

With \( X(\alpha) \) as a point on \( U \), the substitution

\[
r_1 = \alpha_1 + t, \ r_2 = \alpha_2 + t\mu, \ r_3 = \alpha_3 - t(1 + \mu)
\]

into (28) gives a quartic in \( \mu \) which is written as

\[
A\mu^4 + 4B\mu^3 + 6C\mu^2 + 4D\mu + E = 0.
\]

The value of the cross-ratio of the pencil of tangents is given in terms of the roots of this equation by

\[
\lambda = \frac{(\mu_1 - \mu_3)(\mu_2 - \mu_4)}{(\mu_1 - \mu_4)(\mu_2 - \mu_3)}
\]
c.f. (13). The following prescription will allow the calculation of $\lambda$. First find $S$ and $T$ by
\[
12S = A\varepsilon - 4BD + 3C^2, \\
T = ACE + 2BCD - AD^2 - EB^2 - C^3
\]
then $\Lambda$ from
\[
\frac{T^2}{S^3} = \frac{16(4\Lambda + 1)(\Lambda - 2)^2}{(\Lambda + 1)^3}
\]
and finally $\lambda$ from
\[
\lambda = \Lambda(\lambda - 1).
\]
There are six related values for $\lambda$, viz. $\lambda, 1/\lambda, 1 - \lambda, 1/(1 - \lambda), \lambda/(\lambda - 1), (\lambda - 1)/\lambda$ and three for $\Lambda$ which are related by $1/\Lambda_1 + 1/\Lambda_2 + 1/\Lambda_3 = -3$ and $\Lambda_1\Lambda_2\Lambda_3 = -1$. The condition for a harmonic pencil is simply $T = 0$ and $\lambda$ is real when $(4S)^3 \geq T^2$.

Taking $U$ as
\[
U \equiv r_1^3 + r_2^3 + r_3^3 + 6kr_1r_2r_3 = 0,
\]
the above results may be used to show that the six values of $\lambda$ are independent of the position of $X$ on $U$; this is Salmon’s theorem. However, much algebra is involved. Of course, once the result has been established, one may choose $X$ so as to minimise the amount of algebra. The following gives the value of the cross-ratio $\lambda$ (or $\Lambda$) for various cubic curves.

- $ar_1(r_2^2 - r_3^2) + br_2(r_3^2 - r_1^2) + cr_3(r_1^2 - r_2^2) = 0 \Rightarrow \lambda = (a^2 - c^2)/(a^2 - b^2)$.
- McCay’s cubic:
\[
a^2(-a^2 + b^2 + c^2)(c^2r_2^2 - b^2r_3^2)r_1 + b^2(a^2 - b^2 + c^2)(a^2r_3^2 - c^2r_1^2)r_2 \\
+ c^2(a^2 + b^2 - c^2)(b^2r_1^2 - a^2r_2^2)r_3 = 0
\]
$\Rightarrow \lambda = (a^2 - c^2)/(a^2 - b^2)$.
- $ar_1^2(r_2 - r_3) + br_2^2(r_3 - r_1) + cr_3^2(r_1 - r_2) = 0 \Rightarrow \lambda = b(a - c)/(c(a - b))$.
- $r_1^2(ar_2 + br_3) + r_2^2(cr_2 + dr_3) + 2kr_1r_2r_3 = 0
\Rightarrow \Lambda_1 = abcd/[(bc + ad - k^2)^2 - 4abcd)]$.


- In (17), let $X$ be a common point of the cubic curves $U$ and $V$. $Q$ now consists of the three lines $XA, XB, XC$ where $A, B, C$ are the common points of $UX$ and $VX$ other than $X$.
- If $P$ is the common point of the lines $L$ and $M$ then the four lines $L, M, (LM)X$ and $PX$
form a harmonic pencil for any point $X$.
- If $X$ and $Y$ lie on the conic $C$ then the line-pair
\[
CC_{XY} = 2C_XC_Y
\]
is $(XY)^2$.
- With any conic $S$ there is an associated symmetric matrix $S$ such that $S \equiv rSr^T = 0$ and the discriminant of $S$ is $\Delta(S) = |S|$.

The six points of contact of the tangents from $X(\alpha)$ to the cubic curve $U$ lie on the conic $S = UX$. Furthermore, these six tangents
meet \( U \) again at points on another conic \( S' = 4U_XU(\alpha) - 3(U_{XX})^2 \) and
\[
\Delta(S') = 16U(\alpha)^3\Delta(S).
\]
This shows that if \( S \) is a line-pair then so is \( S' \).

- The substitution \( V = L^3 \) (where \( L \) is a line) into (17) gives
\[
Q \equiv U(L_X)^3 - 3U_XL(L_X)^2 + 3U_{XX}L^2L_X - L^3U_{XXX} = 0.
\]
If the line \( L \) meets the cubic \( U \) at \( A, B, C \) then \( Q \) consists of the lines \( AX, XB, XC \).

- With \( X \) and \( Y \) being any points and \( S \) a conic, the conic
\[
Q \equiv (S_{XY})^2S - 2S_{XY}S_XS_Y + S_YYS_X = 0
\]
is a line-pair with common point \( Y \). Let the points of contact of the tangents from \( X \) to \( S \) be \( A \) and \( B \). Further, let \( AY \) meet \( S \) again at \( C \) and \( BY \) meet \( S \) again at \( D \) and let \( E \) be the common point of the lines \( AB \) and \( CD \). Then it is apparent that \( Q \) is the line-pair \( AY, BY \).

A. The Hessian of a Curve.

For any \( n \)-curve \( U \), the Hessian of \( U \) is defined as \( H \equiv \frac{\partial^2 U}{\partial r_i \partial r_j} \). For a line \( (n = 1) \) this is simply zero and for a conic it gives the discriminant, i.e. the determinant of the matrix associated with the conic. For a cubic curve \( (n = 3) \) the Hessian of \( U \) is another cubic curve and this is the only case of interest here.

Let the cubic curve \( U \) be written in coordinate form as
\[
U \equiv \sum_{i,j,k} u_{ijk}r_ir_jr_k = 0
\]
where the value of \( u_{ijk} \) is unchanged by a permutation of the indices. Let \( X(\alpha) \) be a point with the property that the six points of contact of the tangents from \( X \) to \( U \) lie on two straight lines. Then the polar curve \( U_X \) of the point \( X \) in the cubic curve \( U \) is a line-pair. Let these two lines meet at the point \( Y(\beta) \). Theorem 1 implies that
\[
U_{XY} \equiv 0 \Rightarrow \sum_{i,j,k} u_{ijk}r_i\alpha_j\beta_k = 0 \forall r \Rightarrow \sum_{j,k} u_{ijk}\alpha_j\beta_k = 0 \forall i.
\]
Thus
\[
\sum_j \left( \sum_k u_{ijk}\alpha_k \right) \beta_j = 0
\]
is a set of three linear, homogeneous equations for \( \beta \) and for a non-trivial solution we must have
\[
H(\alpha) \equiv \left| \sum_k u_{ijk}\alpha_k \right| = 0
\]
and so $X$ lies on the Hessian of $U$. Likewise

$$\sum_k \left( \sum_j u_{ijk} \beta_j \right) \alpha_k = 0$$

implies that $\beta$ also lies on $H$. Furthermore, the common point of the two lines forming $U_Y$ is $X$.

This result is well known, but this proof (the crucial fact being that $U_{XY}$ is identically zero) is simpler than the customary geometrical argument, see e.g. [5].

Two final comments on the Hessian:

- $U \equiv r_1^3 + r_2^3 + r_3^3 + 6kr_1r_2r_3 = 0 \Rightarrow H \equiv 216[-k^3U + (fg + 8k^3)r_1r_2r_3] = 0$.
- The common points of $U$ and $H$ are the nine points of inflexion of $U$ (and $H$).

**Appendix B. Envelopes.**

**B.1. The Case $n = 2$.** With $Q(r, \alpha)$ quadratic in $r$, the conditions

$$Q(r, \alpha) \equiv Q(\alpha, r) \quad \text{and} \quad \sum_i \alpha_i \frac{\partial Q}{\partial r_i} \equiv 0$$

are sufficient to deduce that the form of $Q$ is

$$Q(r, \alpha) \equiv R^T A R = 0 \quad \text{(29)}$$

where

$$R = r \times \alpha$$

and the $3 \times 3$ symmetric matrix $A$ is independent of $r$ and $\alpha$.

To investigate whether this line-pair defines an envelope, suppose first the the point $X(\alpha)$ is confined to a line $L$. Specifically, replace $\alpha$ by $(\alpha_1 + t, \alpha_2 + t\mu, \alpha_3 - t(1 + \mu))$. The parameter $\mu$ determines the direction of $L$ and $t$ determines the position of $X$ on $L$. The result of this substitution is an expression which is quadratic in $t$. The envelope of these lines as $X$ moves along $L$ is found by requiring this quadratic to have a repeated root and this yields the condition

$$L^2 E(r) = 0.$$ 

That $L$ should form part of the envelope is inevitable. The important facts are that $E$ is quadratic in $r$ and is independent of $\alpha$ and $\mu$. Thus regardless of the position of $X(\alpha)$, each of the lines given by (29) touch the curve $E$. In fact

$$E(r) \equiv r^T \text{adj}(A)r = 0 \quad \text{(30)}$$

Thus if a solution to (29) is interpreted as the coefficients of a line (so that $R \cdot r = 0$) then (29) is a conic-envelope and the corresponding conic-locus is given by (30). The curve $E$ is sometimes called the *reciprocal* of $Q$. 
B.2. The Case $n = 3$. With $Q(r, \alpha)$ being cubic in $r$, the conditions

$$Q(r, \alpha) \equiv -Q(\alpha, r) \quad \text{and} \quad \sum_i \alpha_i \frac{\partial Q}{\partial r_i} \equiv 0$$

are sufficient to deduce that the form of $Q$ is

$$Q(r, \alpha) \equiv q(R) \quad \text{where} \quad R = r \times \alpha$$

and $q(R)$ is any homogeneous cubic in $R = (R_1, R_2, R_3)$. The fact that this set of lines has an envelope is proved in exactly the same way as in the last section. The condition that the equation for $t$ has a repeated root is of course more complicated but still tractable. The result is

$$L^6 E(r) = 0$$

where $L$ is as before but now $E$ has degree six in $r$ and degree four in the coefficients occurring in (31). It is found that in general $E$ comprises 418 terms. However if $Q$ is restricted to

$$Q(r, \alpha) \equiv q(R) \equiv f R_1 + g R_2 + h R_3 + 6 k r_1 r_2 r_3 = 0$$

then

$$E(r) \equiv (gh r_3^3 + hfr_2^3 + gfr_3^3 - 12 k^2 r_1 r_2 r_3)^2 - 4(gh + 8k^3)(fr_2^3 + gr_3^3) + h^3 r_1^3 + 6 kr_1^2 r_2^2 r_3^2).$$

An alternative way of finding $E$ is to eliminate $R$ from the equations

$$r_i = \frac{\partial q}{\partial R_i} \quad (1 \leq i \leq 3), \quad \sum_{i=1}^3 r_i R_i = 0.$$