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THE FIRST VARIATION FORMULA ON A SURFACE

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ABSTRACT. We prove a rule for differentiating the length of a variable arc with respect to arc length on a smooth surface with variable Gaussian curvature also known as the First variation formula.

1. INTRODUCTION

Let $\gamma_1 : \vec{r}_1 = \vec{r}_1(s)$ and $\gamma_2 : \vec{r}_2 = \vec{r}_2(s)$ be smooth curves in $\mathbb{R}^3$, s the arc length parameter, $\alpha_2(s)$ be the angle formed by the vectors $\overrightarrow{\gamma_1(s)\gamma_2(s)}$ and the velocity vector $\frac{d\vec{r_1}(s)}{ds}$ of $\gamma_1(s)$ at the intersection point and $\alpha_2(s)$ be the angle formed by the vectors $\overrightarrow{\gamma_2(s)\gamma_1(s)}$ and the velocity vector $\frac{d\vec{r_2}(s)}{ds}$ of $\gamma_2(s)$ at the intersection point.

Toponogov proved in [8, Problem 1.5.3, Solution 1.5.3,pp. 16-17] that the derivative of the length $l(s)$ of a line segment $\gamma_1(s)\gamma_2(s)$ is given by:

$$\frac{dl}{ds} = \cos\alpha_1(s) + \cos\alpha_2(s).$$

This became known as the First variational formula of a family of smooth curves in R^3.

If the curve $\gamma_2(s)$ degenerates to a point, the first variational formula takes the form

$$\frac{dl}{ds} = \cos\alpha_1(s).$$

Some important derivations of the first variational formula are:

(i) in a C^2 regular surface given by Toponogov in
[8, Lemma 3.5.1, Remark 3.5.1,pp. 164-165]

(ii) in a convex surface given by Aleksandrov in [1, Theorem 5,pp. 115-120]

(iii) in Riemannian manifolds given in many classical geometrical books. For instance, we refer to the classical studies of Milnor, Bishop and Crittensen, DoCarmo, Berger ([7], [4], [6], [3]).

(iv) in a space of non-positive or non-negative curvature given by D. Burago, Y. Burago and S. Ivanov in [5, Theorem 4.5.6, Corollary 4.5.7,pp. 123-125].

In this paper, we introduce a variational method, which is based on the unified cosine law of Berg-Nikolaev given in [2] for the K-plane (Sphere S^2_K,

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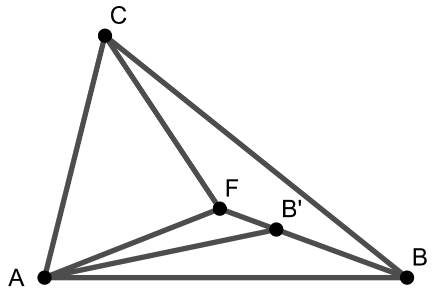

FIGURE 1. The first variational formula on a surface with variable Gaussian curvature: $|K| < \epsilon$

Hyperbolic plane H^2_K and Euclidean Plane $\mathbb{R}^2$) and an approximation of the differential(variation) of the physical parameter, which possess some Landau symbols o, derived by its Taylor expansion, to extend the First variational formula on a C^2surface S with variable Gaussian curvature K : $|K| < \epsilon$, for $\epsilon > 0$, or a finite number of glued surfaces S.

As an example, we may apply the first variation formula on a torus or surfaces of revolution with Gaussian curvature K in $\mathbb{R}^3$, having elliptic points ($K > 0$) hyperbolic points ($K < 0$) and parabolic points $K = 0$.

We denote by φ_Q, the angle formed by the tangent vectors of the geodesic arcs RQ and TQ at Q, by $l_X(Y)$ the length of the geodesic arc XY for X, Y in S, by F a fixed point inside the region W_F having boundary geodesic arcs AB, BC, CA, by B' an interior variable point of the geodesic arc FT in W_F (Figure 1), by dl_R the infinitesimal change of the length $l_R(F)$ to the length $l_R(B')$, such that: $l_R(B') = l_R(F) \pm dl_R$ and we set $dl_T \equiv B'F = l_{B'}(F)$, for $Q, R, T \in \{A, B, C\}$ and $Q \neq R \neq T$.

Our main result states that:

Theorem 1 (The first variation formula on S).

$$\frac{dl_R}{dl_T} \sim \cos(\pi - \varphi_Q). \tag{1.1}$$

2. THE PROOF OF THE THEOREM 1

Proof. Assume that $\triangle ABF$, $\triangle BFC$, $\triangle AFC$ belong to a spherical, hyperbolic or planar region of constant Gaussian curvature k_3, k_1, k_2, for $k_i \in \mathbb{R}$, $i = 1, 2, 3$.

We set $l_A(B') \equiv l_A(F) + dl_A$

and $l_{B'}(F) = dl_B$.

We set

$$\kappa_i \equiv \begin{cases} \sqrt{K_i} & \text{if } K_i > 0, \\ i\sqrt{-K_i} & \text{if } K_i < 0. \end{cases}$$

The unified cosine law for $\triangle AB'F$ is given by:

$$\cos(\kappa_3(l_A(F)+dl_A)) = \cos(\kappa_3 l_A(F))\cos(\kappa_3 dl_B)+\sin(\kappa_3 l_A(F))\sin(\kappa_3 dl_B)\cos(\varphi_C), \tag{2.1}$$

or

$$\cos(\kappa_3 l_A(F))\cos(\kappa_3 dl_A) - \sin(\kappa_3 l_A(F))\sin(\kappa_3 dl_A) = \cos(\kappa_3 l_A(F))\cos(\kappa_3 dl_B) + \sin(\kappa_3 l_A(F))\sin(\kappa_3 dl_B)\cos(\varphi_C), \tag{2.2}$$

By applying Taylor's formula, we obtain:

$$\cos\kappa_3 dl_A = 1 + o((k_3 dl_A)^2), \tag{2.3}$$

$$\sin\kappa_3 dl_A = \kappa_3 dl_A + o((k_3 dl_A)^3), \tag{2.4}$$

$$\cos\kappa_3 dl_B = 1 + o((k_3 dl_B)^2), \tag{2.5}$$

and

$$\sin\kappa_3 dl_B = \kappa_3 dl_B + o((k_3 dl_B)^3). \tag{2.6}$$

By replacing (2.3), (2.4),(2.5),(2.6) in (2.2) and neglecting second order terms, we derive that:

$$\frac{dl_A}{dl_B} \sim \cos(\pi - \varphi_C). \tag{2.7}$$

The unified cosine law for $\triangle CB'F$ is given by:

$$\cos(\kappa_1(l_C(F)+dl_C)) = \cos(\kappa_1 l_C(F))\cos(\kappa_1 dl_B)+\sin(\kappa_1 l_C(F))\sin(\kappa_1 dl_B)\cos(\varphi_A), \tag{2.8}$$

By applying Taylor's formula, we obtain:

$$\cos\kappa_1 dl_C = 1 + o((k_1 dl_C)^2), \tag{2.9}$$

$$\sin\kappa_1 dl_C = \kappa_1 dl_C + o((k_1 dl_C)^3), \tag{2.10}$$

$$\cos\kappa_1 dl_B = 1 + o((k_1 dl_B)^2), \tag{2.11}$$

and

$$\sin\kappa_1 dl_B = \kappa_1 dl_B + o((k_1 dl_B)^3). \tag{2.12}$$

Similarly, by replacing (2.9), (2.10),(2.11),(2.12) in (2.8) and neglecting second order terms, we derive that:

$$\frac{dl_C}{dl_B} \sim \cos(\pi - \varphi_A) \tag{2.13}$$

concluding the proof. □

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