Abstract. In this paper, we study rectifying Gaussian surfaces of Legendre curves on unit tangent bundle of unit 2-sphere, $UTS^2$. We introduce on $UTS^2$ two rectifying developable surfaces of a Legendre curve; one by using the modified Darboux vector and the other by using a new defined modified Darboux vector. Afterwards, we classify these surfaces at their singularity points. Finally, we give an example to illustrate the main idea of the study.

1. Introduction

One of the important notions in differential geometry is the Legendre curve. Legendre curves and their generalizations slant curves have been studied in several papers on unit tangent bundle of unit 2-sphere, $UTS^2$, e.g. [1], [5], [8] and [13]. It is well known from [9] that to a Legendre curve on a tangent bundle of unit 2-sphere, $TS^2$, corresponds a developable ruled surface, and the singularities of this type of surfaces have a deep relation with Legendre curvature functions. Singularities have also been studied by several geometers in Euclidean space, e.g [4] and [11].

A space curve is always a geodesic of the rectifying developable (surface) of itself. The notion rectifying Gaussian (resp., rectifying developable) surface is given in [11] as a ruled surface by using Frenet-Serret frame (resp., Darboux vector) along a regular curve. We generalize these notions along a Legendre curve on $UTS^2$. The rectifying developable surface in [11] is defined by using the modified Darboux vector. However we have defined a new type of rectifying developable surface by using a new defined Darboux vector, which we have called “new modified Darboux vector”, besides the modified Darboux vector given in [11]. On the other hand, we show that the singularities of the surfaces, which we have obtained by generalizing the notions rectifying Gaussian and rectifying developable surfaces along Legendre curves on $UTS^2$, are deeply related with order of contact with helices of component of the Legendre curves. According to this relationship, these surfaces can be cuspidal edge, swallowtail or a cone surface.

Keywords and phrases: Singularity, Rectifying developable surface, Darboux vector, Legendre curve, Unit tangent bundle

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The paper is organized as follows: In Section 2, we recall the concepts curves, Darboux vector, Rectifying developable surfaces and Legendre curves on $UTS^2$ to provide a necessary background material. In Section 3, we generalize the concept rectifying Gaussian surface of a regular curve to a Legendre curve on $UTS^2$. In Section 4, we define rectifying developable surface $RD_D$ by using the modified Darboux vector of a Legender curve on $TS^2$, and we introduce a new rectifying developable surface $RD_{\tilde{D}}$ by using the new Darboux vector. Moreover, we classify these surfaces at their singularity points, and close this section by given an example.

2. Preliminaries

In this section we recall some basic concepts on classical geometry of space curves on 3-dimensional Euclidean space $\mathbb{E}^3$ to give the background necessary to appreciate the rest of the article.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed curve parameterized by arc-length parameter $s$, so $|\alpha'(s)| = 1$ for each $s$ in open interval $I$. Here, $\alpha'(s) = d\alpha(s)/ds$ and $|.|$ denote, respectively, the velocity vector and the norm of $\alpha = \alpha(s)$. If the norm of a vector is equal to 1, we call it as unit vector.

The vector field $t = \alpha'(s)$ is called the unit tangent vector field on $\alpha$. Since $t = t(s)$ has constant length 1, its derivative is $t'(s) = \alpha''(s)$ and we call it as the curvature vector field of $\alpha$ and measures the way the curve is turning on $\mathbb{E}^3$. The real-valued function $\kappa$ such that $\kappa(s) = |t'(s)|$ for all $s \in I$ is called the curvature function of $\alpha$. Thus, $\kappa \geq 0$, and the larger $\kappa$ is, the sharper the turning of $\alpha$. Taking $\kappa(s) \neq 0$ (note that if $\kappa(s) = 0$ for all $s$, then $\alpha$ is a straight line), the unit vector field $n(s) = t'(s)/\kappa(s)$ on $\alpha$ is called the principal normal vector field and tells the direction in which $\alpha$ is turning at each point $s$. The vector field $b(s) = t(s) \times n(s)$ on $\alpha$ is called the binormal vector field of $\alpha$. The real-valued function $\tau$ such that $b'(s) = -\tau(s)n(s)$ for all $s \in I$ is called the torsion function of $\alpha$ and measures the twisting of $\alpha$.\footnote{The curvature of a space curve at a point is defined to be non-negative, but its torsion can be positive, negative or zero. The sign chosen here is somehow arbitrary, and some authors, like Carmo [6], write $b'(s) = \tau(s)n(s)$ instead of $b'(s) = -\tau(s)n(s)$.} If $\tau(s) = 0$ for all $s$, then $\alpha$ is a plane curve.

Lemma 2.1. Let $\alpha$ be a unit-speed curve on $\mathbb{E}^3$ with $\kappa > 0$. Then, the three vector fields $t$, $n$ and $b$ on $\alpha$ are unit vector fields that are mutually orthogonal at each point. We call the set $\{t, n, b\}$ the Frenet-Serret frame field on $\alpha$ and $\{t, n, b, \kappa, \tau\}$ as Frenet-Serret frame apparatus of $\alpha$.

Theorem 2.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed curve with curvature $\kappa(s) > 0$ and torsion $\tau(s)$. Then,

\[ t'(s) = \kappa(s)n(s), \quad n'(s) = -\kappa(s)t(s) + \tau(s)b(s), \quad b'(s) = -\tau(s)n(s) \]

are called the Frenet-Serret formulas.

As a point moves along the regular curve $\alpha$, then the Frenet-Serret frame $\{t, n, b\}$ makes an instant helical motion at each moment $s \in I$ along an axis called the Darboux axis of $\alpha$. The vector which indicates the direction of this axis is called the Darboux vector of $\alpha$ and is expressed as
satisfying the following symmetrical properties:
\[ t' = w \times t, \]
\[ n' = w \times n, \]
\[ b' = w \times b, \]
where \( \times \) denotes the standard cross product on \( \mathbb{E}^3 \). The unit Darboux vector of \( \alpha \) is also
\[ \bar{w} = \frac{w}{|w|} = \frac{\tau t + \kappa b}{\sqrt{\tau^2 + \kappa^2}}. \]

The Darboux vector of a curve provides a concise way of interpreting the curvature and torsion of the curve geometrically: Curvature measures the rotation of the Frenet-Serret frame about the binormal vector along the curve, whereas torsion measures the rotation of the frame about the unit tangent vector along the curve.

Spherical curve and rectifying Gaussian surface of \( \alpha \) are:
\[ d : I \to \mathbb{S}^2; \quad d(s) = \bar{w} = \frac{\tau t(s) + \kappa b(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}} \]
and
\[ RG_\alpha : I \times \mathbb{R} \to \mathbb{R}^3; \quad RG_\alpha(s, u) = \{ ut(s) + b(s) : u \in \mathbb{R}, \ s \in I \}, \]
respectively, see [11].

Tangent bundle over unit 2-sphere \( \mathbb{S}^2 \) is
\[ TS^2 = \{ (\gamma, v) \in \mathbb{E}^3 \times \mathbb{E}^3 : |\gamma| = 1 \text{ and } < \gamma, v > = 0 \} \]
and the unit tangent bundle over \( \mathbb{S}^2 \) is
\[ UTS^2 = \{ (\gamma, v) \in \mathbb{S}^2 \times \mathbb{S}^2 : |\gamma| = |v| = 1 \text{ and } < \gamma, v > = 0 \} \]
where \( <, > \) is the standard inner product on \( \mathbb{E}^3 \).

**Definition 2.1.** Let
\[ \Gamma : I \subset \mathbb{R} \longrightarrow UTS^2 \subset \mathbb{S}^2 \times \mathbb{S}^2 ; \quad \Gamma(s) = (\gamma(s), v(s)) \]
be a smooth curve. If
\[ < \gamma'(s), v(s) > = 0 \quad \text{for all } s \in I, \]
then \( \Gamma \) is called a Legendre curve.

In [13], a special case of Definition 2.1 is given for the Legendre curve condition as: \( \Gamma(s) = (\gamma(s), v(s)) \) is a Legendre curve if \( \gamma(s) \) and \( v(s) \) are \( \Delta \)-dual to each other.

If \( \Gamma = (\gamma, v) \) is a Legendre curve on \( UTS^2 \), then \( < \gamma, v > = 0 \). Thus, we can construct the Frenet-Serret type frame \( \{ v, \eta = \gamma \wedge v, \gamma \} \) along \( v \)-direction curve \( \beta \) (i.e., \( \beta(s) = \int v(s)ds \)).
Theorem 2.2. Let $\Gamma(s) = (\gamma(s), v(s))$ be a Legendre curve on $UTS^2$. Then,

\begin{equation}
\begin{pmatrix}
v' \\
\eta' \\
\gamma'
\end{pmatrix}
= 
\begin{pmatrix}
0 & n & 0 \\
-n & 0 & m \\
0 & -m & 0
\end{pmatrix}
\begin{pmatrix}
v \\
\eta \\
\gamma
\end{pmatrix}.
\end{equation}

are equivalent:

\begin{itemize}
\item $\nu' = n(s)\eta(s)$,
\item $\eta' = -n(s)v(s) - m(s)\gamma(s)$,
\item $\gamma' = m(s)\eta(s)$
\end{itemize}

are called the Frenet-Serret type formulas depending on the orthonormal vectors $\nu$, $\eta$ and $\gamma$. Here, $n = <\nu', \eta>$ and $m = -<\gamma', \eta>$ are the curvature and torsion functions of $\beta$, respectively. The couple $(n, m)$ is called the curvature functions of $\Gamma$.

The Frenet-Serret type formulas of the frame $\{\nu, \eta, \gamma\}$ can be given by matrix product as

\begin{equation}
\begin{pmatrix}
v' \\
\eta' \\
\gamma'
\end{pmatrix}
= 
\begin{pmatrix}
0 & n & 0 \\
-n & 0 & m \\
0 & -m & 0
\end{pmatrix}
\begin{pmatrix}
v \\
\eta \\
\gamma
\end{pmatrix}.
\end{equation}

Thus, the Darboux vector (resp. unit Darboux vector) of the Legendre curve $\Gamma = (\gamma, v)$ (i.e., of the frame $\Gamma = (\nu, v)$ along $\beta$) can be given as

\begin{equation}
D = mv + n\gamma \quad \text{(resp., } D = \frac{mv + n\gamma}{\sqrt{n^2 + m^2}} \text{ for } (m, n) \neq (0, 0)).
\end{equation}

3. Rectifying Gaussian surfaces of Legendre curves

In this section we generalize the rectifying Gaussian surface notion of any regular unit speed curve to a Legendre curve on $UTS^2$.

Definition 3.1. Let $\Gamma(s) = (\gamma(s), v(s))$ be a smooth Legendre curve on $UTS^2$. Then the rectifying Gaussian surface of $\Gamma$ is defined by

\begin{equation}
RG_V : I \times \mathbb{R} \to \mathbb{R}^3; \quad RG_V(s, u) = \{ u\gamma(s) + v(s) : u \in \mathbb{R}, s \in I \}.
\end{equation}

If we take $\Gamma(s) = (\gamma(s), v(s)) = (t(s), b(s))$ in Definition 3.1, we get the Gaussian surface given by Equation (4).

Theorem 3.1. Let $\Gamma(s) = (\gamma(s), v(s))$ be a smooth Legendre curve on $UTS^2$ with Frenet-Serret type frame $\{v(s), \eta(s) = \gamma(s) \wedge v(s), \gamma(s)\}$ and whose curvature functions are $(m(s), n(s))$. Then the following assertions are equivalent:

1. Curves $\gamma(s)$ and $v(s)$ are parts of a circle.
2. Geodesic curvature $\kappa_g$ of $v(s)$ is a constant.
3. Curve $\eta(s)$ is a big circle, i.e. $\kappa_g[\eta] = 0$.
4. Evolute $D_1(s) = \varepsilon_v(s)$ of $v(s)$ is a constant vector, i.e. $D_1(s) = v(s)\cos\theta + \gamma(s)\sin\theta$, where $\cos\theta = m/n$.
5. Surface $F(\nu, \gamma) = RG_V(s, u) = uv(s) + \gamma(s)$ is a cone surface.
6. Surface $F(\nu, D_1) = RG_V(s, u) = uv(s) + D_1(s)$ is a cylinder.
7. Curves $\int v(s)ds$ and $\int \gamma(s)ds$ are helices with axes $D_1(s)$.

Proof. If we use the Frenet-Serret type frame given in Equation (8) for the Legendre curve $\Gamma(s) = (\gamma(s), v(s))$ in $UTS^2$, we get

$$
\kappa_g[\eta] = \frac{\det(\eta, \eta', \eta'')}{|\eta|^3} = \frac{-m^2}{(n^2 + m^2)^{\frac{3}{2}}} \left(\frac{n}{m}\right),
$$

from which we can easily get the equivalence of Assertions (1) to (6). The $v$-direction curve $\beta(s) = \int v(s)ds$ (resp., $\gamma$-direction curve $\alpha(s) = \int \gamma(s)ds$)
is a helix, because $\langle \beta'(s), D_1(s) \rangle = \cos \theta$ (resp., $\langle \alpha'(s), D_1(s) \rangle = \sin \theta$) is a constant. Thus, Assertions (4) and (7) are equivalent.

**Corollary 3.1.** Let $\Gamma_c(s) = (c(s), v(s))$ be a smooth Legendre curve on $UTS^2$ with Frenet-Serret type frame $\{v(s), \eta(s) = c(s) \wedge v(s), c(s)\}$, where $c(s)$ is a constant vector. Then the following assertions are equivalent:

1. Geodesic curvature $\kappa_g$ of $v(s)$ is a constant.
2. Curve $v(s)$ is a big circle.
3. Curve $\eta(s)$ is a big circle, i.e. $\kappa_g[\eta] \equiv 0$.
4. Evolute $D_1(s) = \varepsilon_v(s)$ of $v(s)$ is a constant vector, i.e. $D_1(s) = \gamma(s)$.

**Corollary 3.2.** Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with frame apparatus $\{n, c = n'/|n'|, \bar{w} = n \times c, f, g\}$ (resp., $\{c, d = c'/|c'|, \bar{u} = c \times d, f_1, g_1\}$).

If the Legendre curve $\Gamma(s) = (\bar{w}(s), n(s))$ (resp., $\Gamma(s) = (\bar{u}(s), c(s))$) satisfies one of the assertions given by Theorem 3.1, then $\alpha$ is a slant helix (resp., clad helix), where $\bar{w}$ (resp., $\bar{u}$) is the unit Darboux vector of the Frenet-Serret frame $\{t, n, b\}$ (resp., of the frame $\{n, c, \bar{w}\}$) on $\alpha$.

For more details of the term clad helix and of the frame apparatus given corollary 3.2, we refer the interested readers to [3] and [15]. For the singularity points of a rectifying Gaussian surface of a smooth Legendre curve $\Gamma$ in $UTS^2$, we refer also the interested readers to [2].

### 3.1. Applications by using Darboux frame.

Consider a smooth surface $M = X(U)$ given locally by an embedding $X : U \to \mathbb{R}^3$, where $\mathbb{R}^3$ is the 3-dimensional Euclidean space and $U \subset \mathbb{R}^2$ is an open set. Let $\bar{\gamma} : I \to U$ be an embedding, where $\bar{\gamma}(t) = (u(t), v(t))$ and $U \subset \mathbb{R}$ is an open set. Then, we have a regular curve

$$\gamma = X \circ \bar{\gamma} : I \to M \subset \mathbb{R}^3$$

on the surface $M$. On the surface $M$, we have the unit normal vector field $N$ defined by

$$N(p) = \frac{X_u \times X_v}{|X_u \times X_v|}(u, v),$$

where $p = X(u, v)$. Since the curve $\gamma$ is a space curve on $\mathbb{R}^3$, we adopt the arc-length parameter as usual and write $\gamma(s) = X(u(s), v(s))$. Then, the unit tangent vector field of $\gamma(s)$ is $t(s) = \gamma'(s)$. The normal vector field of $\gamma$ is $n_\gamma = N \circ \gamma(s)$. Moreover, the binormal vector field of $\gamma$ is $b_\gamma(s) = n_\gamma(s) \times t(s)$. The orthonormal frame $\{t, b_\gamma, n_\gamma\}$ is called the Darboux frame along $\gamma$ satisfying the following Frenet-Serret type formulas:

$$
\begin{align*}
\tau' &= \kappa_g b_\gamma + \kappa_n n_\gamma, \\
\kappa'_\gamma &= -\kappa_g t + \tau_g n_\gamma, \\
\kappa''_\gamma &= -\kappa_n t - \tau_g b_\gamma,
\end{align*}
$$

where $\kappa_g$ is the geodesic curvature, $\kappa_n$ is the normal curvature and $\tau_g$ is the geodesic torsion of $\gamma$.

The following vectors

$$
\begin{align*}
D_n &= -\kappa_n b_\gamma + \kappa_g n_\gamma, \\
D_r &= \tau_g t + \kappa_n n_\gamma, \\
D_o &= \tau_g t - \kappa_n b_\gamma
\end{align*}
$$
are called the normal Darboux vector field, the rectifying Darboux vector field and the osculating Darboux vector field along $\gamma$, respectively. The spherical image of each Darboux vector field are, respectively:

$$
\bar{D}_n = \frac{-\kappa_n b_s + \kappa_g n_s}{\sqrt{\kappa_n^2 + \kappa_g^2}} \quad \text{if} \quad (\kappa_n, \kappa_g) \neq (0, 0),
$$

$$
\bar{D}_r = \frac{\tau_n t + \kappa_n n_s}{\sqrt{\tau_n^2 + \kappa_n^2}} \quad \text{if} \quad (\tau_n, \kappa_n) \neq (0, 0),
$$

$$
\bar{D}_o = \frac{\tau_s t - \kappa_s b_s}{\sqrt{\tau_s^2 + \kappa_s^2}} \quad \text{if} \quad (\kappa_s, \kappa_g) \neq (0, 0).
$$

Thus, we have the following three spherical images

$$
\bar{D}_n : I \to S^2 \quad \text{if} \quad (\kappa_n, \kappa_g) \neq (0, 0),
$$

$$
\bar{D}_r : I \to S^2 \quad \text{if} \quad (\tau_n, \kappa_n) \neq (0, 0),
$$

$$
\bar{D}_o : I \to S^2 \quad \text{if} \quad (\kappa_g, \kappa_n) \neq (0, 0),
$$

which are called the spherical normal Darboux image, the spherical rectifying Darboux image and the spherical osculating Darboux image along $\gamma$, respectively.

Curves $\Gamma_1 = (D_n, t)$, $\Gamma_2 = (\bar{D}_r, b_\gamma)$ and $\Gamma_3 = (\bar{D}_o, n_\gamma)$ are Legendre curves on $UTS^2$.

Using Equation (8), the Frenet-Serret type formulas for the Legendre curve $\Gamma_1 = (\bar{D}_n, t)$ are

$$
t' = \kappa n; \quad n' = -\kappa t + \tau b; \quad \bar{D}_n' = -\tau n,
$$

where $t' = D_n \times t$, $n = t' / |t'|$ and $\kappa, \tau$ are the curvature functions defined by Equation (7). Here, $\kappa_g = \tau / \kappa$, $D_n = b$ and the spherical rectifying Darboux image along $\gamma$ is the unit Darboux vector

$$
\bar{w} = \frac{\kappa t + \tau b_\gamma}{\sqrt{\kappa^2 + \tau^2}}.
$$

Thus, for the Legendre curve $\Gamma_1$ we get the following propositions from Theorem 3.1.

**Proposition 3.1.** Consider the smooth Legendre curve $\Gamma_1$ on $UTS^2$ with Frenet-Serret frame apparatus $\{t, n, b, \kappa, \tau\}$ and with geodesic curvature $\kappa_g$. Then, the following assertions are equivalent:

1. Curves $D_n(s) = b(s)$ and $t(s)$ are parts of a circle.

2. Geodesic curvature $\kappa_g = \tau / \kappa$ of $t(s)$ is a constant.

3. Curve $n(s)$ is a big circle, i.e. $\kappa_g [n] \equiv 0$.

4. $\bar{w}$ is a constant vector and $\int t(s)ds$ and $\int D_n(s)ds = \int b(s)ds$ are helices with axes $\bar{w}$.

5. Surface $F(t, D_n) = RG_\Gamma(s, u) = ut(s) + \bar{D}_n(s)$ is a cone surface, where $\Gamma = (t(s), \bar{D}_n(s))$.

The same studies can be done for the Legendre curves $\Gamma_2 = (\bar{D}_r, b)$ and $\Gamma_3 = (\bar{D}_o, n_\gamma)$ using the same methods.
Let $\beta_n$ be the $t$-direction, $\beta_r$ be the $b_\gamma$-direction and $\beta_o$ be the $n_\gamma$-direction curves, i.e.

$$
\beta_n = \int t \, ds, \quad \beta_r = \int b_\gamma \, ds \quad \text{and} \quad \beta_o = \int n_\gamma \, ds.
$$

Then, the curvatures and torsions of these curves are

$$
\kappa = \sqrt{\kappa_n^2 + \kappa_g^2}, \quad \tau_n = \tau = \delta_n, \\
\kappa_r = \sqrt{\tau_g^2 + \kappa_g^2}, \quad \tau_r = \delta_r, \\
\kappa_o = \sqrt{\tau_g^2 + \kappa_n^2}, \quad \tau_o = \delta_o,
$$

respectively. Here,

$$
\delta_n = \tau_g - \frac{\kappa_n\kappa_g' - \kappa'_n\kappa_g}{\kappa_n^2 + \kappa_g^2} \quad \text{if} \quad (\kappa_n, \kappa_g) \neq (0, 0), \\
\delta_r = \kappa_n - \frac{\kappa_g\tau_g' - \kappa_g'\tau_g}{\kappa_g^2 + \tau_g^2} \quad \text{if} \quad (\kappa_g, \tau_g) \neq (0, 0), \\
\delta_o = \kappa_g + \frac{\kappa_n\tau_g' - \kappa_n'\tau_g}{\kappa_n^2 + \tau_g^2} \quad \text{if} \quad (\kappa_n, \tau_g) \neq (0, 0),
$$

see [7].

**Proposition 3.2.** Consider the curves $\beta_n$, $\beta_r$ and $\beta_o$ given by Equation (11). Then,

1. $\beta_n$ is a helix curve with axis $W_n = \bar{w} = (\kappa t + \tau b)/\sqrt{\kappa^2 + \tau^2}$ if and only if the geodesic curvature $\kappa_g = \tau/\kappa$ of $t(s)$ is a constant function.

2. $\beta_r$ is a helix curve with axis $W_r = (\delta_b b_\gamma + \kappa_r \bar{D}_r)/\sqrt{\kappa_r^2 + \tau_r^2}$ if and only if the geodesic curvature $\kappa_g = \delta_r/\kappa_r$ of $b_\gamma(s)$ is a constant function.

3. $\beta_o$ is a helix curve with axis $W_o = (\delta_n n_\gamma + \kappa_o \bar{D}_o)/\sqrt{\kappa_o^2 + \tau_o^2}$ if and only if the geodesic curvature $\kappa_g = \delta_o/\kappa_o$ of $n_\gamma(s)$ is a constant function.

4. **Singularities of rectifying developable surfaces of Legendre curves**

In this section, we present a new idea to define rectifying developable surfaces of Legendre curves on $UTS^2$ as a generalization of the rectifying developable surfaces of regular curves.

Let $\alpha$ be a regular unit speed curve on $\mathbb{E}^3$ with frame apparatus $\{t, n, b, \kappa, \tau\}$. Then, the modified Darboux vector of $\alpha$ is

$$
\tilde{D} = \frac{\tau}{\kappa} t + b
$$

Now, we introduce a different kind Darboux vector of $\alpha$ from $\tilde{D}$ by

$$
\tilde{\tilde{D}} = t + \frac{\tau}{\kappa} b
$$

and we call it as new modified Darboux vector of $\alpha$.

The rectifying developable surface of $\alpha$ depending on $\tilde{D}$ is

$$
RD\tilde{D}(\alpha) : I \times \mathbb{R} \to \mathbb{R}^3
$$
defined by
\[
RD_D(\alpha)(s, u) = \{ \int tds + u\tilde{D} : u \in \mathbb{R}, s \in I \} = \{ \alpha + u\tilde{D} : u \in \mathbb{R} \},
\]
(14)
see [11].

We introduce now a different kind developable surface of \(\alpha\) from \(RD_D(\alpha)\) by
\[
RD_{\tilde{D}}(\alpha)(s, u) = \{ \int bds + u\tilde{\tilde{D}} : u \in \mathbb{R}, s \in I \} = \{ \beta + u\tilde{\tilde{D}} : u \in \mathbb{R} \},
\]
(15)
where \(\beta\) is the \(b\)-direction curve of \(\alpha\). We call this new surface as new rectifying developable surface of \(\alpha\).

The modified Darboux vector and the new modified Darboux vector of the Legendre curve \(\Gamma(\gamma, v)\) are
\[
\tilde{D} = \frac{m}{n}v + \gamma \quad \text{and} \quad \tilde{\tilde{D}} = v + \frac{n}{m}\gamma,
\]
(16)
respectively.

The surface
\[
RD_D(\beta)(s, u) = \{ u\tilde{D}(s) + \beta(s) : u \in \mathbb{R}, s \in I \}, \quad \text{(resp., } RD_{\tilde{D}}(\delta)(s, u) = \{ u\tilde{\tilde{D}}(s) : u \in \mathbb{R}, s \in I \})
\]
(17, 18)
is called the rectifying developable surface of \(\beta = \int vds\) (resp., \(\delta = \int \gamma ds\)) depending on the vector \(\tilde{D}\) (resp., \(\tilde{\tilde{D}}\)) of the Legendre curve \(\Gamma = (\tilde{D}, \beta)\) (resp., \(\Gamma = (\tilde{\tilde{D}}, \delta)\)) on \(UTS^2\).

As a remark, the rectifying developable surface \(RD_D\) is defined in [11] only for the modified Darboux vector of any regular curve. However, in this study we have defined two rectifying developable surfaces for a Legendre curve on \(UTS^2\); one for the modified Darboux vector \(\tilde{D}\) and one, which is a new definition, for the new modified Darboux vector \(\tilde{\tilde{D}}\).

We classify now the rectifying developable surfaces \(RD_D\) and \(RD_{\tilde{D}}\) at their singularity points:

**Theorem 4.1.** Let \(\tilde{D}\) be the modified Darboux vector of \(\Gamma\) and \(\beta\) be the \(v\)-direction curve. Then, we have the following:

1. Surface \(RD_D(\beta)(s, u)\) is locally diffeomorphic to cuspidal edge \(C \times \mathbb{R}\) at \(u_0\tilde{D}(s_0) + \beta(s_0)\) if and only if \(u_0 = -(m/n)'(s_0)^{-1}\), \((m/n)'(s_0) \neq 0\) and \((m/n)''(s_0) \neq 0\).

2. Surface \(RD_{\tilde{D}}(\beta)(s, u)\) is locally diffeomorphic to swallow tail \(SW\) at \(u_0\tilde{\tilde{D}}(s_0) + \beta(s_0)\) if and only if \(u_0 = -(m/n)'(s_0)^{-1}\), \((m/n)'(s_0) \neq 0\), \((m/n)''(s_0) = 0\) and \((m/n)'''(s_0) \neq 0\).
Proof. Using Theorem 2.2 in the study [11] for the curve $\beta$ we get the proof.

Similar to Theorem 4.1, we give now the classification theorem depending on $\tilde{D}$ as follow:

**Theorem 4.2.** Let $\tilde{D}$ be the new modified Darboux vector of $\Gamma$ and $\delta$ be the $\gamma$-direction curve. Then, we have the following:

1. Surface $RD_{\tilde{D}}(\delta)(s,u)$ is locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$ at $u_0\tilde{D}(s_0) + \delta(s_0)$ if and only if $u_0 = -(m/n)'(s_0)^{-1}$, $(m/n)'(s_0) \neq 0$ and $(m/n)''(s_0) \neq 0$.

2. Surface $RD_{\tilde{D}}(\delta)(s,u)$ is locally diffeomorphic to swallow tail SW at $u_0\tilde{D}(s_0) + \delta(s_0)$ if and only if $u_0 = -(m/n)'(s_0)^{-1}$, $(m/n)'(s_0) \neq 0$, $(m/n)''(s_0) = 0$ and $(m/n)'''(s_0) \neq 0$.

**Corollary 4.1.** Let $\tilde{D}$ (resp., $\tilde{D}$) be the modified (resp., new modified) Darboux vector of $\Gamma$ and $\beta$ (resp., $\delta$) be the $v$-direction curve. Then, we have the same classifications for $RD_{\tilde{D}}(\beta)$ (resp., $RD_{\tilde{D}}(\delta)$) of $\beta$ (resp., $\delta$) for $\tilde{D}$ (resp., $\tilde{D}$) of $\Gamma$ and for the ruled surface $\tilde{\Phi}(u,s) = u\beta(s) + \tilde{D}(s)$ (resp., $\tilde{\Phi}(u,s) = u\delta(s) + \tilde{D}(s)$).

**Proof.** For the ruled surface $\tilde{\Phi}(u,s) = u\beta(s) + \tilde{D}(s)$, we have

$$\tilde{\Phi}_u = \beta$$

$$\tilde{\Phi}_s = ((m/n)' + u)v.$$ 

The normal vector field of $\tilde{\Phi}$ is

$$N_{\tilde{\Phi}} = ((m/n)' + u)v \times \beta.$$ 

Thus, the singularity points of the surface $\tilde{\Phi}$ are $u(s_0) = -(m/n)'(s_0)$. And using Theorem 4.1, we get the proof. The proof of the second case can be given using the same methods for the Theorem 4.2.

**Proposition 4.1.** Let $\tilde{D}$ be the modified Darboux vector of $\Gamma$ and $\beta$ be the $v$-direction curve. Then, we have the following:

1. $\beta$ is a rectifying curve, i.e. $(m/n)(s) = as + b$ where $a, b \in \mathbb{R}$ and provided that $a \neq 0$.

2. $\beta$ is a geodesic curve of $RD_{\tilde{D}}(\beta)$.

3. Surface $RD_{\tilde{D}}(\beta)$ is a cone surface.

**Proof.** Let $\beta$ be a rectifying curve. Then, derivating the ruled surface

$$\Phi(s,u) = RD_{\tilde{D}}(\beta)(s,u)$$

and using Equation (8), we get

$$\Phi_s = \left(1 + \frac{m}{n}'u\right)v,$$

$$\Phi_u = \frac{m}{n}v - \gamma = \tilde{D},$$
and thus
\[ \mathbf{N} = \Phi_u \times \Phi_s = \left( 1 + \left(\frac{m}{n}\right)' u \right) \eta. \]
Moreover, we have
\[ \beta'' = n\eta. \]
Hence, \( \beta'' \) is parallel to the normal vector field \( \mathbf{N} \) of \( \Phi \). So, \( \beta \) is a geodesic curve on \( \Phi \). Let \( \sigma(s_0) \) be a singularity point of \( \Phi \), where \( \sigma \) is the striction curve of \( \Phi \) given by
\[ \sigma(s) = \beta - \frac{\langle v, (\frac{m}{n})' v \rangle}{((\frac{m}{n})')^2} \tilde{D} = \beta - \frac{1}{(\frac{m}{n})'} \tilde{D}. \]
Derivating \( \sigma \) with respect to \( s \), we get
\[ \sigma'(s) = (\frac{m}{n})'' \tilde{D}. \]
Thus, \( \sigma'(s) = 0 \), i.e. \( RD_{\tilde{D}}(\beta) \) is a cone surface if and only if \( (m/n)(s) \) is a linear function.

**Proposition 4.2.** Let \( \tilde{D} \) be the modified Darboux vector of \( \Gamma \) and \( \delta \) be the \( \gamma \)-direction curve. Then, we have the following:
1. \( \delta \) is a rectifying curve, i.e. \( (n/m)(s) = \bar{a} s + \bar{b} \) where \( \bar{a}, \bar{b} \in \mathbb{R} \) provided that \( \bar{a} \neq 0 \).
2. \( \delta \) is a geodesic curve of \( RD_{\tilde{D}}(\delta) \).
3. Surface \( RD_{\tilde{D}}(\delta) \) is a cone surface.

Proof of Proposition 4.2 can be given similar to the proof of Proposition 4.1.

**Corollary 4.2.** Let \( \gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^3 \) be a non-null curve with arc-length parameter \( s \), and let \( \{t, n, b, \kappa, \tau\} \) be the Frenet-Serret frame apparatus of \( \alpha \). Then, the surface \( RD_{\tilde{D}}(\alpha) \) (resp., \( RD_{\tilde{D}}(\alpha) \)) is a cone surface if and only if \( \gamma \) (resp., \( \beta = \int b ds \)) is a rectifying curve.

Proof. Using Theorems 4.1 and 4.2, and taking account the Legendre curve \( \Gamma = (t, b) \in UT \mathbb{S}^2 \), we get the proof. For the second case, we take account the Legendre curve \( \Gamma = (b, t) \in UT \mathbb{S}^2 \).

As a remark, the result of the Proposition 3.1 given in [12] is a particular case of the condition \( RD_{\tilde{D}}(\gamma) \).

**Corollary 4.3.** Let \( \gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^3 \) be a unit speed curve with frame apparatus \( \{n, c = n'/|n'|, \bar{w} = n \times c, f, g\} \). Then, the surface \( RD_{\tilde{D}}(\beta) \) (resp., \( RD_{\tilde{D}}(\mu) \)) is a cone surface if and only if \( \beta = \int \bar{w} ds \) (resp., \( \mu = \int n ds \)) is a rectifying curve.

Proof of Corollary 4.3, can be given similar to the proof of Corollary 4.2 using the Legendre curves \( \Gamma = (n, w) \) and \( \Gamma = (w, n) \) on \( UT \mathbb{S}^2 \).

**Example 4.1.** Let \( \gamma, v : I = [0, A] \rightarrow \mathbb{R}^3, \ 0 < A \leq 2\pi, \) be smooth curves defined by
\[ \gamma(s) = \frac{1}{4}(3\cos s - \cos 3s, 3\sin s - \sin 3s, 2\sqrt{3}\cos s), \]
\[ v(s) = \frac{1}{4}(3\sin s + \sin 3s, -3\cos s - \cos 3s, -2\sqrt{3}\sin s). \]
Then,
\[ \eta(s) = \gamma(s) \times v(s) = \frac{1}{2}(\sqrt{3}\cos 2s, \sqrt{3}\sin 2s, -1). \]

The curves \( ^2\gamma(s) \) and \( v(s) \) are slant curves on \( TS^2 \). So, the \( \gamma \)-direction (resp., \( v \)-direction) curve is a slant helix. The curve \( \Gamma(s) = (\gamma(s), v(s)) \) is a Legendre on \( UT^2 \) with curvature functions
\[ m(s) = \sqrt{3}\sin s \quad \text{and} \quad n(s) = \sqrt{3}\cos s. \]

The modified Darboux vector of \( \Gamma \) is
\[ \tilde{D}(s) = \frac{m}{n}(s)v(s) + \gamma(s) \]
\[ = (3\tan s\sin s + \tan s\sin 3s + \frac{3}{4}\cos s - \frac{1}{4}\cos 3s, \]
\[ -3\sin s - \tan s\cos 3s + \frac{3}{4}\sin s - \frac{1}{4}\sin 3s, \]
\[ \frac{\sqrt{3}}{2}\cos s - 2\sqrt{3}\tan s\sin s). \]

The Surface \( RD_{\tilde{D}}(\beta) \) is
\[ RD_{\tilde{D}}(\beta) = \{u\tilde{D}(s) + \beta(s) : u \in \mathbb{R}\} \]
\[ = (\Phi_1(u, s), \Phi_2(u, s), \Phi_3(u, s)), \]
see Figure 1, where
\[ \Phi_1(u, s) = u(3\tan s\sin s + \tan s\sin 3s + \frac{3}{4}\cos s - \frac{1}{4}\cos 3s) - \frac{1}{4}(3\cos s + \frac{1}{3}\cos 3s), \]
\[ \Phi_2(u, s) = u(-3\sin s - \tan s\cos 3s + \frac{3}{4}\sin s - \frac{1}{4}\sin 3s) - 3\sin s - \frac{1}{3}\sin 3s, \]
\[ \Phi_3(u, s) = u\sqrt{3}(\frac{1}{2}\cos s - 2\tan s\sin s) + 2\sqrt{3}\cos s. \]

The curve \( \gamma \) is called the spherical nephroid.

---

*Figure 1. Surface \( RD_{\tilde{D}}(\beta) \).*
The \( v \)-direction curve

\[
\beta(s) = \int v(s) \, ds
\]

\[
= \frac{1}{4} (-3 \cos s - \frac{1}{3} \cos 3s, -3 \sin s - \frac{1}{3} \sin 3s, 2\sqrt{3} \cos s).
\]

is a slant helix. The stirction curve, which corresponds to the singularity points, of \( RD\tilde{D}(\beta) \) is

\[
\bar{\beta}(s) = \beta(s) - \frac{\langle \beta'(s), \tilde{D} \rangle}{\langle \tilde{D}, \tilde{D} \rangle} \tilde{D}
\]

\[
= \beta(s) - \left( \left( \frac{m}{n} \right)'(s) \right)^{-1} \tilde{D}
\]

\[
= (\bar{\beta}_1(s), \bar{\beta}_2(s), \bar{\beta}_3(s)),
\]

see Figure 2, where the functions \( \bar{\beta}_1(s), \bar{\beta}_2(s) \) and \( \bar{\beta}_3(s) \) are

\[
\bar{\beta}_1(s) = (1 + \cot^2 s) \left( \frac{1}{4} \cos 3s - 3 \tan s \sin s - \tan s \sin 3s - \frac{3}{4} \cos s \right) - \frac{1}{4} (3 \cos s + \frac{1}{3} \cos 3s),
\]

\[
\bar{\beta}_2(s) = (1 + \cot^2 s) \left( 3 \sin s + \tan s \cos 3s - \frac{3}{4} \sin s + \frac{1}{4} \sin 3s \right) - \frac{1}{4} \left( 3 \sin s + \frac{1}{3} \sin 3s \right),
\]

\[
\bar{\beta}_3(s) = (1 + \cot^2 s) \left( 2\sqrt{3} \tan s \sin s - \frac{\sqrt{3}}{2} \cos s \right) + \frac{\sqrt{3}}{2} \cos s.
\]

From the rations

\[
\left( \frac{m}{n} \right)'(s) = 1 + \tan^2 s, \quad \left( \frac{m}{n} \right)''(s) = 2 \tan s(1 + \tan^2 s),
\]

\[
\left( \frac{m}{n} \right)'''(s) = 2 + 8 \tan^2 s + 6 \tan^4 s
\]

we get the following:

1. If \( s_0 = \pi/2 \), then \( (m/n)'(\pi/2) \neq 0 \) and \( (m/n)''(\pi/2) \neq 0 \). Thus, the surface \( RD\tilde{D}(\beta) \) is locally diffeomorph to the cuspidal edge \( C \times \mathbb{R} \) at \( u\tilde{D}(\pi/2) + \beta(\pi/2) \).

2. If \( s_0 = \pi \), then \( (m/n)'(\pi) \neq 0 \) and \( (m/n)''(\pi) = 0 \) and \( (m/n)'''(\pi) \neq 0 \). Thus, the surface \( RD\tilde{D}(\beta) \) is locally diffeomorph to the swallow tail \( SW \) at \( u\tilde{D}(\pi) + \beta(\pi) \).

5. Conclusions

In this study, we have given the conditions to be circle, helix, etc., for the components (i.e., for the \( \Delta \)-dual components to each other) of a Legendre curve on \( UTS^2 \). Afterwards, the conditions to be cylinder, cone, etc., are given for the rectifying developable surfaces obtained from these components. We have also introduced the new modified Darboux vector field using these components. Finally, we have given the singularities of the rectifying developable surfaces obtained from this new introduced Darboux vector.
Figure 2. The curve of the singularity points of the surface $RD_D(\beta)$.

References

GAZI UNIVERSITY
06500 ANKARA, TURKEY
E-mail address: muratbekar@gazi.edu.tr

DEPARTMENT OF MATHEMATICS
SAIDA UNIVERSITY
20000 SAIDA, ALGERIA
E-mail address: f.hathout@gmail.com

DEPARTMENT OF MATHEMATICS
ANKARA UNIVERSITY
06100 ANKARA, TURKEY
E-mail address: yayli@science.ankara.edu.tr