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ON AN INDEFINITE METRIC ON A 3-DIMENSIONAL RIEMANNIAN MANIFOLD

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Abstract. In the present paper, our considerations are in a single tangent space of a point on a 3-dimensional Riemannian manifold. Besides the positive definite metric this manifold is endowed with a tensor field of type $(1,1)$, whose third power is the identity. Both structures are compatible and determine an indefinite metric on the manifold. We calculate the length and the area of a circle in a single tangent space of the manifold with respect to this indefinite metric. Also, we calculate the circulation done by a vector force field along the curve and the flux of the curl of this vector force field across the curve. We obtain a relation between last two values which is an analog of the well known Green formula.

1. Introduction

In many papers there are studied the properties of spheres and circles with respect to indefinite metrics (for instance [1, 9, 10, 11, 13, 14]). Some problems concerning the length and the area of circles, calculated with respect to indefinite metrics, are given in [3, 12].

We consider a 3-dimensional Riemannian manifold M equipped with a positive definite metric g and a tensor field Q of type $(1,1)$, whose third power is the identity. The structure Q is an endomorphism in every tangent space T_pM at a point p on M. We note that some similar endomorphisms, which satisfy third-degree equations, are discussed in papers [2, 5, 15]. Both structures g and Q are compatible and they define an indefinite metric $\tilde{g}$ [8]. The metric $\tilde{g}$ determines time-like vectors in T_pM. In a special 2-plane α of T_pM, constructed on time-like vectors, we consider a circle k with respect to $\tilde{g}$. We calculate its length and its area (with respect to $\tilde{g}$) and it turns out they are analogue of these measures of the circles in the Euclidean space. In α we consider the circulation C and the flux T done by a vector force field F and curl$F$ acting on k, respectively.

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The paper is organized as follows. In Section 2, we give some facts, definitions and statements, which are necessary for the present considerations. Some of them are obtained in [4], [6], [7] and [8]. We introduce a special 2-plane α of T_pM and we give the equation of a circle k with respect to $\tilde{g}$. In Section 3 we define the functions cosine and sine with respect to $\tilde{g}$. In Subsections 3.1 and 3.2 we calculate the length and the area of k, respectively. In Section 4, we find the circulation and the flux of a vector force field F and its curl F with respect to k, respectively. We give a relation between them. All values obtained in Sections 3 and Sections 4 are calculated with respect to $\tilde{g}$.

2. Preliminaries

We consider a 3-dimensional Riemannian manifold M with an additional tensor structure Q of type $(1,1)$. In a local coordinate system (x^1,x^2,x^3) the coordinates of Q form the following circulant matrix:

$$Q=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

Thus we get

$$Q^3=\mathrm{id}. \tag{1}$$

Let g be a positive definite metric on M, which satisfies the equality

$$g(Qu,Qv)=g(u,v),\quad u,\ v\in\mathfrak{X}M. \tag{2}$$

Such a manifold (M,g,Q) is introduced in [7].

Further u,v,w,e_1,e_2 will stand for arbitrary smooth vector fields on M or arbitrary vectors in the tangent space T_pM, $p\in M$.

The norm of every vector u and the cosine of the angle $\varphi=\angle(u,Qu)$ are given by the following equalities:

$$\|u\|=\sqrt{g(u,u)},\quad \cos\varphi=\frac{g(u,Qu)}{g(u,u)}. \tag{3}$$

We assume that $\|u\|=1$ and due to (3) we have

$$\cos\varphi=g(u,Qu). \tag{4}$$

In [7], for (M,g,Q) it is verified that $\varphi\in\big(0,\frac{2\pi}{3}\big)$. In this case the vector u forms a basis $\{u,Qu,Q^2u\}$, which is called a Q-basis of T_pM. Also, we have

$$\angle(u,Qu)=\angle(u,Q^2u)=\angle(Qu,Q^2u)=\varphi. \tag{5}$$

The associated metric $\tilde{g}$ on (M,g,Q), determined by

$$\tilde{g}(u,v)=g(u,Qv)+g(Qu,v), \tag{6}$$

is necessary indefinite ([8]).

Using (1), (2), (4), (5) and (6), we establish that $\tilde{g}$ satisfies the following equalities:

$$\tilde{g}(u,u)=\tilde{g}(Qu,Qu)=2\cos\varphi,\quad \tilde{g}(u,Qu)=1+\cos\varphi. \tag{7}$$

3. A circle with respect to the metric $\tilde{g}$

Now we introduce a 2-plane $\alpha = \{u, Qu\}$ with the following conditions for the angle $\varphi = \angle(u, Qu)$:

$$\arccos\left(-\frac{1}{3}\right) < \varphi < \frac{2\pi}{3}. \tag{8}$$

From (7) and (8) we get that the vectors u and Qu satisfy $\tilde{g}(u, u) < 0$ and $\tilde{g}(Qu, Qu) < 0$ respectively, so they are time-like vectors with respect to $\tilde{g}$. We construct a coordinate system P_{xy} on α such that u is on P_x and Qu is on P_y.

A circle k in α of a radius R centered at the origin $p \in T_pM$, with respect to the associated metric $\tilde{g}$ on (M, g, Q), is determined by

$$k: \quad \tilde{g}(v, v) = R^2, \tag{9}$$

where v is the radius vector of an arbitrary point on k. We denote

$$R = ri, \quad r > 0, \quad i^2 = -1. \tag{10}$$

The following statement is valid.

Theorem 3.1. *Let $\tilde{g}$ be the associated metric on (M, g, Q) and let $\alpha = \{u, Qu\}$ be an arbitrary 2-plane in T_pM. Let p_{xy} be a coordinate system such that $u \in p_x$, $Qu \in p_y$. Then the equation of the circle* (9) *is given by*

$$2\cos\varphi x^2 + 2(1 + \cos\varphi)xy + 2\cos\varphi y^2 = -r^2. \tag{11}$$

Proof. Let $N(x, y)$ be a point on k with a radius vector

$$v = x.u + y.Qu. \tag{12}$$

From (8), (9) and (12) we get

$$x^2\tilde{g}(u, u) + 2xy\tilde{g}(u, Qu) + y^2\tilde{g}(Qu, Qu) = R^2.$$

Then, using (7) and (10), we find (11).

We construct a vector system $\{e_1, e_2\}$ on α by the following equalities

$$e_1 = \frac{1}{\sqrt{2(1 + \cos\varphi)}}(u + Qu), \; e_2 = \frac{1}{\sqrt{2(1 - \cos\varphi)}}(-u + Qu). \tag{13}$$

Bearing in mind (2), (4), (7) and (13), we calculate $g(e_1, e_2) = 0$ and $g(e_1, e_1) = g(e_2, e_2) = 1$. Therefore the next statement is valid.

Lemma 3.1. *Let α be a 2-plane of T_pM with a basis $\{u, Qu\}$. The system of vectors $\{e_1, e_2\}$, determined by* (13) *is an orthonormal basis of α with respect to g.*

We construct a coordinate system p_{XY} on α, such that e_1 is on the axes p_X and e_2 is on the axes p_Y. Evidently p_{XY} is an orthonormal coordinate system.

Theorem 3.2. *The relation between coordinates (x, y) with respect to p_{xy} and (X, Y) with respect to p_{XY} of an arbitrary point on α is*

$$\begin{aligned} x &= \frac{X}{\sqrt{2(1 + \cos\varphi)}} - \frac{Y}{\sqrt{2(1 - \cos\varphi)}}, \\ y &= \frac{X}{\sqrt{2(1 + \cos\varphi)}} + \frac{Y}{\sqrt{2(1 - \cos\varphi)}}. \end{aligned} \tag{14}$$

Proof. Let (x, y) be a point on α with a radius vector v. Then v is expressed by $v = xu + yQu$ and also by $v = Xe_1 + Ye_2$. Comparing these two equalities and using (13) we get (14).

Theorem 3.3. *Let $\tilde{g}$ be the associated metric on (M, g, Q) and let $\alpha = \{u, Qu\}$ be an arbitrary 2-plane in T_pM. Let e_1 and e_2 be determined by (13). If p_{XY} is a coordinate system such that $e_1 \in p_X$, $e_2 \in p_Y$, then the equation of the circle (9) is given by*

$$-\frac{1 + 3\cos\varphi}{1 + \cos\varphi} X^2 + Y^2 = r^2. \tag{15}$$

Proof. We substitute (14) into (11), and using (7) we obtain (15).

In the case of (8) we have $1 + 3\cos\varphi < 0$. Therefore (15) is an ellipse in the terms of g.

3.1. Length of a circle with respect to $\tilde{g}$.

Theorem 3.4. *The circle (9) has a length*

$$L = 2\pi R. \tag{16}$$

Proof. From (12) we have that dv is a tangent vector on the circle (11). Let the length L of k with respect to $\tilde{g}$ be determined by

$$dL = \sqrt{\tilde{g}(dv, dv)}.$$

Then, using (7) and (12), we obtain

$$L = \oint_k \sqrt{2\cos\varphi dx^2 + 2(1 + \cos\varphi)dxdy + 2\cos\varphi dy^2}, \tag{17}$$

where k has the equation (11). We substitute (14) into (17) and we find

$$L = i \oint_k \sqrt{-\frac{1 + 3\cos\varphi}{1 + \cos\varphi} dX^2 + dY^2}, \tag{18}$$

where k has the equation (15).

We substitute $X = \sqrt{-\frac{1+\cos\varphi}{1+3\cos\varphi}} r\cos t$, $Y = r\sin t$, $t \in [0, 2\pi]$ into (18) and we find (16).

We note that the length of k is an imaginary number.

3.2. Area of a circle with respect $\tilde{g}$.

Analogously to (3) we define the function cosine of φ with respect to $\tilde{g}$ with $\widetilde{\cos}\varphi = \frac{\tilde{g}(u,Qu)}{\tilde{g}(u,u)}$. Then using (7) we get

$$\widetilde{\cos}\varphi = \frac{1 + \cos\varphi}{2\cos\varphi}. \tag{19}$$

Because of (8) we have $\widetilde{\cos}\varphi \in (-1, -\frac{1}{2})$. Thus, bearing in mind (19) and $\widetilde{\cos}^2\varphi + \widetilde{\sin}^2\varphi = 1$, we obtain the sine of φ with respect to $\tilde{g}$, as follows

$$\widetilde{\sin}\varphi = -\frac{\sqrt{(1 + 3\cos\varphi)(\cos\varphi - 1)}}{2\cos\varphi}. \tag{20}$$

Theorem 3.5. *The area of the circle (9) is*

$$S = \pi R^2. \tag{21}$$

Proof. In the coordinate plane p_{xy}, we construct a parallelogram with locus vectors $dx.u$ and $dy.Qu$. For its area S with respect to $\tilde{g}$ we have

$$dS = \sqrt{\tilde{g}(dx.u, dx.u)}\sqrt{\tilde{g}(dy.Qu, dy.Qu)}\widetilde{\sin}\varphi.$$

We apply (2), (7), (8) and (20) into the above equality and we get

$$dS = -\sqrt{(1+3\cos\varphi)(\cos\varphi - 1)}dxdy. \tag{22}$$

We integrate (22) and calculate

$$S = -\sqrt{(1+3\cos\varphi)(\cos\varphi - 1)}\int\int_D dxdy, \tag{23}$$

where

$$D: \ -2\cos\varphi x^2 - 2(1+\cos\varphi)xy - 2\cos\varphi y^2 \le r^2. \tag{24}$$

We substitute (14) into (23) and (24) and we find

$$S = -\frac{\sqrt{(1+3\cos\varphi)(\cos\varphi - 1)}}{\sin\varphi}\int\int_{D'} dXdY, \tag{25}$$

with

$$D': \ -\frac{(1+3\cos\varphi)}{1+\cos\varphi}X^2 + Y^2 \le r^2.$$

We solve the integral (25) and having in mind (10) finally we get (21).

4. Circulation and flux with respect to the metric $\tilde{g}$

Let

$$F(x,y) = P(x,y)u + H(x,y)Qu \tag{26}$$

be a vector force field on curve k with (11).

For the circulation C of a vector field F along a curve k we accept the following definition

$$C = \oint_k \tilde{g}(F, dv), \tag{27}$$

where v is the radius vector of a point on k.

Theorem 4.1. *The circulation C done by a force* (26) *along the curve k by* (11) *is expressed by*

$$\begin{aligned} C = \oint_k &\big(2\cos\varphi P(x,y) + (1+\cos\varphi)H(x,y)\big)dx \\ &+ \big((1+\cos\varphi)P(x,y) + 2\cos\varphi H(x,y)\big)dy. \end{aligned} \tag{28}$$

Proof. From (27) we have

$$dC = \tilde{g}(F, dv).$$

Then, by virtue of (7), (12), (26) we obtain

$$\begin{aligned} dC = &\big(2\cos\varphi P(x,y) + (1+\cos\varphi)H(x,y)\big)dx \\ &+ \big((1+\cos\varphi)P(x,y) + 2\cos\varphi H(x,y)\big)dy. \end{aligned} \tag{29}$$

Thus (11), (27) and (29) imply (28).

Theorem 4.2. *The circulation C done by a force* (26) *along the curve k by* (15) *is expressed by*

$$C = \oint_k \frac{1+3\cos\varphi}{\sqrt{2(1+\cos\varphi)}}\big(P(X,Y)+H(X,Y)\big)dX - \frac{1-\cos\varphi}{\sqrt{2(1-\cos\varphi)}}\big(H(X,Y)-P(X,Y)\big)dY. \tag{30}$$

Proof.
By substituting (14) into (28), and having in mind (15), we find (30).
We construct a vector w in T_pM as follows

$$w = \frac{(1+\cos\varphi)(u+Qu)-(1+3\cos\varphi)Q^2u}{\sqrt{2(1+3\cos\varphi)(\cos\varphi-1)(1+2\cos\varphi)}}. \tag{31}$$

Using (7) we verify that $\tilde{g}(w,u)=\tilde{g}(w,Qu)=0$ and $\tilde{g}(w,w)=1$.
We suppose that the curl of F, determined by (26), with respect to the basis $\{u, Qu, w\}$ is

$$\mathrm{curl}F = (H_x - P_y)w. \tag{32}$$

Then for the flux T of the curl of F across the curve (11) we accept the following definition

$$T = \int\int_D \tilde{g}(curlF, w)ds, \tag{33}$$

where $D:\ -2\cos\varphi x^2 - 2(1+\cos\varphi)xy - 2\cos\varphi y^2 \le r^2$.

Theorem 4.3. *The flux T of the vector field curlF across the curve* (11) *is expressed by*

$$T = -\sqrt{3\cos^2\varphi - 2\cos\varphi - 1}\int\int_D (H_x - P_y)dxdy, \tag{34}$$

where $D:\ -2\cos\varphi x^2 - 2(1+\cos\varphi)xy - 2\cos\varphi y^2 \le r^2$.

Proof.
Now, bearing in mind (22), (31), (32) and (33), we get (34).

Theorem 4.4. *The flux T of curlF across the curve* (15) *is expressed by*

$$T = -\frac{\sqrt{3\cos^2\varphi-2\cos\varphi-1}}{\sin\varphi}\int\int_{D'}\Big(\frac{\sqrt{2(1+\cos\varphi)}}{2}(H-P)_X - \frac{\sqrt{2(1-\cos\varphi)}}{2}(H+P)_Y\Big)dXdY, \tag{35}$$

where $D':\ -\frac{1+3\cos\varphi}{1+\cos\varphi}X^2 + Y^2 \le r^2$.

Proof. We substitute (14) into $H_x - P_y$ and we find

$$H_X - P_Y = \frac{\sqrt{2(1+\cos\varphi)}}{2}(H-P)_X - \frac{\sqrt{2(1-\cos\varphi)}}{2}(H+P)_Y. \tag{36}$$

Now we calculate $dxdy = \frac{dXdY}{\sin\varphi}$ and then, using (33) and (36), we get (35).

Now, we introduce the following notations:

$$
\begin{aligned}
c_1 &= \oint_k \frac{1+3\cos\varphi}{\sqrt{2(1+\cos\varphi)}}(H+P)dX,\\
c_2 &= -\oint_k \frac{1-\cos\varphi}{\sqrt{2(1-\cos\varphi)}}(H-P)dY,\\
\beta &= \sqrt{\frac{1+3\cos\varphi}{\cos\varphi-1}}.
\end{aligned}
\tag{37}
$$

On the other hand, due to the Green's formula, we have

$$
\begin{aligned}
\int\int_D (H+P)_Y dXdY &= -\oint_k (H+P)dX,\\
\int\int_D (H-P)_X dXdY &= -\oint_k (H-P)dY.
\end{aligned}
$$

Bearing in mind the latter formulas we obtain the following statement.

Theorem 4.5. *The relation between the circulation* (30) *and the flux* (35) *is determined by*

$$
T = \beta c_1 - \frac{1}{\beta}c_2,
$$

where c_1, c_2 and β are given in (37).

References

[1] Abe, N., Nakanishi, Y. and Yamaguchi, S., *Circles and spheres in pseudo-Riemannian geometry*, Aequationes Math. **38(2-3) (1990)**, 134–145.

[2] Crasmareanu, M., *The determinant inner product and the Heisenberg product of Sym(2)*, Int. Electron. J. Geom. **14(1) (2021)**, 145–156.

[3] Mustafaev, Z., *The ratio of the length of the unit circle to the area of the unit disk in Minkowski planes*, Proc. American Math. Soc. **133(4) (2004)**, 1231–1237.

[4] Dokuzova, I. *On 3-dimensional almost Einstein manifolds with circulant structures*, Turkish Journal of Mathematics **44(4) (2020)**, 1484-1497.

[5] Dokuzova, I. *Riemannian manifolds with two circulant structures*, J. Geom. **105(3) (2014)**, 529-538.

[6] Dzhelepov, G., *Spheres and circles with respect to an indefinite metric*, Novi Sad J. Math. **48(1) (2018)**, 143–150,

[7] Dzhelepov, G., Dokuzova, I. and Razpopov, D., *On a three-dimensional Riemannian manifold with an additional structure*, Plovdiv. Univ. Paisii Khilendarski Nauchn. Trud. Mat. **38(3) (2011)**, 17–27.

[8] Dzhelepov, G., Razpopov, D. and Dokuzova, I., *Almost conformal transformation in a class of Riemannian manifolds*, In: Proc. of Anniv. Int. Conf. – REMIA, Plovdiv (2010), 125–128.

[9] Ikawa, T., *On curves and submanifolds in an indefinite-Riemannian manifold*, Tsukuba J. Math. **9(2) (1985)**, 353–371.

[10] Konderak, J. J., *Construction of harmonic maps between pseudo-Riemannian spheres and hyperbolic spaces*, Proc. Amer. Math. Soc. **109(2) (1990)**, 469–476.

[11] Lopez, R., *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, Int. Electron. J. Geom. **7(1) (2014)**, 44–107.

[12] Holmes, R.D. and Thompson, A., *N-dimensional area and content in Minkowski spaces*, Pacific J. Math. **85(1979)**, 77–110.

[13] Reynolds, W.F., *Hyperbolic Geometry on a hyperboloid*, Amer. Math. Monthly **100(5) (1993)**, 442–455.

[14] Wielenberg, N.J., *Three-dimensional hyperbolic spaces*, Rocky Mountain J. Math. **11(4) (1981)**, 659–671.

[15] Yano, K. and Ishihara, S., *Structure defined by f satisfying* $f^3 + f = 0$, Proc. U.S.-Japan Seminar in Diferential Geometry, (Kyoto, 1965), 153-166.

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