INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 11 (2022), No. 4, 5 - 11

REDUCTION OF EXACT CONTACT MANIFOLDS

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Abstract. Let $(\mathrm{M}, \alpha, \omega)$ be a contact manifold in the sense of Okassa. We consider Marsden-Weinstein reduction theorem which induces Jacobi-Liouville theorem as special case. The aim of this paper is to describe the Hamiltonian dynamics, integrability of characteristic distribution on a contact manifold and study the reduction.

1. Introduction

Let M be a paracompact and connected smooth manifold, $\mathcal{C}^\infty(\mathrm{M})$ the algebra of smooth functions on M with unit 1, $\mathfrak{X}(\mathrm{M})$ the $\mathcal{C}^\infty(\mathrm{M})$-module of vector fields on M and $\mathcal{D}(\mathrm{M})$ the $\mathcal{C}^\infty(\mathrm{M})$-module of first order differential operators endowed with a Lie algebra structure. A Lie-Rinehart algebra over $\mathcal{C}^\infty(\mathrm{M})$ is a pair $(\mathcal{G}, \rho)$ where $\mathcal{G}$ is simultaneously a $\mathcal{C}^\infty(\mathrm{M})$-module and a $\mathbb{R}$-Lie algebra with bracket $[,]$, and

$$\rho : \mathcal{G} \longrightarrow \mathcal{D}(\mathrm{M})$$

is simultaneously a morphism of $\mathcal{C}^\infty(\mathrm{M})$-modules and of $\mathbb{R}$-Lie algebras verifying

$$[x,\, f \cdot y] = [\rho(x)(f) - f \cdot \rho(x)(1)] \cdot y + f \cdot [x, y]$$

for any $x, y \in \mathcal{G}$ and any $f \in \mathcal{C}^\infty(\mathrm{M})$ (see [8],[9],[10],[11], [7], [6] for the complete bibliography). A differential operator on M is a $\mathbb{R}$-linear map

$$\varphi : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$

such that

$$\varphi(f \cdot g) = \varphi(f) \cdot g + f \cdot \varphi(g) - f \cdot g \cdot \varphi(1)$$

for any $f, g \in \mathcal{C}^\infty(M)$. The differential operator term will mean further first order differential operator. Let δ the cohomology operator associated with the representation

$$id : \mathcal{D}(\mathrm{M}) \longrightarrow \mathcal{D}(\mathrm{M}).$$

The set $\mathcal{D}(\mathrm{M})$ admits a symplectic Lie-Rinehart-Jacobi algebra iff there exist

Keywords and phrases: Lie-Rinehart algebras, differential operators, contact manifolds, momentum map
(2020)Mathematics Subject Classification: 13N05, 53D05, 53D10
Received: 20.01.2022. In revised form: 12.07.2022. Accepted: 16.05.2022.

a $\mathcal{C}^\infty(\mathrm{M})$-linear form

$$\alpha : \mathcal{D}(\mathrm{M}) \longrightarrow \mathcal{C}^\infty(\mathrm{M})$$

such that $\delta\alpha = \delta 1 \wedge \alpha$ and a nondegenerate skew-symmetric bilinear form

$$\omega : \mathcal{D}(\mathrm{M}) \times \mathcal{D}(\mathrm{M}) \longrightarrow \mathcal{C}^\infty(\mathrm{M})$$

such that $\delta\omega = -\alpha \wedge \omega$. In this case, there exists a unique vector field H on M such that $i_H\omega = -\delta 1$. Moreover the linear form

$$i_1\omega : \mathcal{D}(\mathrm{M}) \longrightarrow \mathcal{C}^\infty(\mathrm{M}),\ \varphi \longmapsto \omega(1, \varphi)$$

is such that $i_1\omega(H) = 1$ [11].

Let G be a compact and connected Lie group acting smoothly on a s-mooth closed and connected manifold M. If $A : \mathrm{G} \times \mathrm{M} \longrightarrow \mathrm{M}$ is the action of G on M, we write simply $A(x, g) = x \cdot g$. If $\mathrm{Diff}^k(\mathrm{M})$ is the group of all $\mathcal{C}^k$-diffeomorphisms of M, the group $\mathrm{Diff}^k_{\mathrm{G}}(\mathrm{M})$ of $\mathcal{C}^k$-equivariant diffeomorphisms is by definition

$$\mathrm{Diff}^k_{\mathrm{G}}(\mathrm{M}) = \left\{ h \in \mathrm{Diff}^k(\mathrm{M}) / h(x \cdot g) = h(x) \cdot g, \forall x \in \mathrm{M}, \forall g \in \mathrm{G} \right\}.$$

When $\mathcal{L}$ is a Lie algebra, is commutator $[\mathcal{L}, \mathcal{L}]$ is defined to be the subset of elements which can be expressed as a finite sum of brackets. The cohomology we consider is the usual Lie algebra cohomology with real coefficients [3]. It is well known fact [3], [2] that the first cohomology group $\mathcal{H}^1(\mathcal{L})$ is the linear space dual to $\mathcal{L}/\left[\mathcal{L}, \mathcal{L}\right]$.

2. Hamiltonian dynamics for contact manifolds

For any $\eta \in \mathcal{L}_{sks}(\mathcal{D}(\mathrm{M}), \mathcal{C}^\infty(\mathrm{M}))$, the cohomology operator δ_α associated with the representation $\rho_\alpha : \mathcal{D}(\mathrm{M}) \longrightarrow \mathcal{D}(\mathrm{M})$ is defined by $\delta_\alpha\eta = \delta\eta + (\alpha - \delta 1) \wedge \eta$. In particular $\delta_\alpha 1 = \alpha$.

Proposition 2.1. [11] *Let the pair $(\mathcal{D}(\mathrm{M}), \rho)$ is a Lie-Rinehart algebra, then the map ρ is always of the form*

$$\rho_\alpha : \mathcal{D}(\mathrm{M}) \longrightarrow \mathcal{D}(\mathrm{M}), \varphi \longmapsto \varphi + (\alpha - \delta 1)(\varphi),$$

where

$$\alpha : \mathcal{D}(\mathrm{M}) \longrightarrow \mathcal{C}^\infty(\mathrm{M})$$

is a linear form such that $\delta\alpha = (\delta 1) \wedge \alpha$.

If $\mathcal{D}(\mathrm{M})$ admits a symplectic Lie-rinehart algebra structure, then there exist a nondegenerate skew-symmetric bilinear form ω and a linear form α satisfying

(i) $\delta_\alpha\omega = \delta\omega - (\delta 1 - \alpha) \wedge \omega$;
(ii) $\delta_\alpha\alpha = \delta\alpha - (\delta 1) \wedge \alpha = 0$.

A smooth function $f : \mathrm{M} \longrightarrow \mathbb{R}$ gives rise to a differential operator φ_f uniquely defined by the equation $i_{\varphi_f}\omega = \delta_\alpha f = \delta f + f \cdot (\alpha - \delta 1)$ called the Hamiltonian differential operator with Hamiltonian f and X_f the unique element of $\ker(i_1\omega/\mathfrak{X}(\mathrm{M}))$ such that

$$i_{X_f}\omega = \delta_\alpha f - [H(f) + f \cdot \alpha(H)] \cdot i_1\omega - f \cdot [1 + \alpha(1)] \cdot \delta 1.$$

Let

$$H_\alpha = \alpha(1)\cdot[H+\alpha(H)]\,;\varphi_f = H(f)+f\cdot\alpha(H)+X_f - f\cdot\alpha(1)\cdot H.$$

If $\{f,g\} = -\omega(\varphi_f,\varphi_g) = -\omega(X_f,X_g) - f\cdot H_\alpha(g) + g\cdot H_\alpha(f)$, for all $f,g\in\mathcal{C}^\infty(\mathrm{M})$, then this bracket is a Jacobi bracket on $\mathcal{C}^\infty(\mathrm{M})$ and M is a Jacobi manifold [11], [6]. Recall that $(\mathcal{C}^\infty(\mathrm{M}),\{,\})$ is an infinite dimensional Lie algebra, it is called a Jacobi algebra of M; it play a major role in several parts of Mathematics and Mechanics. The above bracket relates to the bracket on forms as follows

$$\delta_\alpha\{f,g\} = \{\delta_\alpha f, \delta_\alpha g\}.$$

We point out that $\varphi_{\{f,g\}} = [\varphi_f,\varphi_g]$. Similary $(\mathcal{C}^\infty(\mathrm{M}),\{,\})_c$ denotes the Jacobi algebra of compactly supported functions. According to [11], let setting $H_\alpha = [1+\alpha(1)]\cdot\rho_\alpha(H)$, one obtains

$$\{f,g\} = -\omega(X_f,X_g) - f\cdot H_\alpha(g) + g\cdot H_\alpha(f),$$

and the dimension of M is odd. In [6], the author verifies that $\theta_{\varphi_f}\omega = 0$, where θ_{φ_f} is Lie derivative with respect to the differential operator φ_f. We denote $\mathcal{D}_\omega(\mathrm{M})$ the Lie algebra of Hamiltonian differential operators with the usual Lie bracket for differential operators $[\varphi,\psi] = \varphi\circ\psi - \psi\circ\varphi$. One defines the gradient of $f\in\mathcal{C}^\infty(\mathrm{M})$ as the differential operator φ_f with the following property, for any $\psi\in\mathcal{D}(\mathrm{M}), \omega(\varphi_f,\psi) = (\delta_\alpha)(\psi)$. Let

$$\pi : \mathcal{C}^\infty(\mathrm{M}) \longrightarrow \mathcal{D}_\omega(\mathrm{M})$$

be the gradient mapping. One has easily $\pi(\mathcal{C}^\infty(\mathrm{M}))\subset\mathcal{D}_\omega(\mathrm{M})$. A differential operator is called Hamiltonian if it lies in the image of π and one has $\pi(\mathcal{C}^\infty(\mathrm{M})) = \mathcal{D}_\omega(\mathrm{M})$. Since M is supposed to be connected, two Hamiltonian for the same differential operator only differ by a constant. The map π is a Lie algebra morphism. Let $\mathcal{D}_H(\mathrm{M})$ be the Lie algebra of Hamiltonian differential operators on M, $\mathcal{C}^\infty_H(\mathrm{M})$ be the Lie algebra of smooth functions on M. The compactness of M is established by the next theorem.

Theorem 2.1. *Let* $(\mathrm{M},\alpha,\omega)$ *be an exact contact manifold. Then the following statements are equivalent:*

1) M *is compact;*
2) π *has a right inverse in the category of Lie algebra;*
3) $\mathcal{C}^\infty_H(\mathrm{M}) \neq [\mathcal{C}^\infty_H(\mathrm{M}),\mathcal{C}^\infty_H(\mathrm{M})]$*;*
4) $\mathcal{H}^1(\mathcal{C}^\infty_H(\mathrm{M})) \neq \{0\}$.

Proof. For more details we refer the reader to [3], [1],[2], [4].

Let

$$\mathcal{L}_\omega(\mathrm{M}) = \{\varphi\in\mathcal{D}(\mathrm{M}), \delta_\alpha\,(i_\varphi\omega) = 0\}$$

be the Lie algebra of symplectic differential operators and

$$\mathcal{D}_H(\mathrm{M}) = Ham(\mathrm{M}) = \{\varphi\in\mathcal{D}(\mathrm{M}), i_\varphi\omega\, is\, \delta_\alpha - exact\}\subset\mathcal{L}_\omega(\mathrm{M}).$$

Proposition 2.2. *Suppose that* M *is compact, then* $\mathcal{C}^\infty_H(\mathrm{M})$ *is weakly integrable. For noncompact* M*, then Jacobi algebra* $(\mathcal{C}^\infty(\mathrm{M}),\{,\})_c$ *is weakly integrable.*

Proof. The symplectic gradient mapping π is a surjective Lie algebra homomorphism, with kernel $\mathbb{R}$ (provided that M is connected). The sequence of Lie algebra

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^{\infty}_{H}(\mathrm{M}) \longrightarrow \mathcal{D}_{H}(\mathrm{M}) \longrightarrow \{0\}$$

is exact. According to [4], the above sequence splits iff M is compact. In this case, $\mathcal{C}^{\infty}_{H}(\mathrm{M})$ is isomorphic as Lie algebra with $\mathcal{D}_{H}(\mathrm{M}) \cdot \mathbb{R}$. Hence $\mathcal{C}^{\infty}_{H}(\mathrm{M})$ is weakly integrable if M is compact. In the noncompact case, the proof is obvious.

Lemma 2.1. *The fundamental group $\pi_1(\mathrm{M})$ of M admits a real central extension which acts on $\widehat{\mathrm{M}} \cdot \mathbb{S}^1$ as a subgroup of the group of contact diffeomorphisms* [5] *of* $\left(\widehat{\mathrm{M}} \cdot \mathbb{S}^1, \alpha, \omega\right)$.

Theorem 2.2. *Let $(\mathrm{M}, \alpha, \omega)$ be a contact manifold such that the pullback of ω and α to the universal covering of M is δ_α-exact, then $\mathcal{C}^{\infty}_{H}(\mathrm{M})$ is isomorphic to the Lie algebra $\mathcal{L}$ of differential operators which preserve the contact forms on $\widehat{\mathrm{M}} \cdot \mathbb{S}^1$ and which invariant by the contact action of a central extension of the fundamental group of* M.

Set

$$I_\alpha(\mathrm{M}) = \left\{ f \in \mathcal{C}^{\infty}(M), \alpha(\varphi_f) = 0 \right\}.$$

For the triple $(\mathrm{M}, \alpha, \omega)$ the fundamental vector field φ_1 (see [10], for more details) is the ω-dual of the linear form α, i.e., φ_1 is a differential operator on M defined by $i_{\varphi_1}\omega = \alpha$. Then a smooth f on M is sits in $I_\alpha(\mathrm{M})$ iff $\delta f(\varphi_1)$ vanishes identically on M in view of the equalities $\alpha(\varphi_f) = -i_{\varphi_f} i_{\varphi_1}\omega = -\delta f(\varphi_1)$. We verify that the pair $(I_\alpha(\mathrm{M}), \{,\})$ is a Poisson algebra.

Lemma 2.2. (i) *If $f \in I_\alpha(\mathrm{M})$, then f is constant along the flow of φ_f.*
(ii) *Let G_t be the flow of φ_t. Then $G_t^*\omega = \omega, G_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$ iff $f \in I_\alpha(\mathrm{M})$.*

Proof. The proof presents no difficulty.

3. Integrability of characteristic distribution

Let $T\mathrm{M}$ be the tangent vector bundle of M. The $\mathcal{C}^{\infty}(\mathrm{M})$-module $\mathcal{D}(\mathrm{M})$ is the $\mathcal{C}^{\infty}(\mathrm{M})$-module of sections of the vector bundle $\mathbb{R} \times T\mathrm{M} \longrightarrow \mathrm{M}$. As ω is a nondegenerate skew-symmetric 2-form on $\mathcal{D}(\mathrm{M})$, then for any $x \in \mathrm{M}, \omega_x : (\mathbb{R} \times T_x\mathrm{M}) \times (\mathbb{R} \times T_x\mathrm{M}) \longrightarrow \mathbb{R}$ is a nondegenerate skew-symmetric 2-form on $\mathbb{R} \times T_x\mathrm{M}$. Thus the dimension of M is odd.

Let N be a submanifold of a contact manifold $(\mathrm{M}, \alpha, \omega)$. Let $i : \mathrm{N} \hookrightarrow \mathrm{M}$ be the standard inclusion of N in M. One notes

$$(\mathbb{R} \times T_x\mathrm{N})^{\omega} = \left\{ \varphi_x \in (\mathbb{R} \times T_x\mathrm{M}) / \omega_x(\varphi_x, \psi_x) = 0, \forall\, \psi_x \in (\mathbb{R} \times T_x\mathrm{N}) \right\}.$$

For every $x \in \mathrm{N}$, assume that $\dim (\mathbb{R} \times T_x\mathrm{N})^{\omega} \cap (\mathbb{R} \times T_x\mathrm{N})$ is constant.

By $(\mathbb{R} \times T\mathrm{N})^{\omega} \cap (\mathbb{R} \times T\mathrm{N})$ we denote a characteristic distribution

$$\bigcup_{x \in \mathrm{N}} (\mathbb{R} \times T_x\mathrm{N})^{\omega} \cap (\mathbb{R} \times T_x\mathrm{N})$$

which is a subbundle of tangent bundle to $\mathbb{R} \times \mathrm{N}$.

Proposition 3.1. *The characteristic distribution* $(\mathbb{R} \times T_x\mathrm{N})^{\omega} \cap (\mathbb{R} \times T_x\mathrm{N})$ *is involutive.*

Proof. Let φ, ψ be smooth sections of $(\mathbb{R} \times T\mathrm{N})^{\omega} \cap (\mathbb{R} \times T\mathrm{N})$. Let ϕ be a smooth differential operator on N. We have $\delta\omega(\varphi, \psi, \phi) = -\omega([\varphi, \psi], \phi)$. On the other hand $\delta\omega = (\delta 1 - \alpha) \wedge \omega$. Therefore $\delta\omega(\varphi, \psi, \phi) = 0$. Thus we get $\omega([\varphi, \psi], \phi) = 0$ for every differential operator on N. On the other $[\varphi, \psi]$ is a section of $\mathbb{R} \times T\mathrm{N}$, since φ and ψ are sections of $\mathbb{R} \times TN$. Therefore $[\varphi, \psi]$ is a smooth section of $(\mathbb{R} \times T\mathrm{N})^{\omega} \cap (\mathbb{R} \times T\mathrm{N})$.

4. Moment maps for exact contact manifolds

Let G be a Lie group with Lie algebra $\mathcal{G}$ which acts smoothly on M preserving ω and α. for any $\varphi \in \mathcal{G}$, we associate a differential operator φ_M on M obtained by the infinitesimal action of φ. Assume, for each $\xi \in \mathcal{G}$, a differentiable function J_ξ exists in such a way that the Hamiltonian differential operator Φ_{J_ξ} coincides with ξ_{M}. Let $J : \mathrm{M} \longrightarrow \mathcal{G}^*$ a moment map such that $\langle \xi, J(x) \rangle = J_\xi(x), x \in \mathrm{M}$. Note that J is always G-equivariant.

Let $\mathcal{G}^*_{reg}$ be the set of all regular values of J, and for each $\varrho \in \mathcal{G}^*$, let G_ϱ be the isotopy subgroup of G at ϱ.
Set $\mathrm{M}_\varrho = J^{-1}(\varrho)/\mathrm{G}_\varrho$, and let $\pi_\varrho : J^{-1}(\varrho) \longrightarrow \mathrm{M}_\varrho$ and $i_\varrho : J^{-1}(\varrho) \hookrightarrow \mathrm{M}$ be the projection and the inclusion, respectively.

For all $\xi \in \mathcal{G}$, we will implicitly assume in the whole section that $\alpha(\xi_{\mathrm{M}}) = 0$. If an element $g \in \mathcal{G}$ is regarded as a diffeomorphism of M, we set $\xi^g_{\mathrm{M}} = (g)^{-1}_* \xi_{\mathrm{M}}$.

Proposition 4.1. *The moment map* $J : \mathrm{M} \longrightarrow \mathcal{G}^*, x \longmapsto J(x)$ *is equivariant.*

Proof. By $i_{\xi^g_{\mathrm{M}}}\omega = i_{g^* J_\xi}\omega$, we have $g^* J_\xi = J_{Ad_{(g^{-1})}\xi}$, i.e., this means the equivariance of the moment map J.

We verify that $\left[\Phi_{J_\xi}, \Phi_{J_\varrho}\right] = -\Phi_{J_{[\xi,\varrho]}}$, for all $\xi, \varrho \in \mathcal{G}$.

Lemma 4.1. *Let* $\varrho \in \mathcal{G}^*_{reg}$ *and* $x \in J^{-1}(\varrho)$. *If the action of* M_ϱ *on* $J^{-1}(\varrho)$ *is free and proper, then on the tangent space* $T_x\mathrm{M}$ *of* M *at* x, *the following holds:*

(i) $T_x(\mathrm{G}_\varrho \cdot x) = T_x(\mathrm{G} \cdot x) \cap T_x(J^{-1}(\varrho))$;
(ii) *For every* $\Phi \in T_x(J^{-1}(\varrho))$ *and* $\Psi \in T_x(\mathrm{G} \cdot x)$, *then there exists an element* ξ^Ψ *in* $\mathcal{G}$ *such that* $\omega(\Phi, \Psi) = J_{\xi^\Psi} \cdot \alpha(\Phi)$.

Proof.

(i) Let $\mathcal{G}_\varrho$ be the Lie algebra of the isotopy subgroup G_ϱ. For any $\xi \in \mathcal{G}$, we have $\delta J(\xi_{\mathrm{M}}(x)) = ad(\xi)^*(\varrho)$ since J is equivariant. Hence $\xi_{\mathrm{M}}(x) \in T_x(J^{-1}(\varrho))$ iff $ad(\xi)^*(\varrho) = 0$, i.e., $\xi \in \mathcal{G}_\varrho$.
(ii) For ψ, suppose the existence of $\xi^\psi \in \mathcal{G}$ such that the associated differential operator ξ^ψ_{M} coincides with ψ. Thus $\omega(\Phi, \Psi) = -i_\Phi(i_\Psi\omega) = J_{\xi^\Psi} \cdot \alpha(\Phi)$, as desired.

Remark 4.1. *In particular* $T_x(J^{-1}(\varrho))$ *is the* ω*-orthogonal complement of* $T_x(\mathrm{G} \cdot x)$ *in* $T_x\mathrm{M}$ *iff* $J_{\xi^\Psi} \cdot \alpha(\Phi) = 0$ *for all* $\Phi \in T_x(J^{-1}(\varrho))$ *and* $\Psi \in T_x(\mathrm{G} \cdot x)$.

Proposition 4.2. *Let $f : \mathrm{M} \longrightarrow \mathbb{R}$ be a G-invariant smooth function. Let G_t be the flow of φ_t. For any point $x \in \mathrm{M}$, if either $x \in J^{-1}(0)$ or $f \in I_\alpha(\mathrm{M})$, then $J(\mathrm{G}_t(x)) = J(x)$.*

Proof. The proof of this assertion is obvious.

Theorem 4.1. (i) *Let $\varrho \in \mathcal{G}^*_{reg}$ be such that G_ϱ acts on $J^{-1}(\varrho)$ properly and freely. Assume that $i^*_\varrho \alpha = 0$. Then M_ϱ admits a unique contact structure $(\alpha_\varrho, \omega_\varrho)$ such that $\pi^*_\varrho \omega_\varrho = i^*_\varrho \omega$.*

(ii) *Suppose that $0 \in \mathcal{G}^*_{reg}$ and that the isotopy subgroup G_0 of G at 0 acts on $J^{-1}(0)$ properly and freely. Then M_0 admits a contact structure (α_0, ω_0) satisfying $\pi^*_0(\alpha_0, \omega_0) = i^*_0(\alpha_0, \omega_0)$.*

(iii) *Let $f : \mathrm{M} \longrightarrow \mathbb{R}$ be a G-invariant smooth function and G_t the flow on M of the Hamiltonian differential operator φ_t. Suppose that either $\varrho = 0$ of $f \in I_\alpha(\mathrm{M})$. Then the flow G_t canonically induces a flow $\overline{\mathrm{G}_t}$ on M_ϱ satisfying $\pi_\varrho \circ \mathrm{G}_t = \overline{\mathrm{G}_t} \circ \pi_\varrho$ and $f_\varrho \circ \pi_\varrho = f \circ i_\varrho$ for some $f_\varrho \in \mathcal{C}^\infty(\mathrm{M}_\varrho)$. Moreover f_ϱ is constant along the flow $\overline{\mathrm{G}_t}$ if $f \in I_\alpha(\mathrm{M})$.*

Proof. (i) Let $\varrho \in \mathcal{G}^*_{reg}$ and $\Phi \in T_x(J^{-1}(\varrho))$ for any $x \in \mathrm{M}$ such that $\overline{\Phi} \in T_x(J^{-1}(\varrho))/T_x(\mathrm{G}_\varrho \cdot x)$. If either $\varrho = 0$ or α vanishes on $T_x(J^{-1}(\varrho))$, then on M_ϱ, we define α_ϱ and ω_ϱ by $\alpha_\varrho(\overline{\Phi}) = \alpha(\Phi), \omega_\varrho(\overline{\Phi}, \overline{\Psi}) = \omega(\Phi, \Psi)$ for all $\Phi, \Psi \in T_x(J^{-1}(\varrho))$. We have easily $\pi^*_\varrho(\alpha_\varrho, \omega_\varrho) = i^*_\varrho(\alpha, \omega)$. Since π_ϱ and $\delta\pi_\varrho$ are surjective, we deduce the assertion. By setting $\varrho = 0$, we obtain (ii). For the axiom (iii), we use the above proposition, the fact that f is G-invariant and the surjectivity of π_ϱ.

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