INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 11 (2022), No. 3, 134-152

A geometric inequality in triangles and its applications

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Abstract. In this paper, we establish a new geometric inequality in triangles and give its some applications. We also present some interesting conjectures, which are new sharpened versions of the famous Erdös-Mordell inequality.

1. Introduction

Given a triangle ABC, let a, b, c be the side lengths of ABC, m_a, m_b, m_c the corresponding medians, h_a, h_b, h_c the altitudes, w_a, w_b, w_c the angle-bisectors, and r_a, r_b, r_c the radii of excircles. And, let s, R, r and S be its semi-perimeter, radius of circumcircle, radius of incircle and area, respectively. In addition, denote cyclic sums and products by $\sum$ and $\prod$, respectively.

In a Chinese paper [5], the author gave the following simple acute triangle inequality:

$$r_b + r_c \geq 2m_a, \tag{1.1}$$

which can be obtained from the following identity:

$$(r_b + r_c)^2 - 4m_a^2 = \frac{2(b-c)^2(b^2+c^2-a^2)}{(c+a-b)(a+b-c)}. \tag{1.2}$$

Clearly, equality in (1.1) holds if and only if $b = c$ or $A = \pi/2$.

In the monograph [7], the author gave some applications of inequality (1.1). For example, by using (1.1) the author proved that for the acute triangle ABC the following two inequalities hold (see [7, pp.52-53]):

$$\cos B + \cos C \leq \frac{2r_a}{m_a + r_a}, \tag{1.3}$$

$$\cos B + \cos C \leq \frac{2r_a}{m_b + m_c}. \tag{1.4}$$

We note that the later actually holds for any triangle ABC. This actuates the author to study upper bounds of the single median m_a for any triangle ABC and finds the following result:

Keywords and phrases: Triangle, Median, the fundamental triangle inequality, the Gerretsen inequality, the Erdös-Mordell inequality
(2010)Mathematics Subject Classification: 51M16
Received: 11.01.2022. In revised form: 12.05.2022. Accepted: 07.04.2022.

Theorem 1.1. *In any triangle ABC we have*

$$m_a \le h_a + R\left(\frac{b-c}{a}\right)^2, \tag{1.5}$$

with equality if and only if $b = c$ or $A = \pi/2$.

By the well known relation:

$$R = \frac{abc}{4S}, \tag{1.6}$$

and the formula $h_a = 2S/a$, one can see that inequality (1.5) has the following equivalent forms:

$$m_a \le h_a + \frac{bc(b-c)^2}{4aS}, \tag{1.7}$$

$$m_a \le \frac{8S^2 + bc(b-c)^2}{4aS}, \tag{1.8}$$

$$\frac{m_a}{h_a} \le 1 + \frac{bc(b-c)^2}{8S^2}. \tag{1.9}$$

In fact, we can easily prove inequality (1.5). However, it is worth noticing that a lot of inequalities involving medians of a triangle can be proved by applying inequality (1.5). In Section 3, we shall give some examples.

Inspired by the first proof of (1.5) given in the next section, the author presented some interesting conjectures related to the famous Erdös-Mordell inequality, we shall introduce them in the last section.

2. Two proofs of Theorem 1.1

In this section, we shall give two proofs of Theorem 1.1. The first proof is as follows:

Proof. Firstly, by the simplest arithmetic-geometric mean inequality, we get

$$\frac{1}{4}\left(r_b + r_c + \frac{4m_a^2}{r_b + r_c}\right) \ge m_a. \tag{2.1}$$

Again, by the formula $r_a = S/(s-a)$ we get

$$r_b + r_c = \frac{aS}{(s-b)(s-c)}. \tag{2.2}$$

Consequently, using the known median formula:

$$4m_a^2 = 2(b^2 + c^2) - a^2, \tag{2.3}$$

Heron's formula:

$$S = \sqrt{s(s-a)(s-b)(s-c)} \tag{2.4}$$

and its equivalent form

$$2\sum b^2c^2 - \sum a^4 = 16S^2, \tag{2.5}$$

we have

$$
\begin{aligned}
m_a &\le \frac{aS}{4(s-b)(s-c)} + \frac{(s-b)(s-c)\left[2b^2+2c^2-a^2\right]}{4aS}\\
&= \frac{a^2S^2+(2b^2+2c^2-a^2)(s-b)^2(s-c)^2}{4a(s-b)(s-c)S}\\
&= \frac{a^2s(s-a)+(s-b)(s-c)(2b^2+2c^2-a^2)}{4aS}\\
&= \frac{a^2\left[(b+c)^2-a^2\right]+(2b^2+2c^2-a^2)\left[a^2-(b-c)^2\right]}{16aS}\\
&= \frac{2\sum b^2c^2-\sum a^4+2bc(b^2+c^2)-4b^2c^2}{8aS}\\
&= \frac{16S^2+2bc(b-c)^2}{8aS}\\
&= \frac{2S}{a}+\frac{2bc(b-c)^2}{4aS}\\
&= h_a + R\left(\frac{b-c}{a}\right)^2,
\end{aligned}
$$

where the last step used $h_a = 2S/a$ and relation (1.6). Thus, inequality (1.5) is proved. Note that the equality in (2.1) occurs if and only if

$$r_b + r_c = \frac{4m_a^2}{r_b+r_c},$$

i.e.,$r_b+r_c = 2m_a$. Further, by the identity (1.2), we deduce that the equality in (1.5) holds if and only if $b = c$ or $A = \pi/2$. This completes the proof of Theorem 1.1.

In the above proof, we actually have proved the following identity:

$$\frac{1}{4}\left(r_b+r_c+\frac{4m_a^2}{r_b+r_c}\right) = h_a + R\left(\frac{b-c}{a}\right)^2, \tag{2.6}$$

which together with (2.1) shows that inequality (1.5) holds.

Now, we give the second proof of Theorem 1.1 as follows:

Proof. Firstly, we use the median formula (2.3) and $ah_a = 2S$ to obtain

$$4a^2(m_a^2-h_a^2) = a^2(2b^2+2c^2-a^2)-16S^2. \tag{2.7}$$

But

$$a^2(2b^2+2c^2-a^2)-16S^2 = (b^2-c^2)^2,$$

which is equivalent to identity (2.5). Then, we get

$$m_a^2-h_a^2 = \frac{(b^2-c^2)^2}{4a^2}. \tag{2.8}$$

Using this identity, the known relation $2Rh_a = bc$ and (1.6), we have

$$\begin{aligned}
&\left[h_a + R\left(\frac{b-c}{a}\right)^2\right]^2 - m_a^2 \\
&= h_a^2 - m_a^2 + 2Rh_a\left(\frac{b-c}{a}\right)^2 + R^2\left(\frac{b-c}{a}\right)^4 \\
&= -\frac{(b^2-c^2)^2}{4a^2} + bc\left(\frac{b-c}{a}\right)^2 + R^2\left(\frac{b-c}{a}\right)^4 \\
&= \left(\frac{b-c}{a}\right)^2\left[-\frac{1}{4}(b+c)^2 + bc + R^2\left(\frac{b-c}{a}\right)^2\right] \\
&= \frac{(b-c)^4(4R^2-a^2)}{4a^4}.
\end{aligned}$$

Note that $a = 2R\sin A$, we obtain the following identity:

$$\left[h_a + R\left(\frac{b-c}{a}\right)^2\right]^2 - m_a^2 = \frac{(b-c)^4}{a^4}R^2\cos^2 A, \tag{2.9}$$

which clearly implies that inequality (1.5) holds and its equality if and only if $b = c$ or $\cos A = 0$, i.e., $A = \pi/2$. This completes the proof of Theorem 1.1.

Remark 2.1. *For inequality* (1.5)*, we have the following reverse inequality:*

$$m_a \geq h_a + 2r\left(\frac{b-c}{a}\right)^2, \tag{2.10}$$

which could be proved easily.

Remark 2.2. *For the acute triangle ABC, the author proved the following reverse inequality* (1.5)*:*

$$m_a \geq h_a + \frac{24}{25}R\left(\frac{b-c}{a}\right)^2. \tag{2.11}$$

3. Applications of Theorem 1.1

In this section, we discuss applications of inequality (1.5) and its equivalent forms. The following corollaries are our results of applying inequality (1.5).

For simplicity, we shall omit the details of deducing some identities in a triangle.

Corollary 3.1. *In an acute triangle ABC, we have*

$$m_a \leq \frac{(h_b+h_c)^2}{2a^2-(b-c)^2}R, \tag{3.1}$$

with equality if and only if $b = c$ or $A = \pi/2$.

Proof. By Theorem 1.1, we need to show that the following inequality

$$h_a + R\left(\frac{b-c}{a}\right)^2 \le \frac{(h_b+h_c)^2}{2a^2-(b-c)^2}R \tag{3.2}$$

holds for the acute triangle ABC. Using the formula $h_a = 2S/a$, the previous identity (1.6) and Heron's formula (2.4), we can easily obtain the following identity:

$$\begin{aligned} &\frac{(h_b+h_c)^2}{2a^2-(b-c)^2}R - h_a - R\left(\frac{b-c}{a}\right)^2 \\ &= \frac{(b^2+c^2-a^2)(b-c)^2 D_0}{16abcS(2a^2+2bc-b^2-c^2)}, \end{aligned} \tag{3.3}$$

where

$$D_0 = a^4 - (a^2+2bc)(b-c)^2.$$

Since D_0 can be rewritten as

$$D_0 = \frac{1}{2}(c^2+a^2-b^2)(a^2+b^2-c^2) + \frac{1}{2}(c+a-b)^2(a+b-c)^2,$$

which can be verified by expanding. Thus, $D_0 > 0$ holds for the acute triangle ABC. Hence, inequality (3.2) follows from (3.3) and inequality (3.1) is proved. Also, from (3.3) we see that the equality condition of (3.1) is the same as that of (1.1). This completes the proof of Corollary 3.1.

Remark 3.1. *It is easy to prove that inequality* (3.1) *is stronger than inequality* (1.1)*. In fact, for the acute triangle ABC we have the following inequality chain:*

$$\begin{aligned} m_a &\le \frac{(h_b+h_c)^2}{2a^2-(b-c)^2}R \le \sqrt{\frac{1}{2}(b^2+c^2)}\,\cos\frac{A}{2} \\ &\le R\left(1+\cos A\cos^2\frac{B-C}{2}\right) \le \frac{1}{2}(r_b+r_c), \end{aligned} \tag{3.4}$$

in which the inequality

$$m_a \le R\left(1+\cos A\cos^2\frac{B-C}{2}\right) \tag{3.5}$$

was first proved by Z.Y.Deng in [3].

Corollary 3.2. *For any triangle ABC, inequality* (1.4) *holds.*

Proof. Adding two inequalities similar to (1.8), we get

$$m_b + m_c \le h_b + h_c + \frac{ca(c-a)^2}{4bS} + \frac{ab(a-b)^2}{4cS}. \tag{3.6}$$

Using $h_b = 2S/b, h_c = 2S/c$ and Heron's formula, we further obtain

$$m_b + m_c \le \frac{E_0}{8bcS}, \tag{3.7}$$

where

$$\begin{aligned} E_0 = &-(b+c)a^4 + 2(b^2+c^2)a^3 - 2(b+c)(b-c)^2a^2 \\ &+ 2(b^4+c^4)a - (b-c)^2(b+c)^3. \end{aligned}$$

On the other hand, by using the law of cosine we easily get

$$\cos B + \cos C = \frac{(b+c)(c+a-b)(a+b-c)}{2abc}. \tag{3.8}$$

Thus, to prove inequality (1.4) we need to show that

$$2r_a \geq \frac{(b+c)(c+a-b)(a+b-c)}{2abc} \cdot \frac{E_0}{8bcS}.$$

Since $r_a = 2S/(b+c-a)$, the above inequality is equivalent to

$$64a(bc)^2S^2 - (b+c)(b+c-a)(c+a-b)(a+b-c)E_0 \geq 0.$$

And, by Heron's formula and $s = (a+b+c)/2$, we only need to prove

$$4a(a+b+c)(bc)^2 - (b+c)E_0 \geq 0.$$

Substituting the expression of E_0 into this inequality and arranging gives

$$\begin{aligned} E_1 \equiv\, & (b+c)^2a^4 - 2(b+c)(b^2+c^2)a^3 + 2(b^4+c^4)a^2 \\ & - 2(b-c)^2(b+c)^3a + (b-c)^2(b+c)^4 \geq 0, \end{aligned} \tag{3.9}$$

which is required to prove. But it is easy to verify the following identity:

$$E_1 = (b+c)^2(b-c)^2(b+c-a)^2 + a^2(b^2+c^2-ab-ac)^2, \tag{3.10}$$

which shows that inequality (3.9) is true. Hence, Corollary 3.2 is proved.

Corollary 3.3. *In any triangle ABC, we have*

$$\frac{1}{h_a} - \frac{1}{m_a} \leq \frac{1}{2r} - \frac{1}{R}. \tag{3.11}$$

Proof. The above inequality is equivalent to

$$\frac{1}{m_a} \geq \frac{1}{h_a} - \frac{1}{2r} + \frac{1}{R}.$$

In view of inequality (1.8), $h_a = 2S/a, r = S/s$ and identity (1.6), we only need to prove

$$\frac{4aS}{8S^2 + bc(b-c)^2} \geq \frac{a-s}{2S} + \frac{4S}{abc},$$

that is

$$F_0 \equiv 8bca^2S^2 - \left[(a-s)abc + 8S^2\right]\left[8S^2 + bc(b-c)^2\right] \geq 0,$$

Using Heren's formula and $s = (a+b+c)/2$, we easily get

$$F_0 = \frac{1}{4}(b+c-a)F_1, \tag{3.12}$$

where

$$\begin{aligned} F_1 =\, & a^7 + (b+c)a^6 - 3(b^2-bc+c^2)a^5 - (b+c)(3b^2-4bc+3c^2)a^4 \\ & + (3b^4 - 6b^3c + 10b^2c^2 - 6bc^3 + 3c^4)a^3 + (b+c)(3b^2 - 2bc \\ & + 3c^2)(b-c)^2a^2 - (b^2+c^2)(b^2-bc+c^2)(b-c)^2a \\ & - (b+c)(b^2+c^2)(b-c)^4. \end{aligned}$$

Thus, we only need to show that $F_1 \geq 0$. Letting $b+c-a = 2x, c+a-b = 2y, a+b-2c = z$, then $a = y+z, b = z+x, c = x+y$. Substituting them into F_1, we obtain

$$F_1 = 2F_2, \tag{3.13}$$

where

$$\begin{aligned}F_2 =&(y+z)(y^2+6yz+z^2)x^4+(2y^4+16y^3z-4y^2z^2+16yz^3+2z^4)x^3\\&+(y+z)(y^4+10y^3z-46y^2z^2+10yz^3+z^4)x^2+2(y^4-12y^3z\\&-10y^2z^2-12yz^3+z^4)yzx+(y+z)(y^2+30yz+z^2)y^2z^2.\end{aligned}$$

Through analysis, we obtain the following identity:

$$\begin{aligned}F_2 =&2yz(x+y+z)\left[x(y+z)(y+z-2x)^2+4(xy+xz-2yz)^2\right]\\&+\left[(y+z)x^4+2(y^2+6yz+z^2)x^3+(y+z)(y^2+10yz+z^2)x^2\right.\\&\left.+y^2z^2(y+z)\right](y-z)^2.\end{aligned}\tag{3.14}$$

Since $x, y, z > 0$, we have $F_2 \geq 0$. This completes the proof of Corollary 3.3.

Remark 3.2. *Note that the known identity:*

$$\frac{1}{r_a}+\frac{2}{h_a}=\frac{1}{r},\tag{3.15}$$

we know that inequality (3.11) *is equivalent to*

$$\frac{1}{r_a}+\frac{2}{m_a}\geq\frac{2}{R},\tag{3.16}$$

which was proposed by Liu B.Q. and proved by Zhang X.M. (see [19, p.570]).

Corollary 3.4. *In any triangle ABC, the following inequality holds:*

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}\leq\frac{R}{r}.\tag{3.17}$$

Proof. Using the two inequalities corresponding to (1.8), $h_b = 2S/b$ and $h_c = 2S/c$, we get

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}\leq\frac{8S^2+ca(c-a)^2}{8S^2}\cdot\frac{c}{b}+\frac{8S^2+ab(a-b)^2}{8S^2}\cdot\frac{b}{c}.$$

Further, using the previous formula (2.5) we obtain

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}\leq\frac{G_0}{16bcS^2},\tag{3.18}$$

where

$$\begin{aligned}G_0 =&-(b^2+c^2)a^4+2(b+c)(b^2-bc+c^2)a^3-2(b-c)^2(b+c)^2a^2\\&+2(b+c)(b^4-b^3c+b^2c^2-bc^3+c^4)a-(b^2+c^2)(b-c)^2(b+c)^2.\end{aligned}$$

Thus, to prove (3.17) we need to show that

$$\frac{G_0}{16bcS^2}\leq\frac{R}{r}.$$

Note that Hereon's formula and the following known identity:

$$\frac{R}{r}=\frac{abc}{4(s-a)(s-b)(s-c)}.\tag{3.19}$$

The claimed inequality becomes

$$G_1\equiv 2a(a+b+c)(bc)^2-G_0\geq 0.\tag{3.20}$$

But it is easy to obtain the following identity:

$$G_1 \equiv (b^2+c^2)a^4 - 2(b+c)(b^2-bc+c^2)a^3 + (2b^4-2b^2c^2 + 2c^4)a^2 - 2(b+c)(b^2+bc+c^2)(b-c)^2a + (b^2+c^2)(b-c)^2(b+c)^2 \geq 0. \tag{3.21}$$

Through analysis, we find that G_1 can be rewritten as

$$G_1 = \frac{1}{2}(b-c)^2(b+c-a)^4 + \frac{1}{2}\left[(b+c)a^2 - 2bca - (b+c)(b-c)^2\right]^2, \tag{3.22}$$

which is easily verified by expanding and shows that inequality (3.20) holds. This completes the proof of Corollary 3.4.

Remark 3.3. *Inequality* (3.17) *was proposed as a conjecture by Liu B.Q. in his book [15]. The author [8] first gave a proof, which is more complicated than the above proof. In addition, by using Potlemy's inequality (see [18]) we can prove that inequality* (3.17) *is stronger than the following Panaitopol's inequality (see [16, p.216]):*

$$2\frac{m_a}{h_a} \leq \frac{R}{r}. \tag{3.23}$$

Corollary 3.5. *In any triangle ABC, the following inequality holds:*

$$m_a - h_a \leq \frac{4}{3}(R-2r). \tag{3.24}$$

Proof. By Theorem 1.1, we need to show that

$$R\left(\frac{b-c}{a}\right)^2 \leq \frac{4}{3}r\left(\frac{R}{r}-2\right).$$

Using identity (3.19), we easily know that the claimed inequality is equivalent to

$$4bca^2 - 32a(s-a)(s-b)(s-c) - 3bc(b-c)^2 \geq 0. \tag{3.25}$$

Putting $s-a=x, s-b=y, s-c=z$, then we have $a=y+z, b=z+x, c=x+y (x,y,z>0)$. Substituting them into (3.25) gives the following algebraic inequality:

$$4(z+x)(x+y)(y+z)^2 - 32(y+z)xyz - 3(z+x)(x+y)(y-z)^2 \geq 0,$$

that is

$$(y^2+14yz+z^2)x^2 + (y+z)(y^2-18yz+z^2)x + yz(y^2+14yz+z^2) \geq 0. \tag{3.26}$$

If $y^2-18yz+z^2 \geq 0$, then the above inequality is clearly true. If $y^2-18yz+z^2 < 0$, then it is easy to compute the discriminant F_x of quadratic function (in x) of the left hand side of (3.26), which is given by

$$F_x = (y^2-34yz+z^2)(y-z)^4.$$

And we have $F_x \leq 0$ under the the condition $y^2-18yz+z^2<0$. Therefore, inequality (3.26) holds for all positive real numbers x,y,z. So inequality (3.25) is proved. This completes the proof of Corollary 3.5.

Next, we shall prove another linear inequality similar to (3.24).

Corollary 3.6. *In any triangle ABC, the following inequality holds:*

$$m_b + m_c - (h_b+h_c) \leq 2(R-2r). \tag{3.27}$$

Proof. By Theorem 1.1, to prove (3.27) we need to show that

$$2(R-2r) \geq \frac{ca(c-a)^2}{4bS} + \frac{ab(a-b)^2}{4cS}.$$

Multiplying both sides of this inequality by $4S$, and then using the previous relation (1.6), the known identity:

$$(s-a)(s-b)(s-c) = rS, \tag{3.28}$$

and $s = (a+b+c)/2$, it becomes

$$2abc - 2(b+c-a)(c+a-b)(a+b-c) \geq \frac{ca(c-a)^2}{b} + \frac{ab(a-b)^2}{c}.$$

i.e.,

$$\begin{aligned} H_0 \equiv\, & 2a(bc)^2 - 2bc(b+c-a)(c+a-b)(a+b-c) \\ & - ac^2(c-a)^2 - ab^2(a-b)^2 \geq 0. \end{aligned} \tag{3.29}$$

But H_0 can be rewritten as

$$H_0 = \frac{1}{2}(b+c-a)(b-c)^2\left[(c+a-b)(a+b-c) + (b+c-a)^2\right], \tag{3.30}$$

which implies the claimed inequality (3.29) holds. This completes the proof of inequality (3.27).

Corollary 3.7. *In any triangle ABC, the following inequality holds:*

$$\sum \frac{r_a}{m_b + m_c} a^2 \geq \frac{1}{2} \sum a^2. \tag{3.31}$$

Proof. According to inequality (3.27), to prove (3.31) it is enough to show that

$$\sum \frac{r_a}{h_b + h_c + 2(R-2r)} a^2 \geq \frac{1}{2} \sum a^2. \tag{3.32}$$

Also, we easily prove the following two identities:

$$\prod [h_b + h_c + 2(R-2r)] = \frac{M_1}{2R^2}, \tag{3.33}$$

$$\sum a^2 r_a [h_c + h_a + 2(R-2r)][h_a + h_b + 2(R-2r)] = \frac{4s^2}{R} N_1, \tag{3.34}$$

where

$$\begin{aligned} M_1 &= Rs^4 + (8R^3 - 16R^2 r + 6Rr^2 - 2r^3)s^2 + (R-2r)(2R-r)^4, \\ N_1 &= (R-r)Rs^2 + 4R^4 - 12R^3 r + 15R^2 r^2 - 6Rr^3 + 2r^4. \end{aligned}$$

It follows from (3.33) and (3.34) that

$$\sum \frac{r_a}{h_b + h_c + 2(R-2r)} a^2 = \frac{8Rs^2 N_1}{M_1}. \tag{3.35}$$

By identity (3.33) and Euler's inequality:

$$R \geq 2r, \tag{3.36}$$

one sees that that $M_1 > 0$. Hence, by the well known identity:

$$\sum a^2 = 2(s^2 - 4Rr - r^2), \tag{3.37}$$

to prove inequality (3.32) we have to prove that

$$8Rs^2N_1 - (s^2 - 4Rr - r^2)M_1 \geq 0.$$

Using the expressions of M_1 and N_1, we further know that the above inequality is equivalent to

$$\begin{aligned} I_0 \equiv & - Rs^6 + (12R^2 - 5Rr + 2r^2)rs^4 + (2R + 3r)(2R - r)^3 Rs^2 \\ & + (4R + r)(R - 2r)(2R - r)^4 r \geq 0. \end{aligned} \tag{3.38}$$

Now, recall that for any triangle ABC we have the following fundamental triangle inequality (see [1] and [16, pp.1-10]):

$$t_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \geq 0, \tag{3.39}$$

and Gerretsen's inequality (see [1]):

$$g_2 \equiv 4R^2 + 4Rr + 3r^2 - s^2 \geq 0. \tag{3.40}$$

Based on the above two inequalities, we rewrite I_0 as follows:

$$I_0 = m_1 t_0 + g_2 m_2 + m_3, \tag{3.41}$$

where

$$\begin{aligned} m_1 &= Rs^2 + 4R^3 + 8R^2 r + 3Rr^2 - 2r^3, \\ m_2 &= 4(12R^4 + 35R^3 r + 2R^2 r^2 - 11Rr^3 + r^4)r, \\ m_3 &= 8(R - 2r)(16R^5 - 4R^4 r + 14R^3 r^2 - 9R^2 r^3 - 6Rr^4 + r^5)r. \end{aligned}$$

By Euler's inequality (3.36), one sees that $m_1 > 0, m_2 > 0$ and $m_3 \geq 0$ are valid. Thus, from (3.39)-(3.41), we conclude that (3.38) holds. This completes the proof of Corollary 3.7.

Remark 3.4. *Inequalities* (3.31), (3.66) *and* (3.78) *below were given in the monographs [7], where the author only proved that these three inequalities are valid for the acute triangle ABC.*

The following acute inequality (3.42) was established by the author in [6]. Here, we shall use Theorem 1.1 to give a new proof.

Corollary 3.8. *In the acute triangle ABC, we have*

$$\frac{\sum h_a}{\sum m_a} \geq \frac{1}{2} + \frac{r}{R}. \tag{3.42}$$

Proof. By Theorem 1.1 we have

$$\sum m_a \leq \sum h_a + R \sum \left(\frac{b-c}{a}\right)^2. \tag{3.43}$$

Thus, to prove inequality (3.42) we need to show

$$\sum h_a \geq \left(\frac{1}{2} + \frac{r}{R}\right) \left[\sum h_a + R \sum \left(\frac{b-c}{a}\right)\right]^2,$$

i.e.,

$$J_0 \equiv \left(\frac{1}{2} - \frac{r}{R}\right) \sum h_a - \left(\frac{1}{2} + \frac{r}{R}\right) R \sum \left(\frac{b-c}{a}\right)^2 \geq 0, \tag{3.44}$$

But we have the following known identity:

$$\sum h_a = \frac{s^2+4Rr+r^2}{2R}, \tag{3.45}$$

and it is easy to prove the following identity:

$$\sum \left(\frac{b-c}{a}\right)^2 = \frac{-s^4+2(6R^2+2Rr-r^2)s^2-r(4R+r)^3}{4R^2s^2}. \tag{3.46}$$

Substituting (3.45) and (3.46) into J_0, we further obtain

$$J_0 = \frac{J_1}{8s^2R^2}, \tag{3.47}$$

where

$$J_1 = (3R-2r)s^4 - 4(3R^2+5Rr+5r^2)Rs^2 + (R+2r)(4R+r)^3 r.$$

Also, we can rewrite J_1 as

$$J_1 = 2rt_0 + 3R\left[s^2-(2R+r)^2\right]\left[s^2-(2R^2+8Rr+3r^2)\right] + J_2, \tag{3.48}$$

where t_0 is the same as in (3.39) and

$$\begin{aligned} J_2 = {} & (6R^3+8R^2r-48Rr^2+4r^3)s^2 - 24R^5 - 56R^4r \\ & + 166R^3r^2 + 144R^2r^3 + 40Rr^4 + 4r^5. \end{aligned}$$

Note that in the acute (non-obtuse) triangle ABC we have Ciamberlini's inequality (see [2]):

$$s \geq 2R + r \tag{3.49}$$

(with equality if and only if $\triangle ABC$ is a right triangle) and Walker's inequality (cf. [16] and [9]):

$$s^2 \geq 2R^2 + 8Rr + 3r^2, \tag{3.50}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles. Also, for any triangle ABC we have the fundamental inequality (3.39). Hence, by (3.48) it remains to show that $J_2 \geq 0$ holds for the acute triangle ABC. We consider two cases to finish the proof of this inequality.

Case 1. $6R^3 + 8R^2r - 48Rr^2 + 4r^3 \geq 0$.

In this case, by (3.49) and Euler's inequality we have

$$\begin{aligned} J_2 \geq {} & (6R^3+8R^2r-48Rr^2+4r^3)(2R+r)^2 - 24R^5 \\ & - 56R^4r + 166R^3r^2 + 144R^2r^3 + 40Rr^4 + 4r^5 \\ = {} & 4r^2(3R^3 - 6R^2r + 2Rr^2 + 2r^3) > 0. \end{aligned}$$

Case 2. $6R^3 + 8R^2r - 48Rr^2 + 4r^3 < 0$.

In this case, by Gerretsen's inequality (3.40) and Euler's inequality we have

$$\begin{aligned} J_2 \geq {} & (6R^3+8R^2r-48Rr^2+4r^3)(4R^2+4Rr+3r^2) - 24R^5 \\ & - 56R^4r + 166R^3r^2 + 144R^2r^3 + 40Rr^4 + 4r^5 \\ = {} & 8(R-2r)(3R^2+5Rr-r^2)r^2 \geq 0. \end{aligned}$$

Combining the arguments of the above two cases, we conclude that $J_2 \geq 0$ holds for all acute triangles. This completes the proof of inequality (3.42).

Remark 3.5. *In [6], the author established the following linear inequality:*

$$\sum m_a - \sum h_a \leq 2(R - 2r). \tag{3.51}$$

Comparing this inequality with (3.43)*, the author finds that the following inequality*

$$\sum \left(\frac{b-c}{a}\right)^2 \geq 2\left(1 - \frac{2r}{R}\right) \tag{3.52}$$

is equivalent to the fundamental triangle inequality (3.39)*. In fact, this statement can be showed by using identity* (3.46)*. The author also gave two other equivalent forms of the fundamental triangle inequality in the recent paper [14].*

Corollary 3.9. *In any triangle ABC, the following inequality holds:*

$$\sum \frac{1}{m_a} \geq \frac{5}{2R + r}. \tag{3.53}$$

Proof. We set

$$\begin{cases} q_a = h_a + R\left(\dfrac{b-c}{a}\right)^2, \\ q_b = h_b + R\left(\dfrac{c-a}{b}\right)^2, \\ q_c = h_c + R\left(\dfrac{a-b}{c}\right)^2. \end{cases} \tag{3.54}$$

By Theorem 1.1, to prove (3.53) we only need to show that

$$\sum \frac{1}{q_a} \geq \frac{5}{2R + r}. \tag{3.55}$$

It is not difficult to prove the following two identities:

$$\prod q_a = \frac{M_2}{4Rs^2}, \tag{3.56}$$

$$\sum q_b q_c = \frac{N_2}{2Rs^2}, \tag{3.57}$$

where

$$\begin{aligned} M_2 =& (R + 2r)^2 s^4 + 4(R^2 - 3Rr - r^2)(R + r)^2 s^2 - (4R + r)^3 R^2 r, \\ N_2 =& (R + 2r)s^4 + 2(R + r)(3R^2 - Rr - r^2)s^2 - (4R + r)^3 Rr. \end{aligned}$$

It follows from (3.56) and (3.57) that

$$\sum \frac{1}{q_a} = \frac{2N_3}{M_2}. \tag{3.58}$$

Therefore, to prove (3.55) we need to show

$$\frac{2N_2}{M_2} - \frac{5}{2R + r} \geq 0,$$

i.e.,

$$2(2R + r)N_2 - 5M_2 \geq 0.$$

Using the expression of M_2 and N_2 gives

$$-(R+8r)(R+2r)s^4+4(R+r)(R^3+11R^2r+17Rr^2+4r^3)s^2$$
$$+Rr(R-2r)(4R+r)^3\geq 0,$$

which is required to prove. Dividing both sides of this inequality by s^2 and applying the known result (see [1]):

$$(4R+r)^2\geq 3s^2, \tag{3.59}$$

we only need to prove

$$-(R+8r)(R+2r)s^2+4(R+r)(R^3+11R^2r+17Rr^2+4r^3)$$
$$+3Rr(R-2r)(4R+r)\geq 0,$$

which can be written as

$$(R+8r)(R+2r)(4R^2+4Rr+3r^2-s^2)$$
$$+16(R-2r)(R^2+Rr+r^2)r\geq 0.$$

By Euler's inequality $R\geq 2r$ and Gerretsen's inequality (3.40), one sees that the above inequality holds. This completes the proof of inequality (3.53).

Remark 3.6. *Ciamberlini's inequality* (3.49) *shows that for the acute triangle ABC inequality* (3.53) *is better than the following Janous's inequality (see [4]):*

$$\sum\frac{1}{m_a}>\frac{5}{s}, \tag{3.60}$$

which is valid for any triangle ABC.

Corollary 3.10. *In any triangle ABC, the following inequality holds:*

$$\sum\frac{r_b+r_c}{m_a+r_a}\geq 3. \tag{3.61}$$

Proof. Applying Cauchy's inequality, we have

$$\sum\frac{r_b+r_c}{m_a+r_a}\geq\frac{\left[\sum(r_b+r_c)\right]^2}{\sum(r_b+r_c)(m_a+r_a)}=\frac{4\left(\sum r_a\right)^2}{\sum(r_b+r_c)(m_a+r_a)}.$$

Thus, to prove (3.61) we need to show that

$$4\left(\sum r_a\right)^2-3\sum(r_b+r_c)(m_a+r_a)\geq 0,$$

i.e.,

$$4\left(\sum r_a\right)^2-3\sum(r_b+r_c)m_a-6\sum r_br_c\geq 0.$$

By Theorem 1.1 and the following two known identities:

$$\sum r_a=4R+r, \tag{3.62}$$

$$\sum r_br_c=s^2, \tag{3.63}$$

we only need to prove

$$4(4R+r)^2-6s^2-3\sum(r_b+r_c)q_a\geq 0, \tag{3.64}$$

where q_a, q_b, q_c are given by (3.54). One can easily prove the following identity:

$$\sum (r_b + r_c) q_a = s^2 + 8R^2 - 2Rr - r^2. \tag{3.65}$$

Thus, it remains to show that

$$4(4R+r)^2 - 6s^2 - 3(s^2 + 8R^2 - 2Rr - r^2) \geq 0,$$

i.e.,

$$2(2R+5r)(R-2r) + 9(4R^2 + 4Rr + 3r^2 - s^2) \geq 0,$$

which follows from Euler's inequality $R \geq 2r$ and Gerretsen's inequality (3.40). This completes the proof of inequality (3.61).

Corollary 3.11. *In any triangle ABC, the following inequality holds:*

$$\sum (m_a + r_a) \cos A \leq \sum h_a. \tag{3.66}$$

Proof. Since $1 + \cos A = 2\cos^2 \frac{A}{2}$, we see that inequality (3.67) is equivalent to

$$2\sum m_a \cos^2 \frac{A}{2} + \sum r_a \cos A \leq \sum m_a + \sum h_a.$$

By Theorem 1.1, to prove this inequality we need to prove

$$2\sum \left[h_a + R\left(\frac{b-c}{a}\right)^2 \right] \cos^2 \frac{A}{2} + \sum r_a \cos A - \sum h_a \leq \sum m_a,$$

which is equivalent to

$$\begin{aligned} \sum m_a \geq {} & \sum (r_a + h_a) \cos A - \sum h_a \\ & + 2R \sum \left(\frac{b-c}{a}\right)^2 \cos^2 \frac{A}{2}. \end{aligned} \tag{3.67}$$

Again, it is easy to prove the following identities:

$$\sum (r_a + h_a) \cos A = \frac{3s^2 - 8R^2 - 6Rr - r^2}{2R}, \tag{3.68}$$

$$\sum \left(\frac{b-c}{a}\right)^2 \cos^2 \frac{A}{2} = \frac{8R^2 - 2Rr - r^2 - s^2}{4R^2}. \tag{3.69}$$

Using these two identities and (3.45), inequality (3.67) can be simplified to

$$\sum m_a \geq \frac{s^2 - 4Rr - r^2}{R}. \tag{3.70}$$

From the previous identity (3.37), one sees this inequality is equivalent to the known result (see [16, p.213]):

$$\sum m_a \geq \frac{1}{2R} \sum a^2. \tag{3.71}$$

Therefore, inequality (3.66) is proved.

Corollary 3.12. *In any triangle ABC, the following inequality holds:*

$$\sum \frac{a^2}{m_a r_a} \geq 4. \tag{3.72}$$

Proof. By Theorem 1.1, we need to show that

$$\sum \frac{a^2}{r_a q_a} \geq 4. \tag{3.73}$$

where q_a, q_b, q_c are given by (3.54). It is easy to obtain the following identities:

$$\prod r_a q_a = \frac{r}{4R} M_3, \tag{3.74}$$

$$\sum a^2 r_b r_c q_b q_c = \frac{r}{R} N_3, \tag{3.75}$$

where

$$\begin{aligned} M_3 =&(R+2r)^2 s^4 + 4(R^2 - 3Rr - r^2)(R+r)^2 s^2 - (4R+r)^3 R^2 r, \\ N_3 =&(R^2 + 5Rr + 3r^2)s^4 + (4R^4 - 4R^3 r - 48R^2 r^2 - 26Rr^3 \\ &- 4r^4)s^2 - (R^2 - Rr - r^2)(4R+r)^3 r. \end{aligned}$$

Thus, it follows from (3.74) and (3.75) that

$$\sum \frac{a^2}{r_a q_a} = \frac{4N_3}{M_3}. \tag{3.76}$$

And, we see that inequality (3.73) is equivalent to $N_3 - M_3 \geq 0$, which can be simplified to

$$(R-r)s^4 - 6(4R+r)s^2 Rr + (R+r)(4R+r)^3 r \geq 0.$$

In view of the previous inequality (3.59), we only need to show

$$(R-r)s^4 - 6(4R+r)s^2 Rr + 3(R+r)(4R+r)rs^2 \geq 0,$$

i.e.,

$$(R-r)s^2 \left[s^2 - 3r(4R+r)\right] \geq 0.$$

This is true since we have $R \geq 2r$ and the following known inequality (see [1]):

$$s^2 \geq 3(4R+r)r. \tag{3.77}$$

Consequently, inequality (3.72) is proved.

Corollary 3.13. *In any triangle ABC, the following inequality holds:*

$$\sum \frac{r_a}{(m_b + m_c)\sin^2 A} \geq 2. \tag{3.78}$$

Proof. By Theorem 1.1, we only need to prove that

$$\sum \frac{r_a}{(q_b + q_c)\sin^2 A} \geq 2, \tag{3.79}$$

where q_a, q_b, q_c are given by (3.54). But we can prove the following identity:

$$\sum r_a (q_c + q_a)(q_a + q_b)(\sin B \sin C)^2 = \frac{rN_4}{256 R^6 s^4}, \tag{3.80}$$

$$\prod (q_b + q_c)\sin^2 A = \frac{r^2 M_4}{32 R^6 s^2}, \tag{3.81}$$

where

$$\begin{aligned}M_4 =&(R+2r)s^8 + (16R^3 + 32R^2r + 12Rr^2 - 2r^3)s^6\\&+ 2(R+r)(32R^4 - 32R^3r - 56R^2r^2 - 20Rr^3\\&- r^4)s^4 - 2r(8R^3 + 8R^2r - 2Rr^2 - r^3)(4R+r)^3s^2\\&+ Rr^2(4R+r)^6,\\N_4 =&s^{12} + (16R^2 + 4Rr + 2r^2)s^{10} + (64R^4 - 64R^3r - 64R^2r^2\\&+ 20Rr^3 - r^4)s^8 - 4(64R^5 - 224R^4r - 240R^3r^2 - 64R^2r^3\\&+ 6Rr^4 + r^5)rs^6 + (64R^4 + 192R^3r + 128R^2r^2 - 44Rr^3\\&- r^4)(4R+r)^3rs^4 - 2(8R^2 + 14Rr - r^2)(4R+r)^6r^2s^2\\&+ (4R+r)^9r^3.\end{aligned}$$

Thus, it follows from (3.80) and (3.81) that

$$\sum \frac{r_a}{(q_b+q_c)\sin^2 A} = \frac{N_4}{8rs^2M_4}. \tag{3.82}$$

To prove inequality (3.79) we need to prove that

$$K_0 \equiv N_4 - 16rs^2M_4 \geq 0. \tag{3.83}$$

Simplifying gives the following equivalent inequality:

$$\begin{aligned}K_0 \equiv\, &s^{12} + (16R^2 - 12Rr - 30r^2)s^{10} + (64R^4 - 320R^3r - 576R^2r^2\\&- 172Rr^3 + 31r^4)s^8 - 4(2R+r)(160R^4 - 192R^3r - 376R^2r^2\\&- 148Rr^3 - 7r^4)rs^6 + (64R^4 + 448R^3r + 384R^2r^2 - 108Rr^3\\&- 33r^4)(4R+r)^3rs^4 - 2(8R^2 + 22Rr - r^2)(4R+r)^6r^2s^2\\&+ (4R+r)^9r^3 \geq 0.\end{aligned} \tag{3.84}$$

This inequality can be proved by applying the following Gerretsen's inequality (see [1] and [16]):

$$g_1 \equiv s^2 - 16Rr + 5r^2 \geq 0, \tag{3.85}$$

the fundamental inequality (3.39), and the previous the Gerretsen inequality (3.40). In fact, after analysing we obtain the following identity:

$$\begin{aligned}K_0 =\, &g_1^6 + x_1g_1^5 + x_2g_1^4 + x_3g_1^3 + x_4g_1^2 + x_5g_1 + x_6g_2\\&+ x_7t_0s^2 + x_8,\end{aligned} \tag{3.86}$$

where g_2 and t_0 are the same as in (3.39) and (3.40) respectively, and

$$\begin{aligned}x_1 =&16R^2 + 84Rr - 60r^2,\\x_2 =&64R^4 + 960R^3r + 1904R^2r^2 - 4672Rr^3 + 1156r^4,\\x_3 =&64Rr(44R^4 + 314R^3r + 1173r^4),\end{aligned}$$

$$\begin{aligned}x_4 =&32r(128R^7+2144R^6r+6472R^5r^2-30762R^4r^3\\&+64174R^3r^4-542R^2r^5-7277Rr^6-2707r^7),\\x_5 =&128r^2(512R^8+5952R^7r+8656R^6r^2-109718R^5r^3\\&+156768R^4r^4+9122R^2r^6+3653r^8),\\x_6 =&384Rr^7(28927R^2+1105r^2),\\x_7 =&32r^3(34R^3+3975R^2r+331r^3),\\x_8 =&128(R-2r)(2048R^8+26624R^7r+76448R^6r^2\\&-32808R^5r^3-413606R^4r^4+333201R^3r^5\\&-18404R^2r^6+9257Rr^7+2824r^8)r^3.\end{aligned}$$

By Euler's inequality $R \geq 2r$, one sees that $x_1, x_2, x_5 >$ hold. Moreover, with the help of software Maple we easily obtain the following identities:

$$\begin{aligned}x_4 =&32r(128e^7+3936e^6r+42952e^5r^2+198438e^4r^3\\&+491678e^3r^4+764550e^2r^5+763011er^6+362475r^7),\end{aligned}\tag{3.87}$$

$$\begin{aligned}x_8 =&128er^3(2048e^8+59392e^7r+678560e^6r^2+4038488e^5r^3\\&+13593674e^4r^4+26523169e^3r^5+29338482e^2r^6\\&+16776741er^7+3770610r^8),\end{aligned}\tag{3.88}$$

where $e = R - 2r \geq 0$. Thus, we have $x_4 > 0$ and $x_8 \geq 0$. Therefore, according to identity (3.86), Gerretsen's inequalities (3.40), (3.85) and the fundamental inequality (3.39) we conclude that $K_0 \geq 0$ holds. This completes the proof of Corollary 3.13.

4. Some conjectures related to the Erdös-Mordell inequality

In this section, we shall introduce some interesting inequalities as open problems which, though unproved, have been checked by the computer.

Let R_1, R_2, R_3 be the distances from an interior point P of the triangle ABC to the vertices A, B, C of the triangle ABC, respectively. Let r_1, r_2, r_3 be the distances from P to the sides BC, CA, AB, respectively. Then

$$\sum R_1 \geq 2\sum r_1.\tag{4.1}$$

This is the famous Erdös-Mordell inequality (cf. [16, pp.319-319]).

The author has already established and presented some sharpened versions of the Erdös-Mordell inequality (see [5] and [10]-[13]). For example, the following conjecture was proposed in [12]:

$$\frac{\sum R_1}{\sum r_1} \geq \frac{1}{2}\sum \frac{a^2}{m_b m_c}.\tag{4.2}$$

Here, we shall introduce some new similar conjectures.

Note that the previous inequality (2.1) is equivalent to

$$\frac{2m_a}{r_b+r_c} + \frac{r_b+r_c}{2m_a} \geq 2.\tag{4.3}$$

Inspired and motivated by this inequality and the Erdös-Mordell inequality, the author proposes the following sharpened version of the Erdös-Mordell inequality:

Conjecture 4.1. *For any interior point P of the triangle ABC, we have*

$$\frac{\sum R_1}{\sum r_1} \geq \frac{2m_a}{r_b + r_c} + \frac{r_b + r_c}{2m_a}. \tag{4.4}$$

Four similar conjectures are as follows:

Conjecture 4.2. *For any interior point P of the triangle ABC, we have*

$$\frac{\sum R_1}{\sum r_1} \geq \frac{m_b + m_c}{m_a + r_a} + \frac{m_a + r_a}{m_b + m_c}. \tag{4.5}$$

Conjecture 4.3. *For any interior point P of the triangle ABC, we have*

$$\frac{\sum R_1}{\sum r_1} \geq \frac{m_a h_a}{r_b r_c} + \frac{r_b r_c}{m_a h_a}. \tag{4.6}$$

Conjecture 4.4. *For any interior point P of the triangle ABC, we have*

$$\frac{\sum R_1}{\sum r_1} \geq \frac{4m_b m_c}{2a^2 + bc} + \frac{2a^2 + bc}{4m_b m_c}. \tag{4.7}$$

Conjecture 4.5. *For any interior point P of the triangle ABC, we have*

$$\frac{\sum R_1}{\sum r_1} \geq \frac{\sum r_a}{\sum m_a} + \frac{\sum m_a}{\sum r_a}. \tag{4.8}$$

Finally, we give a sharpened version of the Erdös-Mordell inequality again, which is inspired by the equivalent form (3.52) of the fundamental triangle inequality.

Conjecture 4.6. *For any interior point P of the triangle ABC, we have*

$$\frac{\sum R_1}{\sum r_1} \geq \frac{4r}{R} + \sum \left(\frac{b - c}{a}\right)^2. \tag{4.9}$$

Inequality (3.52) implies that the above inequality is stronger than the Erdös-Mordell inequality.

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