INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 11 (2022), No. 3, 124 - 133

SPECIAL CIRCLE CHAINS[1]

PETER THURNHEER

Abstract. W. Clifford and F. Morley used results of J. Steiner and A. Miquel about the circumcentres of the triangles in a complete quadrilateral as "anchoring" for the construction of two chains of special and interesting circles and points in a general plane n-line. In this text we consider particular n-lines with $n-1$ lines passing through a point A. We show first of all that analogous chains can be found in these starting on the one hand from the circumcentres of the six triangles contained in such a special 5-line as well as from the in- and excentres of the three triangles in such a special 4-line. In the second case the number of such circle chains grows exponentially with the number of lines considered and the centres of their circles have the additional property that they lie on two lines intersecting perpendicularly at the point A.

1 Introduction

In the following, for $n \geq 3$ we understand by an *n-configuration* n lines of the plane in general position i.e. not two parallel and not three through a point and by an *n-situation* n lines passing through a point A and intersecting a line a not through A in the points $B_j, j = 1, 2, \ldots, n$. We denote such an n-situation by $\{AB_1B_2 \ldots B_n\}$ and call the points $B_j, j = 1, 2, \ldots, n$, the base points, A the apex of the n-situation. W. Clifford [1] and F. Morley [4] used results of J. Steiner [5] and A. Miquel [3] about the circumcentres of the triangles in a 4-configuration, i.e. in the complete quadrilateral, as "anchoring" for the construction of two chains of special and interesting circles and points in an n-configuration. Based on statements about special circle chains in the Clifford-Morley sense (Theorem 1), we will show that analogous circle chains can be found starting on the one hand from the circumcentres of the six triangles contained in a 4-situation (Theorem 2) as well as from the in- and excentres of the three triangles in a 3-situation (Theorem 3). In the second case - as for the circle chains found in general n-configurations in [6] - the number of these chains grows exponentially with the number of lines considered, but the centres of their circles have the additional property that they lie on two lines intersecting perpendicularly at the point A (Theorem 3). Finally we show that there are other special circles in such n-situations (Theorems 3,4).

2 Preliminaries

In what follows we need the mentioned statements about the complete quadrilateral of J. Steiner [5] and A. Miquel [3] (Proposition 1a)) and about circle chains based on these of W. Clifford [1] and F. Morley [4] (Proposition 1b),c),d)). Of the many interesting results in this connection (see for example [2], [8]), we summarise only those which we will need later.

[1]**Keywords and phrases**: Special Clifford Morley circle chains, circle chains in special plane n-lines.
(2020) **Mathematics Subject Classification**: 51A05
Received: 14.01.2022. In revised form: 12.05.2022. Accepted: 03.04.2022.

Proposition 1. *a) A 4-configuration contains four triangles. Their circumcentres lie on a circle, the Steiner circle of the 4-configuration, and the circumcircles intersect at a point - the Miquel or Clifford point C of the configuration - on the Steiner circle.*

b) A 5-configuration contains five 4-sub configurations. The centres of the five corresponding Steiner circles lie on a circle, the Steiner circle of the 5-configuration. And so on: The centres of the n Steiner circles of the n $(n-1)$-sub configurations contained in an n-configuration lie on a circle, the Steiner circle of the n-configuration.

c) The Steiner circles of the n $(n \geq 5)$ $(n-1)$-sub configurations contained in an n-configuration intersect at a point, the centre point of the n-configuration.

d) The five Clifford points of the five 4-sub configurations contained in a 5-configuration lie on a circle, the Clifford circle of the 5-configuration. The six Clifford circles of the six 5-sub configurations contained in a 6-configuration intersect at a point, the Clifford point of the 6-configuration. And so on: An n-configuration contains n $(n-1)$-sub configurations. There is a Clifford point at which the n Clifford circles of the $(n-1)$-sub configurations intersect if n is even and a Clifford circle on which the n Clifford points of the $(n-1)$-sub configurations lie if n is odd.

We call the incentre and the excentres of the three triangles defined by a 3-situation $\{AB_1B_2B_3\}$ (Figure 1) the U-, V-, W-points and denote them by $U_j, V_j, W_j, j = 1, 2, 3, 4$. These twelve points are each intersections of three angle bisectors of the lines a, AB_1, AB_2, AB_3. The set of these angle bisectors consists of six pairs of two lines intersecting perpendicularly, once each in B_1, B_2, B_3 and three times in A. In Proposition 2 we recall Theorem 1 in [7].

Proposition 2 (Figure 1). *a) Eight circles $D_1, D_2, ..., D_8$ - the δ_3-circles - pass through the apex A of a 3-situation, each containing a U-, a V-, and a W-point. Through each of the twelve U-, V-, W-points pass two of these circles.*

b) Two lines d_{31}, d_{32} intersecting perpendicularly in A contain each four of the eight δ_3-circle centres.

By Proposition 2 the eight δ_3-circles intersect at sixteen points different from the apex A. Twelve of these are U-,V-, or W-points, four are not. We call the latter the Z-points $Z_j, j = 1, 2, 3, 4$ (Figure 1).
In the following, we name circles by their centre. The circle with centre M is the M-circle or the circle M, the centre of the circle M is the point M.
In an n-situation we consider again and again n-configurations consisting of an angle bisector of the lines a and $AB_j, j = 1, 2, \ldots, n$, at each of the base points. We call such special n-configurations H_n-configurations of the n-situation.
For each of the eight possible H_3-configurations of a 3-situation, the triple of the intersections of the three angle bisectors in the configuration consists of a U-, a V- and a W-point on one of the eight δ_3-circles (Figure 1). This means that statement a) in the following Proposition 3 holds. The evidence of the statements b) and c) is shown by Figure 1 and statement d) is a reformulation of Theorem 2a) in [7], illustrated in [7], Figure 2 and Figure 5, and by one example in Figure 1 below.

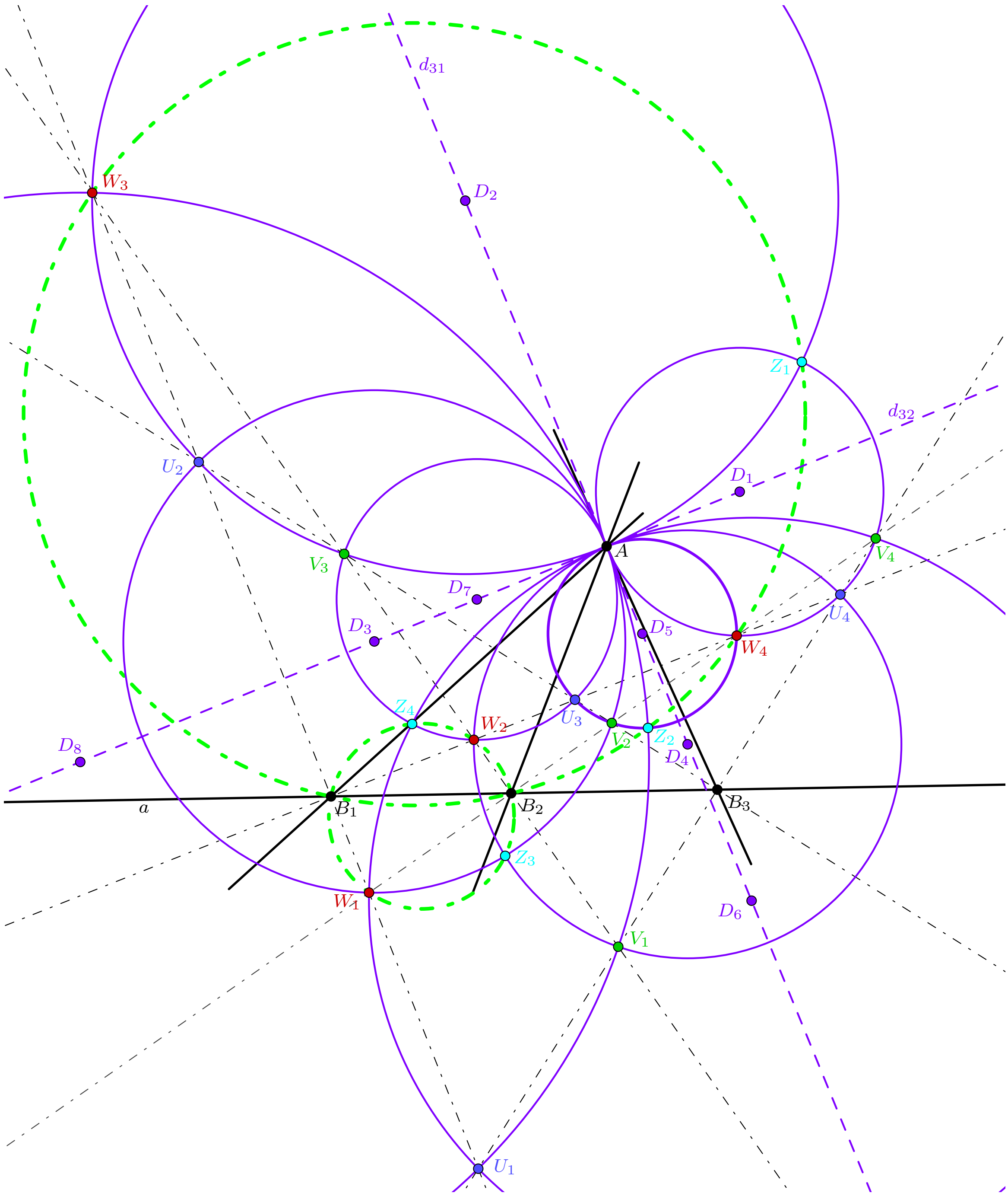

Figure 1

Proposition 3. *a) The circumcircle of each triangle generated by a H_3-configuration is a δ_3-circle.*

b) Each δ_3-circle contains exactly one Z-point.

c) The intersection different from the apex A of two δ_3-circles defined by two H_3-configurations is a Z-point if and only if the angle bisectors of the second H_3-configuration are perpendicular to those of the first one.

d) Through two base points B_j, B_k of a 3-situation pass two circles - F-circles - containing each two Z-points and a pair of U-,V- or W-points from the triangle AB_jB_k.

3 Special circle chains

We start with statements about circle chains of a special kind. We use the analytic method with which F. Morley ([4], p.97-100) analysed in detail situations of the kind and which we summarise as short as possible. Curves, in particular lines, are defined by coordinate equations in the complex plane. For a line g it is of the form (for $z \in \mathbb{C}$ let $\overline{z}$ be the complex conjugate)

$$g : zt + \overline{z} = \overline{x}.$$

Hereby x is the point defining the line g and $t = \overline{x}/x$. The line g is the perpendicular bisector of the line segment delimited by the origin and the point x. For an n-configuration, $n \geq 3$, consisting of the n lines g_j defined by the points $x_j, j = 1, 2, ..., n$ the characteristic constants $a_k, k = 1, 2, ..., n$, are defined by

$$a_k = \frac{x_1 t_1^{n-k}}{(t_1 - t_2)(t_1 - t_3)....(t_1 - t_n)} + \frac{x_2 t_2^{n-k}}{(t_2 - t_3)(t_2 - t_4)....(t_2 - t_n)(t_2 - t_1)} + ...$$

$$.... + \frac{x_n t_n^{n-k}}{(t_n - t_1)(t_n - t_2)....(t_n - t_{n-1})}.$$

A possible common denominator of the terms in this sum - with $N = \{1, 2, ..., n\}$ - is

$$d = \prod_{\substack{r,s \in N \\ s > r}} (t_r - t_s).$$

Then with $N_j = \{1, 2, ..., n\} \setminus \{j\}$ and

$$f_j = \prod_{\substack{r,s \in N_j \\ s > r}} (t_r - t_s), j = 1, 2, ..., n \tag{1}$$

a_k is of the form

$$a_k = \frac{1}{d} \sum_{j=1}^{n} (-1)^{j+1} x_j t_j^{n-k} f_j, k = 1, 2, ..., n. \tag{2}$$

For simplicity let us call the circumcircle of the triangle generated by a 3-configuration the Steiner circle of the 3-configuration. The following equation then maps the unit circle on the Steiner circle of the n-configuration, $n \geq 3$, in the sense that z runs through the Steiner circle when t runs through the unit circle,

$$z = a_1 - a_2 t. \tag{3}$$

Theorem 1. *If the n lines $g_j, j = 1, 2, ..., n$, $n \geq 3$, generating an n-configuration are defined by points x_j of the form $x_j = b_j - i, b_j \in \mathbb{R}, j = 1, 2, ..., n$ then the origin O lies on the Steiner circle of the configuration.*
For all $n \geq 3$ in such a particular n-configuration the centre point ($n \geq 5$) and the Clifford point (n even) of the configuration both fall on the origin and its Clifford circle (n odd) reduces to a point, the origin.

4 Circles in the n-situation

Theorem 2. *a) The circumcentres of the three triangles contained in a 3-situation lie on a circle passing through the apex A of the situation.*

b) *A 4-situation contains six triangles. There are four circles passing through the apex A and containing each tree of the six corresponding circumcentres. The centres of these four circles lie on a circle - the γ_4-circle of the 4-situation - passing through the apex A.*

c) *And so on: The centres of the n γ_{n-1}-circles of the n $(n-1)$-sub situations contained in an n-situation lie on a circle - the γ_n-circle of the n-situation - passing through the apex A.*

Theorem 3. a) *A 4-situation contains four 3-sub situations, i.e. a total of thirty-two δ_3-circles (see Proposition 2) and sixteen Z-points. Through the apex A pass sixteen circles - the δ_4-circles - each containing four δ_3-circle centres and eight circles - the ζ-circles - each containing four Z-points, in both cases one out of every 3-sub situation. Through the apex A pass two perpendicularly intersecting lines, d_{41}, d_{42}, each containing eight of the sixteen δ_4-circle centres and four of the eight ζ-circle centres.*

b) *And so on: An n-situation, $n \geq 5$, contains n $(n-1)$-sub situations, i.e. a total of $n2^{n-1}$ δ_{n-1}-circles. There exist 2^n circles through the apex A - the δ_n-circles - each containing n δ_{n-1}-circle centres, one out of every $(n-1)$-sub situation. Through A pass two perpendicularly intersecting lines d_{n1}, d_{n2}, each containing 2^{n-1} δ_n-circle centres.*

Theorem 4. *An n-situation, $n \geq 3$, contains $\binom{n}{3}$ 3-sub situations, i.e. a total of $4\binom{n}{3}$ Z-points. There exist $n(n-1)$ circles - the σ-circles - each containing two base points, a pair of U-,V-, or W-points out of a 3-sub situation and $2(n-2)$ Z-points. Through every Z-point pass three such σ-circles.*

5 Proofs

Theorem 1. We first show that under the special conditions of Theorem 1 we have $a_{n-1} = a_n$. With (2) and

$$t_j - 1 = \frac{b_j + i}{b_j - i} - 1 = \frac{2i}{x_j}$$

we find

$$a_{n-1} - a_n = \frac{1}{d}\sum_{j=1}^{n}(-1)^{j+1}x_j(t_j - 1)f_j = \frac{2i}{d}\sum_{j=1}^{n}(-1)^{j+1}f_j.$$

We call the last sum S and show that it is zero. This can be seen as follows. Let us multiply out all the brackets in a f_j-product (1). This yields a sum - which we name f_j^* - of products of at most $n-1$ different t-variables. Every such product contains at least $n-2$ different t-variables. This can be seen by direct calculation for $n = 3, n = 4$, even $n = 5$ and by an induction in general. For reasons of symmetry it is enough to consider for instance f_{n-1}, which can be written in the form

$$f_{n-1} = \{(t_1 - t_2)(t_1 - t_3)\ldots(t_1 - t_{n-2})(t_2 - t_3)\ldots(t_{n-4} - t_{n-2})(t_{n-3} - t_{n-2})\}\prod_{k=1}^{n-2}(t_k - t_n).$$

On the product in the curly bracket we apply the assumption of the induction, whereas the other product contributes a factor containing or all $n-2$ t-variables $t_1, t_2, \ldots t_{n-2}$ or at least one additional factor t_n.
Next, we study products in f_j^* containing all $n-1$ t-variables different from t_j. We call such products q-terms. We show that in every f_j^* all these q-terms emerge in pairs of two adding up to zero. Again it is enough to consider f_n^*. Every q-term in f_n^* can be characterised by choosing in every round bracket of the product f_n a corresponding t-variable - one in every bracket. Let

u, v, w be in $N_n = \{1, 2, ..., n-1\}$ with $u < v < w$. For *every* such triple every q-term in f_n^* contains the product $t_u t_v t_w$ as a factor. But in the three brackets $(t_u^* - t_v^{"})(t_u^{"} - t_w^*)(t_v^* - t_w^{"})$ we can choose or the t^*- or the $t^{"}$-variables - keeping fixed the choice in the remaining brackets. This shows that for every q-term there exists a corresponding one such that the two add up to zero.

So every f_j^*-term contains actually only products of exactly $n-2$ different t-variables. But $n-2$ t-variables all together appear in exactly two different f_j-terms. Let us consider the $n-2$ t-variables $t_1, t_2, \ldots, t_{n-2}$. All together they appear in the terms f_{n-1} and f_n which with respect to $t_1, t_2, \ldots, t_{n-2}$ have exactly the same structure:

$$(t_1 - t_2), \ldots, (t_1 - \boldsymbol{t_\alpha})(t_2 - t_3) \ldots (t_2 - \boldsymbol{t_\alpha}) \ldots (t_{n-3} - t_{n-2})(t_{n-2} - \boldsymbol{t_\alpha}),$$

one with $\boldsymbol{t_\alpha} = t_{n-1}$ the other with $\boldsymbol{t_\alpha} = t_n$. So obviously all products containing exactly all t-variables $t_1, t_2, \ldots, t_{n-2}$ in both these terms are the same. Since f_{n-1} and f_n in the sum S have different sign, all the products containing the t-variables $t_1, t_2, \ldots, t_{n-2}$ in S add up to zero. Next we consider the t-variables $t_1, t_2, \ldots, t_{n-3}, t_{n-1}$. They all together appear in the two terms

$$\begin{aligned} f_n = {} & (t_1 - t_2) \ldots (t_1 - t_{n-3})(t_1 - t_{n-2})(t_1 - t_{n-1})(t_2 - t_3) \ldots (t_2 - t_{n-3})(t_2 - t_{n-2})(t_2 - t_{n-1}) \ldots \\ & \ldots (t_{n-4} - t_{n-3})(t_{n-4} - t_{n-2})(t_{n-4} - t_{n-1})(t_{n-3} - t_{n-2})(t_{n-3} - t_{n-1})(t_{n-2} - t_{n-1}) \\ = {} & -(t_1 - t_2) \ldots (t_1 - t_{n-3})(t_1 - t_{n-1})(t_1 - \boldsymbol{t_{n-2}})(t_2 - t_3) \ldots (t_2 - t_{n-3})(t_2 - t_{n-1})(t_2 - \boldsymbol{t_{n-2}}) \ldots \\ & \ldots (t_{n-4} - t_{n-3})(t_{n-4} - t_{n-1})(t_{n-4} - \boldsymbol{t_{n-2}})(t_{n-3} - t_{n-1})(t_{n-3} - \boldsymbol{t_{n-2}})(t_{n-1} - \boldsymbol{t_{n-2}}) \end{aligned}$$

and

$$\begin{aligned} f_{n-2} = {} & (t_1 - t_2) \ldots (t_1 - t_{n-3})(t_1 - t_{n-1})(t_1 - \boldsymbol{t_n})(t_2 - t_3) \ldots (t_2 - t_{n-3})(t_2 - t_{n-1})(t_2 - \boldsymbol{t_n}) \ldots \\ & \ldots (t_{n-4} - t_{n-3})(t_{n-4} - t_{n-1})(t_{n-4} - \boldsymbol{t_n})(t_{n-3} - t_{n-1})(t_{n-3} - \boldsymbol{t_n})(t_{n-1} - \boldsymbol{t_n}), \end{aligned}$$

which with respect to the t-variables $t_1, t_2, \ldots, t_{n-3}, t_{n-1}$ have the same structure but a different sign. Since the terms f_{n-2} and f_n in the sum S have the same sign all products containing exactly the t-variables $t_1, t_2, \ldots, t_{n-3}, t_{n-1}$ in S also sum up to zero. Repeated application of this reasoning shows that the sum S is zero.

So indeed $a_{n-1} = a_n$ and with [4] (p.100, Formula (5)) we find $|a_1| = |a_n| = |a_{n-1}| = |a_2|$ such that for $t = a_1/a_2$ we have $|t| = 1$. Hence we can choose $t = a_1/a_2$ in (3) which shows that the point $z = 0$ lies on the Steiner circle of the configuration. (Since also the sum $\sum_{j=1}^{n} (-1)^{j+1} t_j^{n-2} f_j$ is zero in this special case we actually have even $a_1 = a_2$).

The other statements in Theorem 1 follow from the fact that according to the first part proved above, the circumcircles of all triangles contained in such a special 5-configuration pass through the origin. So the Clifford points of all its 4-sub configurations fall on O and the Clifford circle of the 5-configuration reduces to this point which is also its centre point and so on. □

Theorem 2. In the plain of any n-situation, $n \geq 3, \{AB_1B_2....B_n\}$ we can choose the complex coordinate system such that the apex A is the origin and the points B_j are of the form $B_j = b_j - i, b_j \in \mathbb{R}, j = 1, 2, ..., n$. Then the perpendicular bisectors of the line segments AB_j are lines of the form of the lines g_j considered in Theorem 1. The intersections of these perpendicular bisectors are the circumcentres of the triangles contained in the n-situation, which together with Proposition 1 profs Theorem 2 at once. □

The δ_n-circles. If for the n-situation, $n \geq 3, \{AB_1B_2....B_n\}$ the complex coordinate system is chosen as above then the angle bisectors of the lines AB_j and the line a carrying the base points are again lines of the form of the lines g_j considered in Theorem 1 (with some numbers $c_j \in \mathbb{R}$

in place of $b_j, j = 1, 2, \dots, n$). Since the intersections of these angle bisectors are the in- and excentres of the triangles contained in the n-situation and since in every n-situation there exist 2^n H_n-configurations formed of such angular bisectors this profs together with Proposition 1 and Proposition 3a) the existence of the 2^n δ_n-circles at once. (And it gives another, analytic proof of the Propositions 2a) and 3a), which in [7] were found using geometric arguments). □

In what follows we use again and again the theorems Eucl. III. 20.

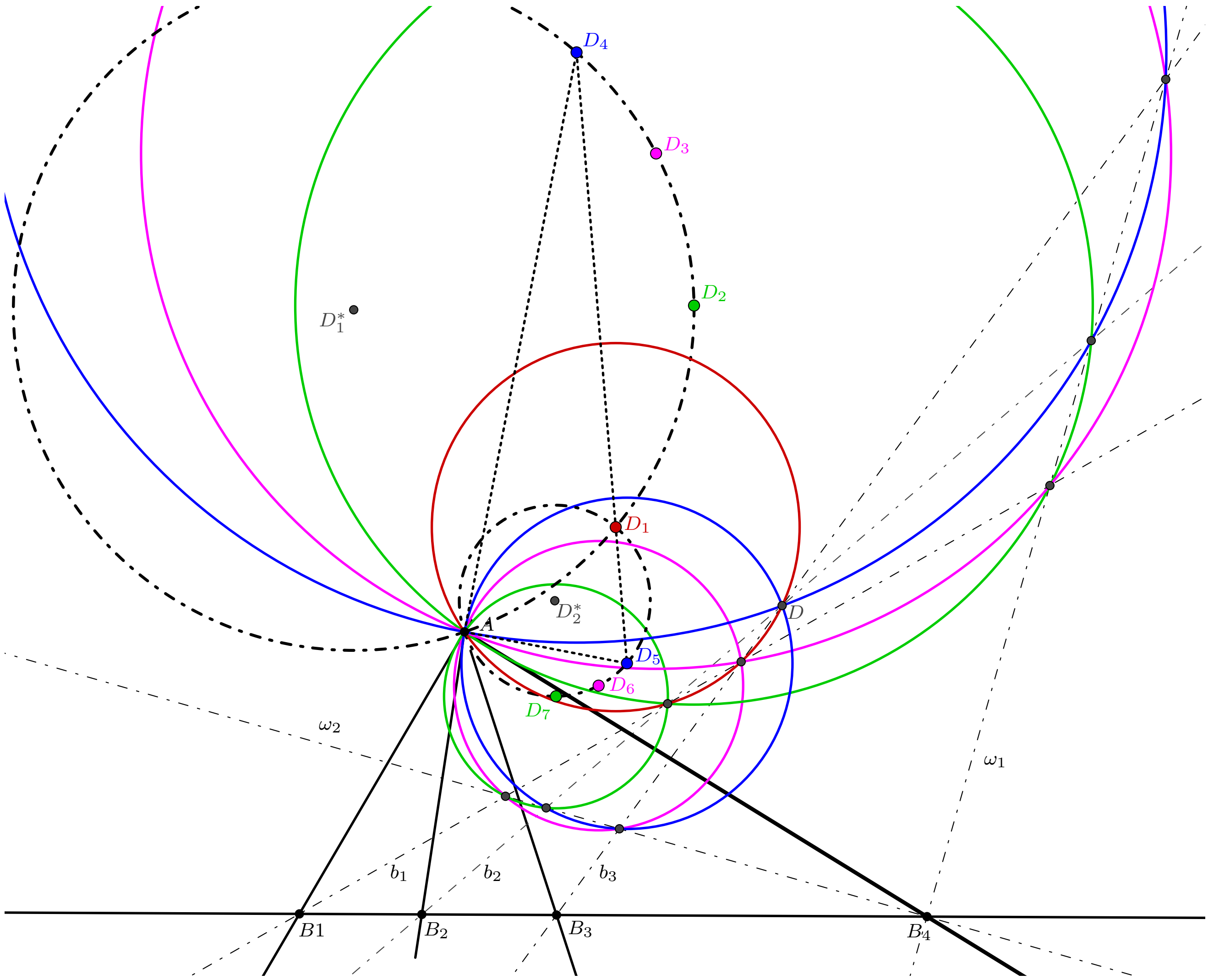

Figure 2

The lines d_{41}, d_{42} (first part). In a 4-situation (Figure 2) we consider two H_4-configurations of the form $\{b_1, b_2, b_3, \omega_1\}$ and $\{b_1, b_2, b_3, \omega_2\}$, where b_j is an angle bisector in the base point $B_j, j = 1, 2, 3$, and ω_1, ω_2 are the angle bisectors in B_4. The four H_3-sub configurations of these two H_4-configurations define two groups of four δ_3-circles, whereby the δ_3-circle determined by the H_3-sub configuration $\{b_1, b_2, b_3\}$ belongs to both circle quadruples. In Figure 2, these are the δ_3-circles D_1, and $D_j, j = 2, ..., 7$. Their centres lie on the δ_4-circles D_1^* and D_2^* respectively, which intersect at the δ_3-circle centre D_1. For every δ_3-circle, different from D_1, in one group there is one out of the same 3-sub situation in the other group: For instance, the circles D_4 and D_5 from the 3-sub situation $\{AB_2B_3B_4\}$ generated by the angle bisectors $\{b_2, b_3, \omega_1\}$ and $\{b_2, b_3, \omega_2\}$. They pass through the intersection D of the angle bisectors b_2, b_3, so they do not touch, but intersect perpendicularly in A according to Proposition 2b), so that $\sphericalangle D_4AD_5 = 90°$ holds. The circle D_1 is defined by the angle bisectors $\{b_1, b_2, b_3\}$ and thus also passes through D. The three circles have the common chord AD. Their centres D_4, D_5, D_1 lie on a line. In addition we have

$$\sphericalangle D_1^*AD_1 = 90° - \sphericalangle D_1D_4A \qquad \sphericalangle D_1AD_2^* = 90° - \sphericalangle D_1D_5A$$
$$\sphericalangle D_1^*AD_2^* = 180° - \{\sphericalangle D_1D_4A + \sphericalangle D_1D_5A\}.$$

The angle sum in the curly bracket is, as seen, equal to 90° and we have $\sphericalangle D_1^*AD_2^* = 90°$. This shows that two H_4-configurations identical up to the angle bisectors in one base point generate two δ_4-circles with centres lying on two lines perpendicularly intersecting in the apex A. This yields the statement to be proved for the δ_4-circle centres. □

The lines $d_{n1}, d_{n2}, n \geq 5$. The preceding statement can be used as an "anchor" for an induction.

Assumption: In every $(n-1)$-situation there exist two lines $d_{(n-1)1}, d_{(n-1)2}$ intersecting perpendicularly in A and containing the centres of 2^{n-2} δ_{n-1}-circles passing through the apex A.

Then one can argue in the same way as before. In an n-situation we consider two H_n-configurations of the form $\{b_1, b_2, ..., b_{n-1}, \omega_1\}$ and $\{b_1, b_2, ..., b_{n-1}, \omega_2\}$, b_j angular bisectors in $B_j, j = 1, 2, \ldots, n-1$, and ω_1, ω_2 the angular bisectors in B_n. These two H_n-configurations define two δ_n-circles - we call them again D_1^* and D_2^* - which intersect at the centre D_1 of the δ_{n-1}-circle D_1 generated by the H_{n-1}-sub configuration $\{b_1b_2....b_{n-1}\}$ and which each contain a group of $n-1$ further δ_{n-1}-circle centres. All these δ_{n-1}-circles pass through A and again for each circle of one group there exists one out of the same $(n-1)$-sub situation in the other group, for instance the δ_{n-1}-circles generated by the H_{n-1}-configurations $\{b_1, b_2, \ldots, b_{n-2}, \omega_1\}$ and $\{b_1, b_2, \ldots, b_{n-2}, \omega_2\}$. we call them again D_4, D_5. They both contain the centre D of the δ_{n-2}-circle D generated by the H_{n-2}-configuration $\{b_1, b_2, \ldots, b_{n-2}\}$ and thus do not touch but intersect perpendicularly in A according to the induction assumption. But $\{b_1, b_2, ..., b_{n-2}\}$ is also a H_{n-2}-sub configuration of the H_{n-1}-configuration generating the δ_{n-1}-circle D_1. The δ_{n-1}-circle D_1 also contains the point D. Thus for the points, respectively the circles $D_1^*, D_2^*, D_1, D_4, D_5, D$ we have exactly the situation shown in Figure 2 and with the same argumentation it follows that the δ_n-circle centres D_1^*, D_2^* lie on two lines intersecting perpendicularly in A. Repeated application of this argumentation shows that again the lines AD_1^*, AD_2^* intersecting perpendicularly in A each contain half of the 2^n δ_n-circle centres. These two lines are the lines d_{n1}, d_{n2}. □

The eight ζ-circles, the lines d_{41}, d_{42}. We start considering in a coordinate system with origin A a trapezium with its nonparallel sides lying on the axis. Its vertices are points - we name them in view of the application - of the following form $D_j = (a, 0), D_j'' = (ka, 0), D_{j+4}'' = (0, b), D_{j+4} = (0, kb)$ and its diagonals intersect at the point $Z = \frac{k}{k+1}(a, b)$. Let $D_1^* = \frac{1}{2}(a, b)$ and $D_2^* = \frac{k}{2}(a, b)$ be the midpoints of the line segments D_jD_{j+4}'' respectively $D_j''D_{j+4}$. These two points, as well as the point Z lie on a line through the origin A. We will need the following observation:

The diagonal D_jD_{j+4} intersects the line AD_1^* at the point Z, which is the image of the point D_1^* under a dilation with centre A and factor $\frac{2k}{k+1}$.

In a 4-situation (Figure 3) we now consider a H_4-configuration defining a δ_4-circle D_1^* which contains the centres of the four δ_3-circles D_j, with radii r_j, each of which contains a Z-point $Z_j, j = 1, 2, 3, 4$, (Proposition 3b) as well as the H_4 configuration whose angle bisectors are perpendicular to those of the first choice. The four H_3-sub configurations of this second H_4-configuration define the δ_3-circles D_j, with radii $r_j, j = 5, 6, 7, 8$, whose centres lie on the

δ_4-circle D_2^*. They are, according to Proposition 3c), those δ_3-circles which also pass through the Z-points $Z_j, j = 1, 2, 3, 4$. The two circles D_1^*, D_2^* have no δ_3-circle centre in common, so they touch each other in the apex A according to the last paragraph and thus emerge from each other by a central dilation with centre A and factor let us say k.

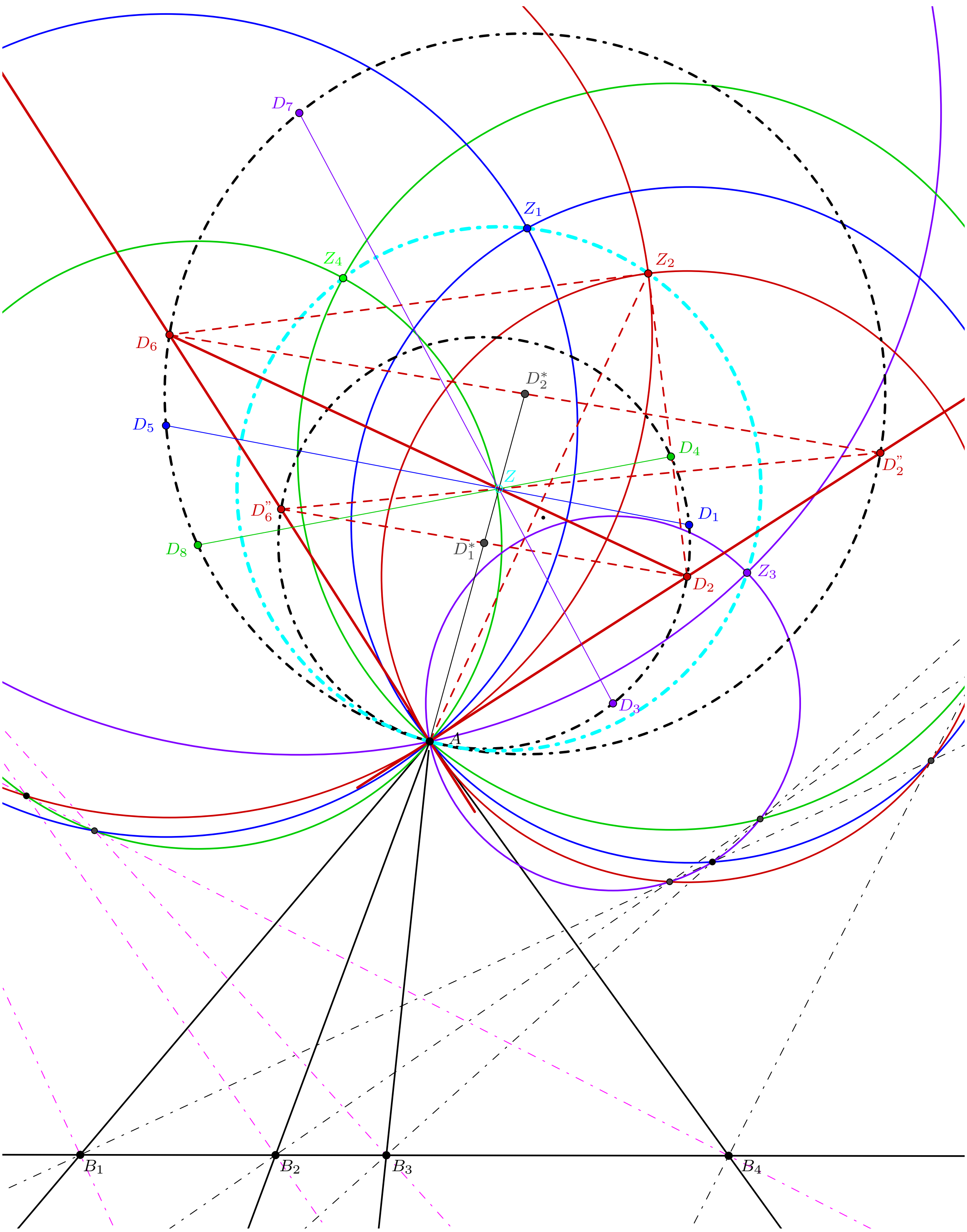

Figure 3

The following reasoning, which we illustrate in Figure 3 for $j = 2$, is exactly the same for all $j = 1, 2, 3, 4$. In particular, k is in the four cases the same factor of the dilation just introduced. Let $D_j^{"}$ and $D_{j+4}^{"}$ be the intersections of the lines AD_j and AD_{j+4} with the circles D_2^* respectively

D_1^*. Since according to Proposition 2 the δ_3-circles D_j and D_{j+4} intersect perpendicularly in A, we have $\sphericalangle D_j A D''_{j+4} = \sphericalangle D_{j+4} A D''_j = 90°$. So all lines $D_j D''_{j+4}$ and $D_{j+4} D''_j$ pass through the circle centres D_1^* respectively D_2^* and these points are the midpoints of the corresponding line segments. Hence according to the preceding observation, all lines $D_j D_{j+4}, j = 1, 2, 3, 4$ pass through the same point Z on the line AD_1^*. But these lines are the perpendicular bisectors of the line segments AZ_j since we have $|D_j Z_j| = |D_j A| = r_j$ and $|D_{j+4} Z_j| = |D_{j+4} A| = r_{j+4}$. So for the point Z on the line AD_1^* we have $|AZ| = |ZZ_j|, j = 1, 2, 3, 4$. The point Z is the centre of one of the eight ζ-circles that can be found using the eight possible pairs of H_4-configurations with pairwise perpendicular angle bisectors. The lines AD_1^* and AD_2^* introduced in the penultimate paragraph are the lines d_{41}, d_{42}. □

The σ-circles. The two F-circles passing through two base points B_j, B_k of a 3-situation according to Proposition 3d) are defined solely by the triangle AB_jB_k. They thus are F-circles of each of the $n-2$ 3-sub situations in an n-situation to which the base points B_j, B_k belong and so contain in total $2(n-2)$ Z-points, two out of each. These two F-circles are σ-circles of the n-situation. There are $\binom{n}{2}$ pairs of basic points in an n-situation and thus $2\binom{n}{2} = n(n-1)$ σ-circles. Through each Z-point in an n-situation thus pass $\frac{2(n-2)n(n-1)}{4\binom{n}{3}} = 3$ σ-circles. □

AKNOWLEDGEMENT

The author would like to thank Professor Norbert Hungerbühler (ETH Zürich) for all his helpful and encouraging comments.

REFERENCES

[1] Clifford W., *Synthetic proof of Miquel's theorem,* Cambridge and Dublin Messenger of Mathematics, **5 (1870)**, 124-141.

[2] Hungerbühler N, Pohle C., *Communicating harmonic pencils of lines,* International Journal of Geometry, **Vol. 9 (2020), No. 2**, 15-38.

[3] Miquel A., *Théorèmes de Géométrie,* Journal de mathématiques pures et appliquées, **1ère série, 3 (1838)**, 485-487.

[4] Morley F., *On the metric geometry of the plane n-line,* Trans. Amer. Math. Soc., **1(2) (1900)**, 97-115.

[5] Steiner J., *Questions proposées, Théorème sur le quadrilatère complet,* Annales de Gergonne, **18 (1827-1828)**, 302-304.

[6] Thurnheer P., *Neue Clifford Morley Kreisketten,* Elemente der Mathematik (to appeare).

[7] Thurnheer P., *Dreiecke, Teildreiecke und Kreise,* Elemente der Mathematik (to appeare).

[8] Weiss G., *Geometrie – Tochter der Kunst, Mutter der Mathematik, 2014,* `https://studylibde.com/doc/3101320/`

INDEPENDENT RESEARCHER
Entlisbergstrasse 29 / CH-8038 Zürich / Switzerland
E-mail address: tpeter@retired.ethz.ch