



MIDPOINT DIAGONAL QUADRILATERALS

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Abstract. A convex quadrilateral, Q , is called a midpoint diagonal quadrilateral if the intersection point of the diagonals of Q coincides with the midpoint of at least one of the diagonals of Q . A parallelogram, \mathbb{P} , is a special case of a midpoint diagonal quadrilateral since the diagonals of \mathbb{P} bisect one another. We prove two results about ellipses inscribed in midpoint diagonal quadrilaterals, which generalize properties of ellipses inscribed in parallelograms. First, Q is a midpoint diagonal quadrilateral if and only if each ellipse inscribed in Q has tangency chords which are parallel to one of the diagonals of Q . Second, Q is a midpoint diagonal quadrilateral if and only if each ellipse inscribed in Q has a pair of conjugate diameters parallel to the diagonals of Q . Finally, we show that there is a unique ellipse, E_I , of minimal eccentricity inscribed in a midpoint diagonal quadrilateral, Q , and also that the unique pair of conjugate diameters parallel to the diagonals of Q are the equal conjugate diameters of E_I .

1. INTRODUCTION

Given a diameter, l , of an ellipse, E_0 , there is a unique diameter, m , of E_0 such that the midpoints of all chords parallel to l lie on m . In this case we say that l and m are conjugate diameters of E_0 , or that m is a diameter of E conjugate to l . l and m are called **equal** conjugate diameters if $|l| = |m|$.

We say that E_0 is *inscribed* in a convex quadrilateral, Q , if E_0 lies inside Q and is tangent to each side of Q . A tangency chord is any chord connecting two points where E_0 is tangent to two different sides of Q .

There are two interesting properties (probably mostly known) of ellipses inscribed in parallelograms, \mathbb{P} , which involve tangency chords and conjugate diameters:

(P1) Each ellipse inscribed in \mathbb{P} has tangency chords which are parallel to one of the diagonals of \mathbb{P} .

(P2) Each ellipse inscribed in \mathbb{P} has a pair of conjugate diameters which are parallel to the diagonals of \mathbb{P} .

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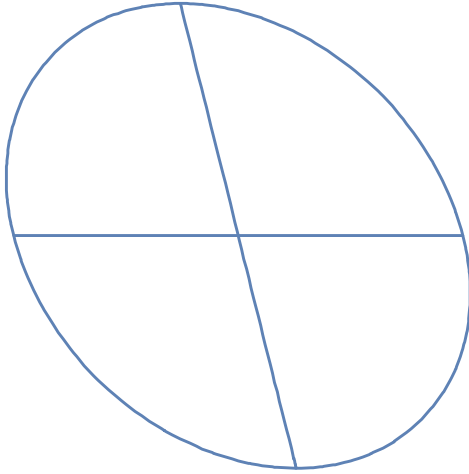


Figure 1: Unequal conjugate diameters

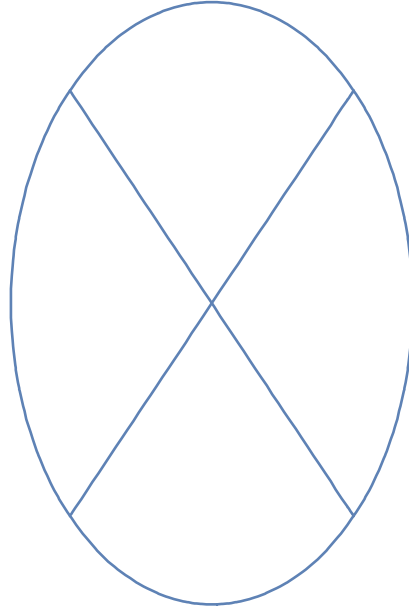


Figure 2: Equal conjugate diameters

Note that when we say that two lines are parallel, we include the possibility that they are equal, which does, in fact, occur for (P2).

This author is not sure if P1 is known at all, while P2 appears to be known([6]) only if E_0 is the ellipse of maximal area inscribed in \mathbb{P} . One of the purposes of this paper is to examine P1 and P2 for a larger class of convex quadrilaterals, which we call midpoint diagonal quadrilaterals(defined below).

Definition: A convex quadrilateral, Q , is called a midpoint diagonal quadrilateral if the intersection point of the diagonals of Q coincides with the midpoint of at least one of the diagonals of Q .

We show that not only do P1 and P2 each hold for midpoint diagonal quadrilaterals, but that if Q is **not** a midpoint diagonal quadrilateral, then no ellipse inscribed in Q satisfies

P1 or P2. Hence each of these properties completely characterizes the class of midpoint diagonal quadrilaterals(see Theorems 3.1 and 4.1 below), and thus they are a generalization of parallelograms in this sense. A parallelogram, \mathbb{P} , is a special case of a midpoint diagonal quadrilateral since the diagonals of \mathbb{P} bisect one another. Equivalently, if Q is not a parallelogram, then Q is a midpoint diagonal quadrilateral if and only if the line, L_Q , thru the midpoints of the diagonals of Q contains one of the diagonals of Q . The line L_Q plays an important role for ellipses inscribed in quadrilaterals due to the following well-known result(see [1] for a proof).

Theorem 1.1. (Newton): Let M_1 and M_2 be the midpoints of the diagonals of a quadrilateral, Q . If E_0 is an ellipse inscribed in Q , then the center of E_0 must lie on the open line segment, $\overline{M_1M_2}$, connecting M_1 and M_2 .

Remark 1. If Q is a parallelogram, then the diagonals of Q intersect at the midpoints of the diagonals of Q , and thus $\overline{M_1M_2}$ is really just one point.

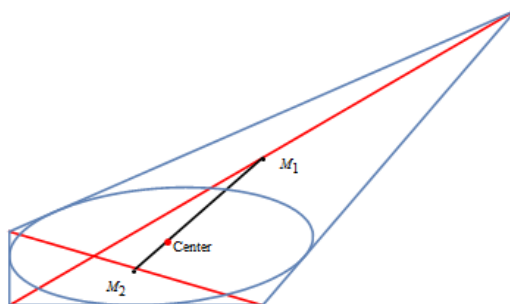


Figure 3

Remark 2. By Theorem 1.1, Q is a midpoint diagonal quadrilateral if and only if the center of any ellipse inscribed in Q lies on one of the diagonals of Q .

If E_0 is an ellipse which is not a circle, then E_0 has a unique set of conjugate diameters, l and m , where $|l| = |m|$. These are called equal conjugate diameters of E_0 . By Theorem 4.1(i), each ellipse inscribed in a midpoint diagonal quadrilateral, Q , has conjugate diameters parallel to the diagonals of Q . In particular, Theorem 4.1(i) applies to the unique ellipse of minimal eccentricity, E_I , inscribed in Q . However, we prove (Theorem 5.1) a stronger result: The **equal** conjugate diameters of E_I are parallel to the diagonals of Q .

2. USEFUL RESULTS ON ELLIPSES AND QUADRILATERALS

We now state a result, without proof, about when a quadratic equation in x and y yields an ellipse. The first condition ensures that the conic is an ellipse, while the second condition ensures that the conic is nondegenerate.

Lemma 2.1. *The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, with $A, C > 0$, is the equation of an ellipse if and only if $\Delta > 0$ and $\delta > 0$, where*

$$(1) \quad \begin{aligned} \Delta &= 4AC - B^2 \text{ and} \\ \delta &= CD^2 + AE^2 - BDE - F\Delta. \end{aligned}$$

The following lemma allows us to express the eccentricity and center of an ellipse as a function of the coefficients of an equation of that ellipse.

Lemma 2.2. *Suppose that E_0 is an ellipse with equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Let a and b denote the lengths of the semi-major and semi-minor axes, respectively, of E_0 . Let (x_0, y_0) denote the center of E_0 and let Δ and δ be as in (1). Then*

$$(2) \quad \frac{b^2}{a^2} = \frac{A + C - \sqrt{(A - C)^2 + B^2}}{A + C + \sqrt{(A - C)^2 + B^2}} \text{ and}$$

$$(3) \quad x_0 = \frac{BE - 2CD}{\Delta}, y_0 = \frac{BD - 2AE}{\Delta}.$$

Proof. Let $\mu = \frac{4\delta}{\Delta^2}$. By [7],

$$(4) \quad \begin{aligned} a^2 &= \frac{1}{2}\mu(A + C + \sqrt{(A - C)^2 + B^2}), \\ b^2 &= \frac{1}{2}\mu(A + C - \sqrt{(A - C)^2 + B^2}). \end{aligned}$$

Note that $\mu > 0$ by Lemma 2.1. (2) then follows immediately from (4). We omit the details for (3).

Throughout the paper we let L_Q denote the *line* thru the midpoints of a given quadrilateral, Q , and we define an affine transformation, $T : R^2 \rightarrow R^2$ to be the map $T(\hat{x}) = A\hat{x} + \hat{b}$, where A is an invertible 2×2 matrix. Note that affine transformations map lines to lines, parallel lines to parallel lines, and preserve ratios of lengths along a given line. Also, the family of ellipses, tangent lines to ellipses, and four-sided convex polygons are preserved under affine transformations.

A quadrilateral which has an incircle, i.e., one for which a single circle can be constructed which is tangent to all four sides, is called a tangential quadrilateral. A quadrilateral which has perpendicular diagonals is called an orthodiagonal quadrilateral. The following lemmas show that affine transformations preserve the class of midpoint diagonal quadrilaterals and send conjugate diameters to conjugate diameters. The proofs follow immediately from the properties of affine transformations.

Lemma 2.3. *Let $T : R^2 \rightarrow R^2$ be an affine transformation and let Q be a midpoint diagonal quadrilateral. Then $Q' = T(Q)$ is also a midpoint diagonal quadrilateral.*

Lemma 2.4. *Let $T : R^2 \rightarrow R^2$ be an affine transformation and suppose that l and m are conjugate diameters of an ellipse, E_0 . Then $l' = T(l)$ and $m' = T(m)$ are conjugate diameters of $E'_0 = T(E_0)$.*

The following lemma shows that the scaling transformations preserve the eccentricity of ellipses, as well as the property of the equal conjugate diameters of an ellipse being parallel to the diagonals of Q .

Lemma 2.5. *Let T be the nonsingular affine transformation given by $T(x, y) = (kx, ky)$, $k \neq 0$.*

(i) *Then E_0 and $T(E_0)$ have the same eccentricity for any ellipse, E_0 .*

(ii) *If E_0 is an ellipse which is not a circle and if the equal conjugate diameters of E_0 are parallel to the diagonals of Q , then the equal conjugate diameters of $T(E_0)$ are parallel to the diagonals of $T(Q)$.*

Proof. (i) follows immediately and we omit the proof. To prove (ii), suppose that l and m are equal conjugate diameters of an ellipse, E_0 , which are parallel to the diagonals of Q . By Lemma 2.4, $l' = T(l)$ and $m' = T(m)$ are conjugate diameters of $E'_0 = T(E_0)$. Since $|T(P_1)T(P_2)| = k|P_1P_2|$ for any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, l' and m' are equal conjugate diameters of $T(E_0)$. Since affine transformations take parallel lines to parallel lines, l' and m' are parallel to the diagonals of $T(Q)$. That proves (ii).

The following lemma shows when a trapezoid can be a midpoint diagonal quadrilateral.

Lemma 2.6. *Suppose that Q is a midpoint diagonal quadrilateral which is also a trapezoid. Then Q is a parallelogram.*

Proof. We use proof by contradiction. So suppose that Q is a midpoint diagonal quadrilateral which is a trapezoid, but which is *not* a parallelogram. By affine invariance, we may assume that Q is the trapezoid with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, t)$, $0 < t \neq 1$. The diagonals of Q are then the open line segments $D_1: y = tx$ and $D_2: y = 1 - x$, each with $0 < x < 1$. The midpoints of the diagonals are $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{1}{2}, \frac{t}{2}\right)$, and the diagonals intersect at $P = \left(\frac{1}{1+t}, \frac{t}{1+t}\right)$. Now $M_1 = P \iff t = 1$ and $M_2 = P \iff t = 1$, which contradicts the assumption that $t \neq 1$. Hence Q is not a midpoint diagonal quadrilateral.

Remark 3. *We use the notation $Q(A_1, A_2, A_3, A_4)$ to denote the quadrilateral with vertices A_1, A_2, A_3 , and A_4 , starting with $A_1 =$ lower left corner and going clockwise. Denote the sides of $Q(A_1, A_2, A_3, A_4)$ by S_1, S_2, S_3 , and S_4 , going clockwise and starting with the leftmost side, S_1 . Denote the lengths of the sides of $Q(A_1, A_2, A_3, A_4)$ by $a = |A_1A_4|$, $b = |A_1A_2|$, $c = |A_2A_3|$, and $d = |A_3A_4|$. Finally, denote the diagonals of $Q(A_1, A_2, A_3, A_4)$ by $D_1 = \overline{A_1A_3}$ and $D_2 = \overline{A_2A_4}$.*

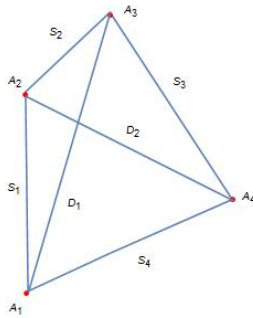
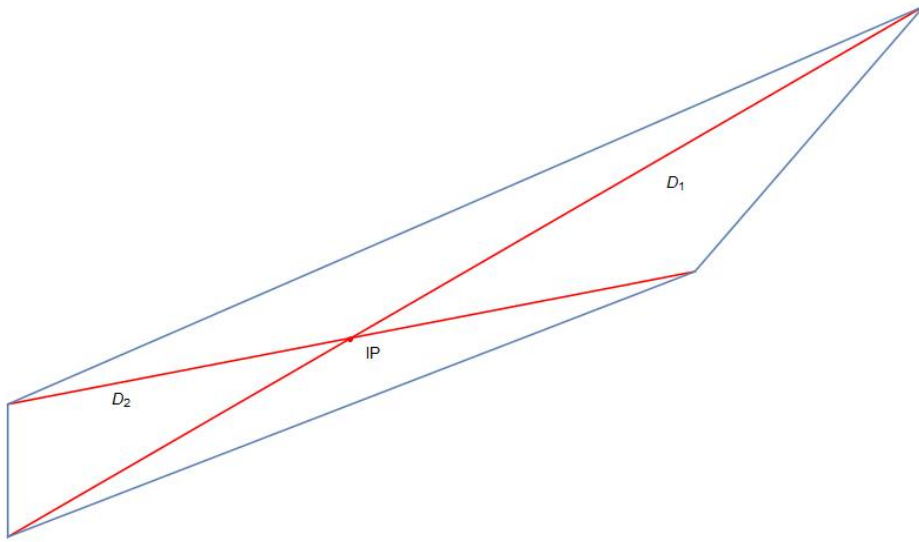
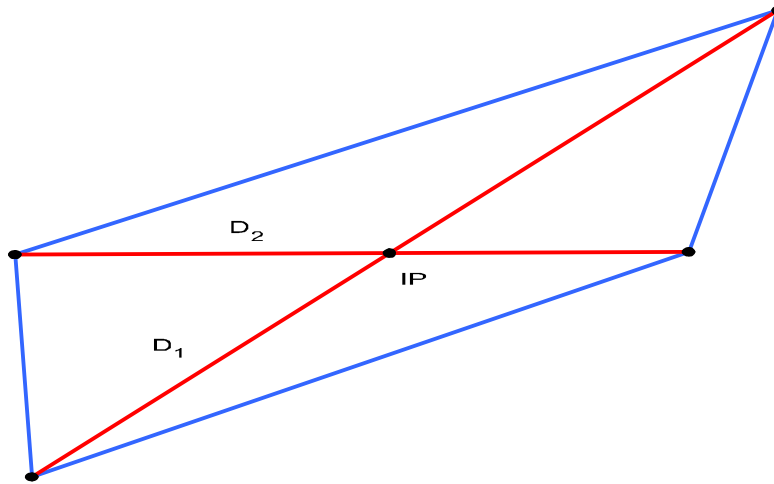


Figure 4: $Q(A_1, A_2, A_3, A_4)$

We note here that there are two types of midpoint diagonal quadrilaterals: Type 1, where the diagonals intersect at the midpoint of D_2 and Type 2, where the diagonals intersect at the midpoint of D_1 .

Figure 5: Diagonals intersect at the midpoint of D_2 Figure 6: Diagonals intersect at the midpoint of D_1

Notation: The lines containing the diagonal line segments, D_1 and D_2 , of any quadrilateral are denoted by $\overleftrightarrow{D_1}$ and $\overleftrightarrow{D_2}$.

Given a convex quadrilateral, $Q = Q(A_1, A_2, A_3, A_4)$, which is **not** a parallelogram, it will simplify our work below to consider quadrilaterals with a special set of vertices. In particular, there is an affine transformation which sends A_1, A_2 , and A_4 to the points $(0, 0)$, $(0, 1)$, and $(1, 0)$, respectively. It then follows that $A_3 = (s, t)$ for some $s, t > 0$. We thus let $Q_{s,t}$ denote the quadrilateral with vertices $(0, 0)$, $(0, 1)$, (s, t) , and $(1, 0)$. Since $Q_{s,t}$ is convex, $s + t > 1$. Also, if Q has a pair of parallel vertical sides, first rotate counterclockwise by 90° , yielding a quadrilateral with parallel horizontal sides. Since we are assuming that Q is not a parallelogram, we may then also assume that $Q_{s,t}$ does not have parallel vertical sides and thus $s \neq 1$. Summarizing, we have:

Suppose that Q is a convex quadrilateral which is **not** a parallelogram. Then there is an affine transformation which sends Q to the quadrilateral

$$\begin{aligned} Q_{s,t} &= Q(A_1, A_2, A_3, A_4), \\ A_1 &= (0, 0), A_2 = (0, 1), A_3 = (s, t), A_4 = (1, 0), \\ (5) \quad &\text{with } (s, t) \in G, \text{ where} \end{aligned}$$

$$(6) \quad G = \{(s, t) : s, t > 0, s + t > 1, s \neq 1\}.$$

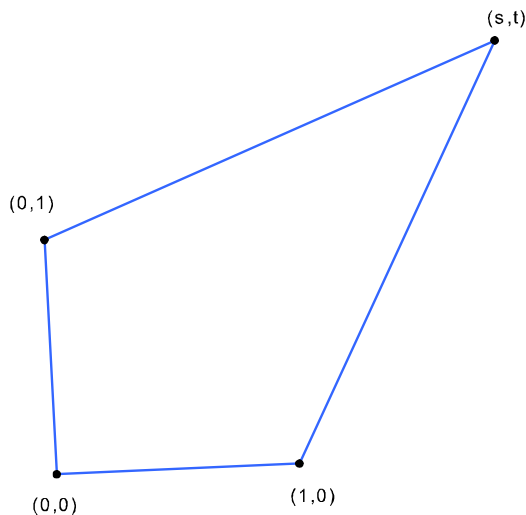


Figure 7

Since the midpoints of the diagonals of $Q_{s,t}$ are $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{s}{2}, \frac{t}{2}\right)$, by Theorem 1.1, the center of any ellipse, E_0 , inscribed in $Q_{s,t}$ must lie on the open line segment $\{(h, L_Q(h)) : h \in I\}$, where

$$(7) \quad \begin{aligned} L_Q(x) &= \frac{1}{2} \frac{s - t + 2x(t - 1)}{s - 1}, \\ I &= \begin{cases} \left(\frac{s}{2}, \frac{1}{2}\right) & \text{if } s < 1 \\ \left(\frac{1}{2}, \frac{s}{2}\right) & \text{if } s \geq 1. \end{cases} \end{aligned}$$

We now answer the following important question: How does one find the equation of an ellipse, E_0 , inscribed in $Q_{s,t}$ and the points of tangency of E_0 with $Q_{s,t}$? We sketch the derivation of the equation and points of tangency now. First, since E_0 has center $(h, L_Q(h))$, $h \in I$, one may write the equation of E_0 in the form

$$(8) \quad (x - h)^2 + B(x - h)(y - L_Q(h)) + C(y - L_Q(h))^2 + F = 0.$$

Throughout the rest of the paper we denote the open unit interval by

$$J = (0, 1).$$

Now suppose that E_0 is tangent to $Q_{s,t}$ at the points $P_q = (q, 0)$ and $P_v = (0, v)$, where $q, v \in J$. Differentiating (8) with respect to x and plugging in

P_q and P_v yields

$$(9) \quad \begin{aligned} q - h &= \frac{BL_Q(h)}{2}, \\ v - L_Q(h) &= \frac{Bh}{2C}. \end{aligned}$$

Plugging in P_q and P_v into (8) yields $(q-h)^2 - BL_Q(h)(q-h) + C(L_Q(h))^2 + F = 0$ and $h^2 - Bh(v - L_Q(h)) + C(v - L_Q(h))^2 + F = 0$. By (9), we have $F = \frac{h^2}{4C}(B^2 - 4C)$ and $F = \frac{L_Q^2(h)}{4}(B^2 - 4C)$. Using both expressions for F gives

$$(10) \quad C = \frac{h^2}{L_Q^2(h)}.$$

Now by (9) again

$$(11) \quad B = \frac{2(q-h)}{L_Q(h)}.$$

(9), (11), and (10) then imply that

$$(12) \quad v = \frac{qL_Q(h)}{h}.$$

Substituting (11) and (10) into $F = \frac{h^2}{4C}(B^2 - 4C)$ yields $F = q^2 - 2qh$. (8) then becomes

$$(13) \quad (x-h)^2 + \frac{2(q-h)}{L_Q(h)}(x-h)(y-L_Q(h)) + \frac{h^2}{L_Q^2(h)}(y-L_Q(h))^2 + q^2 - 2qh = 0.$$

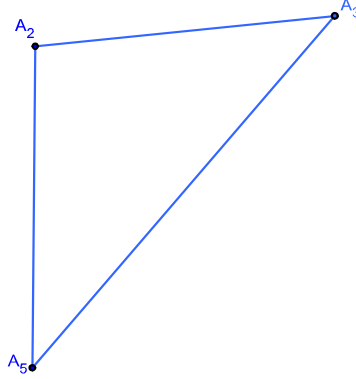
Remark 4. Using Lemma 2.1, it is not hard to show that (13) defines the equation of an ellipse for any $h \in I$.

Finally, we want to find h in terms of q , which makes the final equation simpler than expressing everything in terms of h . One way to do this is to use the following well-known theorem of Marden(see [5]).

Theorem 2.1. (Marden): Let $F(z) = \frac{t_1}{z-z_1} + \frac{t_2}{z-z_2} + \frac{t_3}{z-z_3}$, $\sum_{k=1}^3 t_k = 1$, and let Z_1 and Z_2 denote the zeros of $F(z)$. Let L_1, L_2, L_3 be the line segments connecting $z_2 \text{ } \mathcal{E} \text{ } z_3$, $z_1 \text{ } \mathcal{E} \text{ } z_3$, and $z_1 \text{ } \mathcal{E} \text{ } z_2$, respectively. If $t_1 t_2 t_3 > 0$, then Z_1 and Z_2 are the foci of an ellipse which is tangent to L_1, L_2 , and L_3 at the points $\zeta_1 = \frac{t_2 z_3 + t_3 z_2}{t_2 + t_3}$, $\zeta_2 = \frac{t_1 z_3 + t_3 z_1}{t_1 + t_3}$, and $\zeta_3 = \frac{t_1 z_2 + t_2 z_1}{t_1 + t_2}$, respectively.

Applying Marden's Theorem to the triangle $\Delta A_2 A_3 A_5$, where

$A_5 = \left(0, -\frac{t}{s-1}\right)$, one can show that E_0 is tangent to $Q_{s,t}$ at the point $\left(\frac{s-2h}{2(t-1)h+s-t}, 0\right)$. Many of the details of this can be found in [2].

Figure 8: $\Delta A_2 A_3 A_5$

Hence $q = \frac{s - 2h}{2(t - 1)h + s - t}$, which implies that

$$(14) \quad h = \frac{1}{2} \frac{q(t - s) + s}{q(t - 1) + 1}.$$

Substituting for h in (13) using (14) and (7), (13) becomes

$$(15) \quad t^2 x^2 + (4q^2(t - 1)t + 2qt(s - t + 2) - 2st)xy + ((1 - q)s + qt)^2 y^2 - 2qt^2 x - 2qt((1 - q)s + qt)y + q^2 t^2 = 0.$$

One point of tangency is of course given by $(q, 0)$. Using (15), it is then not difficult to find the other points of tangency, which is given in the following proposition (we have relabeled P_q and P_v).

Proposition 2.2. *Suppose that E_0 is an ellipse inscribed in $Q_{s,t}$. Then E_0 is tangent to the four sides of $Q_{s,t}$ at the points $q_1 = \left(0, \frac{qt}{(t - s)q + s}\right) \in S_1$, $q_2 = \left(\frac{(1 - q)s^2}{(t - 1)(s + t)q + s}, \frac{t(s + q(t - 1))}{(t - 1)(s + t)q + s}\right) \in S_2$, $q_3 = \left(\frac{s + q(t - 1)}{(s + t - 2)q + 1}, \frac{(1 - q)t}{(s + t - 2)q + 1}\right) \in S_3$, and $q_4 = (q, 0) \in S_4$, $q \in J$.*

Remark 5. *It is not hard to show that each of the denominators of the q_j above are non-zero.*

Finally we state the analogy of Proposition 2.2 for parallelograms. A slightly different version was proven in [4]. We omit the details of the proof.

Proposition 2.3. *Let P be the parallelogram with vertices $A_1 = (-l - d, -k)$, $A_2 = (-l + d, k)$, $A_3 = (l + d, k)$, and $A_4 = (l - d, -k)$, where $l, k > 0$, $d < l$. If E_0 is an ellipse inscribed in P , then E_0 is tangent to the four sides of P at the points $q_1 = (-l + dv, kv) \in S_1$, $q_2 = (-lv + d, k) \in S_2$, $q_3 = (l - dv, -kv) \in S_3$, and $q_4 = (lv - d, -k) \in S_4$.*

3. TANGENCY CHORDS PARALLEL TO THE DIAGONALS

The following lemma gives necessary and sufficient conditions for the quadrilateral $Q_{s,t}$ given in (5) to be a midpoint diagonal quadrilateral.

Lemma 3.1. (i) $Q_{s,t}$ is a type 1 midpoint diagonal quadrilateral if and only if $s = t$.

(ii) $Q_{s,t}$ is a type 2 midpoint diagonal quadrilateral if and only if $s+t = 2$.

Proof. The diagonal lines of $Q_{s,t}$ are $\overleftrightarrow{D_1}: y = \frac{t}{s}x$ and $\overleftrightarrow{D_2}: y = 1 - x$, and they intersect at the point $P = \left(\frac{s}{s+t}, \frac{t}{s+t}\right)$. The midpoints of the diagonal line segments D_1 and D_2 are $M_1 = \left(\frac{s}{2}, \frac{t}{2}\right)$ and $M_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$, respectively. Now $M_2 = P \iff \frac{s}{s+t} = \frac{1}{2}$ and $\frac{t}{s+t} = \frac{1}{2}$, both of which hold if and only if $s = t$. That proves (i). $M_1 = P \iff \frac{s}{s+t} = \frac{1}{2}s$ and $\frac{t}{s+t} = \frac{1}{2}t$, both of which hold if and only if $s+t = 2$. That proves (ii).

Now recall property P1 from the introduction: (P1) Each ellipse inscribed in \mathbb{P} has tangency chords which are parallel to one of the diagonals of \mathbb{P} . The following theorem shows that P1 completely characterizes the class of midpoint diagonal quadrilaterals.

Theorem 3.1. Suppose that E_0 is an ellipse inscribed in a convex quadrilateral $Q = Q(A_1, A_2, A_3, A_4)$. Let $q_j \in S_j, j = 1, 2, 3, 4$ denote the points of tangency of E_0 with Q , and let $D_1 = \overline{A_1A_3}$ and $D_2 = \overline{A_2A_4}$ denote the diagonals of Q .

(i) If Q is a type 1 midpoint diagonal quadrilateral, then $\overleftrightarrow{q_2q_3}$ and $\overleftrightarrow{q_1q_4}$ are parallel to D_2 .

(ii) If Q is a type 2 midpoint diagonal quadrilateral, then $\overleftrightarrow{q_1q_2}$ and $\overleftrightarrow{q_3q_4}$ are parallel to D_1 .

(iii) If Q is **not** a midpoint diagonal quadrilateral, then neither $\overleftrightarrow{q_1q_2}$ nor $\overleftrightarrow{q_3q_4}$ are parallel to D_1 , and neither $\overleftrightarrow{q_2q_3}$ nor $\overleftrightarrow{q_1q_4}$ are parallel to D_2 .

Proof. Case 1: Q is not a parallelogram.

Then by Proposition 2, we may assume that $Q = Q_{s,t}$ with diagonal lines $\overleftrightarrow{D_1}: y = \frac{t}{s}x$ and $\overleftrightarrow{D_2}: y = 1 - x$. Using Proposition 2.2, after some simplification we have:

$$\begin{aligned} \text{slope of } \overleftrightarrow{q_1q_2} &= \frac{t(2q(t-1)+s)}{s((t-s)q+s)}, \text{ so that the slope of } \overleftrightarrow{q_1q_2} = \frac{t}{s} \iff \\ \frac{2(t-1)q+s}{(t-s)q+s} &= 1 \iff (s+t-2)q = 0 \iff s+t = 2 \text{ since } q = 0 \notin J. \\ \text{slope of } \overleftrightarrow{q_3q_4} &= \frac{t}{(s+t-2)q+s}, \text{ so that the slope of } \overleftrightarrow{q_3q_4} = \frac{t}{s} \iff \\ \frac{1}{(s+t-2)q+s} &= \frac{1}{s} \iff (s+t-2)q = 0 \iff s+t = 2 \text{ since } q = 0 \notin J. \end{aligned}$$

slope of $\overleftrightarrow{q_2q_3} = -\frac{t(2(t-1)q+s-t+1)}{(s^2+t^2-s-t)q-s^2+st+s}$, so that the slope of $\overleftrightarrow{q_2q_3} = -1 \iff$
 $t(2(t-1)q+s-t+1) = (s^2+t^2-s-t)q-s^2+st+s \iff$
 $(s+t-1)(s-t)(q-1) = 0 \iff s = t$ since $q = 1 \notin J$ and $s+t \neq 1$.
 slope of $\overleftrightarrow{q_1q_4} = \frac{t}{(s-t)q-s}$, so that the slope of $\overleftrightarrow{q_1q_4} = -1 \iff$
 $(s-t)q-s = -t \iff (q-1)(s-t) = 0 \iff s = t$ since $q = 1 \notin J$.

Theorem 3.1 then follows from Lemma 3.1.

Case 2: Q is a parallelogram.

As noted in the introduction, Theorem 3.1 is probably known in this case, and there are undoubtedly other ways to prove it for parallelograms.

Using Proposition 2.3, it follows easily that the slope of $\overleftrightarrow{q_1q_2}$ = the slope of $\overleftrightarrow{q_3q_4} = \frac{k}{l+d}$ and the slope of $\overleftrightarrow{q_2q_3}$ = the slope of $\overleftrightarrow{q_1q_4} = \frac{k}{d-l}$. Since the diagonal lines Q are $\overleftrightarrow{D_1}D_1: y = k + \frac{k}{l+d}(x-l-d)$ and $\overleftrightarrow{D_2}: y = k + \frac{k}{d-l}(x+l-d)$, and a parallelogram is a special case of a midpoint diagonal quadrilateral, that proves Theorem 3.1 for case 2. Note that one could map Q to the unit square and then use a simplified version of Proposition 2.3, but that does not simplify the proof very much.

Now recall that the lengths of the sides of $Q(A_1, A_2, A_3, A_4)$ are denoted by $a = |A_1A_4|$, $b = |A_1A_2|$, $c = |A_2A_3|$, and $d = |A_3A_4|$.

Lemma 3.2. *Suppose that $Q = Q(A_1, A_2, A_3, A_4)$ is both a tangential and a midpoint diagonal quadrilateral. Then Q is an orthodiagonal quadrilateral.*

Remark 6. *We actually prove more—that Q is a **kite**. That is, that two pairs of adjacent sides of Q are equal.*

Proof. Since Q is tangential, there is a circle, E_0 , inscribed in Q . Let $q_j \in S_j, j = 1, 2, 3, 4$ denote the points of tangency of E_0 with Q . Define the triangles $T_1 = \Delta q_4A_1q_1$ and $T_2 = \Delta A_4A_1A_2$, and define the lines $L_1 = \overleftrightarrow{q_1q_4}$ and $L_2 = \overleftrightarrow{q_2q_3}$. Suppose first that Q is a type 1 midpoint diagonal quadrilateral. Then $L_1 \parallel D_2$ by Theorem 3.1(i), which implies that T_1 and T_2 are similar triangles. Also, since E_0 is a circle, $|A_1q_4| = |A_1q_1|$, which implies that T_1 is isosceles. Hence T_2 is also isosceles with $b = a$. In a similar fashion, one can show that $c = d$ using the fact that $L_2 \parallel D_1$. Thus $a^2 + c^2 = b^2 + d^2$, which implies that Q is an orthodiagonal quadrilateral. The proof when Q is a type 2 midpoint diagonal quadrilateral is similar and we omit the details.

We now prove a result somewhat similar to Lemma 3.2.

Lemma 3.3. *Suppose that $Q = Q(A_1, A_2, A_3, A_4)$ is both a tangential and an orthodiagonal quadrilateral. Then Q is a midpoint diagonal quadrilateral.*

Proof. Since Q is tangential, there is a circle, E_0 , inscribed in Q and $a + c = b + d$, which implies that $d = a + c - b$. Since Q is orthodiagonal, $a^2 + c^2 = b^2 + d^2$. Hence

$b^2 + (a+c-b)^2 - a^2 - c^2 = 0$, which implies that $2(b-c)(b-a) = 0$, and so $a = b$ and/or $b = c$. We prove the case when $a = b$. Let $q_j \in S_j, j = 1, 2, 3, 4$

denote the points of tangency of E_0 with Q . Then the triangle $T_1 = \Delta q_4 A_1 q_1$ is isosceles since $|A_1 q_4| = |A_1 q_1|$, and the triangle $T_2 = \Delta A_4 A_1 A_2$ is isosceles since $a = b$. Thus T_1 and T_2 are similar triangles, which implies that the line $\overleftrightarrow{q_1 q_4}$ is parallel to D_2 . By Theorem 3.1(iii), Q is a midpoint diagonal quadrilateral.

4. CONJUGATE DIAMETERS PARALLEL TO THE DIAGONALS

Recall property P2 from the introduction: (P2) Each ellipse inscribed in \mathbb{P} has a pair of conjugate diameters which are parallel to the diagonals of \mathbb{P} . The following theorem shows that P2 completely characterizes the class of midpoint diagonal quadrilaterals.

Theorem 4.1. (i) *Suppose that Q is a midpoint diagonal quadrilateral. Then **each** ellipse inscribed in Q has a unique pair of conjugate diameters parallel to the diagonals of Q .*

(ii) *Suppose that Q is **not** a midpoint diagonal quadrilateral. Then **no** ellipse inscribed in Q has conjugate diameters parallel to the diagonals of Q .*

Proof. Let E_0 be an ellipse inscribed in Q and let D_1 and D_2 denote the diagonals of Q . Use an affine transformation, T , to map E_0 to a circle, E'_0 , inscribed in the tangential quadrilateral, $Q' = T(Q)$. Let L_1 be a diameter of E_0 parallel to D_1 . T maps L_1 to a diameter, L'_1 , of E'_0 parallel to one of the diagonals of Q' , which we call D'_1 . Let D'_2 be the other diagonal of Q' . Let L'_2 be the diameter of E'_0 perpendicular to L'_1 , which implies that L'_1 and L'_2 are conjugate diameters since E'_0 is a circle. By Lemma 2.4, T^{-1} maps L'_2 to L_2 , a diameter of E_0 conjugate to L_1 . To prove (i), suppose that Q is a midpoint diagonal quadrilateral. By Lemma 2.3, Q' is also a midpoint diagonal quadrilateral. By Lemma 3.2, Q' is an orthodiagonal quadrilateral, which implies that $D'_1 \perp D'_2$. Since $L'_1 \parallel D'_1$, $L'_1 \perp L'_2$, and $D'_1 \perp D'_2$, L'_2 must be parallel to D'_2 , which implies that L_2 is parallel to D_2 since T^{-1} is an affine transformation. That proves (i).

To prove (ii), suppose that Q is **not** a midpoint diagonal quadrilateral. Since Q' is tangential, if Q' were also an orthodiagonal quadrilateral, then by Lemma 3.3, Q' would be a midpoint diagonal quadrilateral. Hence Q' cannot be an orthodiagonal quadrilateral, which implies that D'_1 is not perpendicular to D'_2 . Now if L'_2 were parallel to D'_2 , then it would follow that $D'_1 \perp D'_2$ since $L'_1 \parallel D'_1$ and $L'_1 \perp L'_2$, a contradiction. Hence $L'_2 \not\parallel D'_2$, which implies that $L_2 \not\parallel D_2$ since T is an affine transformation. That proves (ii).

5. EQUAL CONJUGATE DIAMETERS AND THE ELLIPSE OF MINIMAL ECCENTRICITY

By Theorem 4.1(i), each ellipse inscribed in a midpoint diagonal quadrilateral, Q , has conjugate diameters parallel to the diagonals of Q . In particular, this holds for the unique ellipse of minimal eccentricity (whose existence we prove below), E_I , inscribed in Q . However, in Theorem 5.1(ii) below, we prove a stronger result for E_I .

Theorem 5.1. (i) *There is a unique ellipse of minimal eccentricity, E_I , inscribed in a midpoint diagonal quadrilateral, Q .*

(ii) Furthermore, the unique pair of conjugate diameters parallel to the diagonals of Q are **equal** conjugate diameters of E_I .

Remark 7. Suppose that Q is a type 1 midpoint diagonal quadrilateral and let CD_1 and CD_2 be the equal conjugate diameters in Theorem 5.1(ii) parallel to the diagonals, D_1 and D_2 . Let $\overleftrightarrow{CD_1}$ and $\overleftrightarrow{CD_2}$ denote the lines containing CD_1 and CD_2 , respectively. Since $\overleftrightarrow{CD_1}$ is parallel to $\overleftrightarrow{D_1}$ and $\overleftrightarrow{D_1} = L$, $\overleftrightarrow{CD_1}$ is parallel to L . Since L and $\overleftrightarrow{CD_1}$ each pass through the center of E_I , $\overleftrightarrow{CD_1} = L$. Similarly, for type 2 midpoint diagonal quadrilaterals, $\overleftrightarrow{CD_2} = L$.

Remark 8. Theorem 5.1(ii) cannot hold if Q is **not** a midpoint diagonal quadrilateral, since in that case no ellipse inscribed in Q has conjugate diameters parallel to the diagonals of Q by Theorem 4.1(ii). But Theorem 5.1(ii) implies the following weaker result: The smallest nonnegative angle between equal conjugate diameters of E_I equals the smallest nonnegative angle between the diagonals of Q when Q is a midpoint diagonal quadrilateral. This was proven in [4] for parallelograms. We do not know if this property of E_I can hold if Q is **not** a midpoint diagonal quadrilateral.

Before proving Theorem 5.1, we need several preliminary results. We omit the details for the proof of Theorem 5.1 when Q is a parallelogram. So suppose that Q is a midpoint diagonal quadrilateral which is **not** a parallelogram. By using an **isometry** of the plane, we may assume that Q has vertices $(0, 0)$, $(0, u)$, (s, t) , and (v, w) , where $s, v, u > 0, t > w$. To obtain this isometry, first, if Q has a pair of parallel vertical sides, first rotate counterclockwise by 90° , yielding a quadrilateral with parallel horizontal sides. Since we are assuming that Q is not a parallelogram, we may then also assume that Q does not have parallel vertical sides. One can now use a translation, if necessary, to map the lower left hand corner vertex of Q to $(0, 0)$. Finally a rotation, if necessary, yields vertices $(0, 0)$, $(0, u)$, (s, t) , and (v, w) . Note that such a rotation leaves Q without parallel vertical sides. In addition, by Lemma 2.5 with $T(x, y) = \frac{1}{u}(x, y)$, we may also assume that one of the vertices of Q is $(0, 1)$. So we now work with the quadrilateral

$$(16) \quad \begin{aligned} Q_{s,t,v,w} &= Q(A_1, A_2, A_3, A_4), A_1 = (0, 0), \\ &A_2 = (0, 1), A_3 = (s, t), A_4 = (v, w), \end{aligned}$$

where

$$(17) \quad s, v > 0, t > w, s \neq v.$$

The sides of $Q_{s,t,v,w}$, going clockwise, are given by $S_1 = \overline{(0, 0)(0, u)}$, $S_2 = \overline{(0, 1)(s, t)}$, $S_3 = \overline{(s, t)(v, w)}$, and $S_4 = \overline{(0, 0)(v, w)}$. By Lemma 2.3, $Q_{s,t,v,w}$ is a midpoint diagonal quadrilateral, which implies, by Lemma 2.6, that $Q_{s,t,v,w}$ is **not** a trapezoid since $Q_{s,t,v,w}$ is **not** a parallelogram. We find it useful to define the following expressions, each of which depend on s, t, v ,

and w :

$$(18) \quad \begin{aligned} f_1 &= v(t-1) + (1-w)s, \\ f_2 &= vt - ws, \\ f_3 &= ws - v(t-1). \end{aligned}$$

• Since $Q_{s,t,v,w}$ is convex, (s, t) must lie above $\overrightarrow{(0, 1) (v, w)}$ and (v, w) must lie below $\overrightarrow{(0, 0) (s, t)}$, which implies that

$$(19) \quad f_1 > 0 \text{ and } f_2 > 0.$$

Since no two sides of $Q_{s,t,v,w}$ are parallel, $S_2 \nparallel S_4$, which implies that

$$(20) \quad f_3 \neq 0.$$

$M_1 = \left(\frac{1}{2}v, \frac{1}{2}(w+1)\right)$ and $M_2 = \left(\frac{1}{2}s, \frac{1}{2}t\right)$ are the midpoints of the diagonals of $Q_{s,t,v,w}$ and the equation of the line thru M_1 and M_2 is

$$(21) \quad \begin{aligned} y &= L(x) = \frac{t}{2} + \frac{w+1-t}{v-s} \left(x - \frac{s}{2}\right), \\ x &\in I = \begin{cases} (v/2, s/2) & \text{if } v < s \\ (s/2, v/2) & \text{if } s < v \end{cases}. \end{aligned}$$

The diagonal line segments of $Q_{s,t,v,w}$ are $D_1 = \overline{(0, 0) (s, t)}$ and $D_2 = \overline{(0, 1) (v, w)}$. Now let E_0 be an ellipse inscribed in $Q_{s,t,v,w}$ and suppose that E_0 is tangent to $Q_{s,t,v,w}$ at the points $P_q = \left(q, \frac{w}{v}q\right) \in S_4$ and $P_r = (0, r) \in S_1$, $0 < q < v$, $r \in J$. Using these points of tangency, it is not hard to show that $q = \frac{svr}{(s-f_2)r + f_2}$. That leads to Proposition 5.2 below, which gives necessary and sufficient conditions for the general equation of an ellipse inscribed in $Q_{s,t,v,w}$. It is useful for us to emphasize the dependence of the coefficients of the general equation on the parameter r in our notation.

Proposition 5.2. *Suppose that (17), (19), and (20) hold.*

(i) *Let E_0 be an ellipse inscribed in $Q_{s,t,v,w}$. Then for some $r \in J$, the general equation of E_0 is given by $\psi(x, y) = 0$, where*

$$(22) \quad \psi(x, y) = A(r)x^2 + B(r)xy + C(r)y^2 + D(r)x + E(r)y + F(r),$$

and

$$(23) \quad \begin{aligned} A(r) &= (s^2 + v^2t^2 + w^2s^2 - 2tvs(w+1) + 2ws(2v-s))r^2 + \\ &\quad 2v(st - 2ws - t^2v + tsw)r + t^2v^2, \\ B(r) &= -2vs(2r^2(v-s) + rs(w+1) + v(t-rt-2r)), \\ C(r) &= v^2s^2, D(r) = 2srv(-rs(w+1) + 2ws + tv(r-1)), \\ E(r) &= -2rv^2s^2, F(r) = r^2s^2v^2. \end{aligned}$$

(ii) *Conversely, if for some $r \in J$ the general equation of E_0 is given by $\psi(x, y) = 0$, where (22) and (23) hold, then E_0 is an ellipse inscribed in $Q_{s,t,v,w}$.*

Proof. Using Lemma 2.1, it is not hard to show that $\psi(x, y) = 0$ defines the equation of an ellipse for any $r \in J$. Using standard calculus techniques, it is also not difficult to show that the ellipse defined by $\psi(x, y) = 0$ is inscribed in $Q_{s,t,v,w}$ for any $r \in J$. The converse result, that any ellipse inscribed in $Q_{s,t,v,w}$ has equation given by $\psi(x, y) = 0$ for some $r \in J$ can be proven in a similar fashion to the proof of Proposition 2.2. We leave the details to the reader.

Remark 9. *We know have an alternate method of proving Proposition 5.2 which does not require Marden's Theorem. Instead it uses the following well-known property of tangents to an ellipse: If the tangents to an ellipse at two points A and B meet at a point P , then the line joining P to the center O of the ellipse bisects the chord AB . The advantage of this alternate approach is that it also gives the center of E_0 and the points of tangency of E_0 with $Q_{s,t,v,w}$.*

The following lemma gives necessary and sufficient conditions for $Q_{s,t,v,w}$ to be a midpoint diagonal quadrilateral.

Lemma 5.1. *Suppose that (17), (19), and (20) hold.*

(i) $Q_{s,t,v,w}$ is a type 1 midpoint diagonal quadrilateral if and only if

$$(24) \quad vt = (w + 1)s.$$

(ii) $Q_{s,t,v,w}$ is a type 2 midpoint diagonal quadrilateral if and only if

$$(25) \quad (t - 2)v = (w - 1)s.$$

Proof. \overleftrightarrow{D}_1 has equation $y = \frac{t}{s}x$. Using (21), $\overleftrightarrow{D}_1 = L \iff$

$$(26) \quad \frac{w + 1 - t}{v - s} = \frac{t}{s} \text{ and} \\ \frac{t}{2} - \frac{s}{2} \frac{w + 1 - t}{v - s} = 0.$$

It follows easily that (26) holds if and only if (24) holds, which proves (i). The proof of (ii) follows in a similar fashion.

6. PROOF OF THEOREM 5.1

Proof. We assume first that Q is a *tangential* quadrilateral. Then Q is an orthodiagonal quadrilateral by Lemma 3.2, and so the diagonals of Q are perpendicular. Also, there is a unique circle, Φ , inscribed in Q , which implies that Φ is the unique ellipse of minimal eccentricity inscribed in Q since Φ has eccentricity 0. Since any pair of perpendicular diameters of a circle are equal conjugate diameters, in particular the unique pair which are parallel to the diagonals of Q are equal conjugate diameters of Φ , and Theorem 5.1 holds. So assume now that Q is **not** a tangential quadrilateral. It suffices to assume that $Q = Q_{s,t,v,w}$ and that (17), (19), and (20) hold. Let E_0 be an ellipse inscribed in $Q_{s,t,v,w}$. By Proposition 2, the general equation of E_0 is given by $\psi(x, y) = 0$, where ψ is given by (22) and (23) for some $r \in J$. Let a and b denote the lengths of the semi-major and semi-minor axes, respectively, of E_0 . Now $\frac{b^2}{a^2}$ is really a function of $r \in J$ if we allow E_0 to

vary over all ellipses inscribed in $Q_{s,t,v,w}$. By (2) in Lemma 2.2, $\frac{b^2}{a^2} = G(r)$,

where $G(r) = \frac{A(r) + C(r) - \sqrt{(A(r) - C(r))^2 + (B(r))^2}}{A(r) + C(r) + \sqrt{(A(r) - C(r))^2 + (B(r))^2}}$. Since the square of the eccentricity of E_0 equals $1 - \frac{b^2}{a^2}$, it suffices to maximize $\frac{b^2}{a^2}$. Letting

$$(27) \quad \begin{aligned} O(r) &= A(r) + C(r), \\ M(r) &= (A(r) - C(r))^2 + (B(r))^2, \end{aligned}$$

we have $G(r) = \frac{O(r) - \sqrt{M(r)}}{O(r) + \sqrt{M(r)}}$. Since Q is not a tangential quadrilateral, it follows easily that $Q_{s,t,v,w}$ is also not a tangential quadrilateral. If $M(r_0) = 0$ for some $r_0 \in J$, then $A(r_0) - C(r_0) = 0$ and $B(r_0) = 0$, which implies that the ellipse inscribed in $Q_{s,t,v,w}$ corresponding to r_0 is a circle. But that contradicts the assumption that $Q_{s,t,v,w}$ is not a tangential quadrilateral. Thus $M(r) \neq 0$ for all $r \in J$. Since M is non-negative, it follows that

$$(28) \quad M(r) > 0, r \in J.$$

Define the quartic polynomial

$$N(r) = O^2(r) - M(r).$$

After some simplification, N factors as

$$(29) \quad N(r) = 16s^2v^2r(1-r)((s-v)r+v)((s-v)r+f_2),$$

and N has roots

$$(30) \quad r_1 = 0, r_2 = 1, r_3 = \frac{f_2}{v-s}, r_4 = \frac{v}{v-s}.$$

Note that $r_3 = r_4 \iff f_3 = 0$, which cannot hold by (20), and $r_3 \neq 0 \neq r_4$ since $v \neq 0 \neq f_2$ by (17) and (19). $r_3 = 1 \iff f_1 = 0$, which cannot hold by (19), and $r_4 = 1 \iff s = 0$, which cannot hold by (17). Thus all roots listed in (30) are *distinct*. A simple computation yields

$$(31) \quad G'(r) = \frac{p(r)}{(O(r) + \sqrt{M(r)})^2 \sqrt{M(r)}},$$

where the quartic polynomial p is given by

$$(32) \quad p(r) = 2M(r)O'(r) - O(r)M'(r).$$

To finish the proof of Theorem 5.1, the following lemmas will be used to show that p has a unique root in J .

Lemma 6.1. $N(r) > 0$ on J .

Proof. First define the linear function of r , $L(r) = (s-v)r + f_2$. $L(0) = f_2 > 0$ and $L(1) = f_1 > 0$ by (19) and thus $L > 0$ on J . Similarly, $(s-v)r + v > 0$ on J . By (29), $N > 0$ on J since $r(1-r) > 0$ on J .

Lemma 6.2. $O(r) > 0$ on J .

Proof. If $O(r_0) = 0$ for some $r_0 \in J$, then $N(r_0) = -M(r_0) \leq 0$ since $M(r) \geq 0$ on J . That contradicts Lemma 6.1. Hence $O(r)$ is nonzero on J . Since $O(0) = (s^2 + t^2)v^2 > 0$, that proves Lemma 6.2.

Now assume that $Q_{s,t,v,w}$ is a type 1 midpoint diagonal quadrilateral. In [2] we proved that there is a unique ellipse of minimal eccentricity inscribed in any convex quadrilateral, Q . The uniqueness for midpoint diagonal quadrilaterals would then follow from that result. However, the proof here, specialized for midpoint diagonal quadrilaterals, is self-contained, uses different methods, and does not require the result from [2]. Use (24) to substitute $\frac{s(w+1)}{v}$ for t in Proposition 5.2. Some simplification then yields $A(r) = s(4w(v-s)r^2 - 4vwr + s(w+1)^2)$, $B(r) = 2sv(2(s-v)r^2 + 2vr - s(1+w))$, $C(r) = s^2v^2$, $D(r) = 2rs^2v(w-1)$, $E(r) = -2rs^2v^2$, and $F(r) = r^2s^2v^2$. Using (27) and (32), it then follows that

$$p(r) = -16v^2s^4(2(s-v)r + v)\alpha(r),$$

where

$$(33) \quad \alpha(r) = 2(s-v)(v^2 + w^2 + 1)r^2 + 2v(v^2 + w^2 + 1)r - s(v^2 + (w+1)^2).$$

(17), (19), and (20) now become $s, v > 0$ (already assumed) and $2s-v > 0$. Now $\alpha(0) = -s(v^2 + (w+1)^2) < 0$ and $\alpha(1) = s(v^2 + (w-1)^2) > 0$, which implies that α has precisely one root in J since α is a quadratic. By (31), (28), and Lemma 6.2, G is differentiable on J . The linear function $2(s-v)r + v$ is nonzero at $r = 0$ since $v > 0$ and at $r = 1$ since $2s-v > 0$. Thus the other factors of p are nonzero and hence p has precisely one root, $r_1 \in J$, which is also the unique root of $G'(r)$ in J by (31). Since $G(r) > 0$ on J , $G(0) = G(1) = 0$, and G is positive in the interior of I and vanishes at the endpoints of I , $G(r_1)$ must yield the global maximum of G on J . That proves Theorem 5.1(i). Note that the equation of E_I is obtained by letting $r = r_1$ in Proposition 5.2. To prove Theorem 5.1(ii), by Theorem 4.1(i), E_I has conjugate diameters, CD_1 and CD_2 , parallel to the diagonals, D_1 and D_2 , of $Q_{s,t,v,w}$. $\overleftrightarrow{D_1}$ has equation $y = \frac{t}{s}x = \frac{w+1}{v}x$ and $\overleftrightarrow{D_2}$ has equation $y = 1 + \frac{w-1}{v}x$. Since $Q_{s,t,v,w}$ is type 1, $\overleftrightarrow{D_1} = L$, the line thru the midpoints of D_1 and D_2 , and thus L has equation $y = \frac{w+1}{v}x$. It then follows that $\overleftrightarrow{CD_1} = L$ (see Remark 7) and hence $\overleftrightarrow{CD_1}$ has equation $y = \frac{w+1}{v}x$. For convenience, we let

$$\begin{aligned} \beta &= (s-v)r_1 + v, \\ \zeta &= (s-v)r_1 + s. \end{aligned}$$

By Proposition 5.2, $\frac{B(r)E(r) - 2C(r)D(r)}{4A(r)C(r) - B^2(r)} = \frac{1}{2} \frac{sv}{(s-v)r + v}$. Thus by (3) of Lemma 2.2, E_I has center (x_0, y_0) , where $x_0 = \frac{B(r_1)E(r_1) - 2C(r_1)D(r_1)}{4A(r_1)C(r_1) - B^2(r_1)} = \frac{1}{2} \frac{sv}{\beta}$. Since E_I has center $(x_0, L(x_0))$ by Newton's Theorem and $L\left(\frac{1}{2} \frac{sv}{\beta}\right) =$

$\frac{1}{2} \frac{w+1}{v} \frac{sv}{\beta} = \frac{1}{2} \frac{s(w+1)}{\beta}$, E_I has center $\left(\frac{1}{2} \frac{sv}{\beta}, \frac{1}{2} \frac{s(w+1)}{\beta}\right)$. Since $\overleftrightarrow{CD_2}$ passes through the center of E_I and has the same slope as $\overleftrightarrow{D_2}$, which is $\frac{w-1}{v}$, $\overleftrightarrow{CD_2}$ has equation $y - \frac{1}{2} \frac{s(w+1)}{\beta} = \frac{w-1}{v} \left(x - \frac{1}{2} \frac{sv}{\beta}\right)$, which simplifies to $y = \frac{w-1}{v}x + \frac{s}{\beta}$.

Proof. Now suppose that CD_1 intersects E_I at the two distinct points $P_1 = (x_1, y_1) = \left(x_1, \frac{w+1}{v}x_1\right)$ and $P_2 = (x_2, y_2) = \left(x_2, \frac{w+1}{v}x_2\right)$. Since P_1 and P_2 lie on E_I , by Proposition 5.2, $A(r_1)x_j^2 + B(r_1)x_jy_j + C(r_1)y_j^2 + D(r_1)x_j + E(r_1)y_j + I(r_1) = 0, j = 1, 2$. Substituting $y_j = \frac{w+1}{v}x_j$ yields $A(r_1)x_j^2 + B(r_1)x_j^2 \left(\frac{w+1}{v}\right) + C(r_1)x_j^2 \left(\frac{w+1}{v}\right)^2 + D(r_1)x_j + E(r_1) \left(\frac{w+1}{v}\right)x_j + I(r_1) = 0, j = 1, 2$, which simplifies to

$$(34) \quad \begin{aligned} & \left(A(r_1) + \left(\frac{w+1}{v}\right)B(r_1) + \left(\frac{w+1}{v}\right)^2 C(r_1)\right)x_j^2 + \\ & \left(D(r_1) + \left(\frac{w+1}{v}\right)E(r_1)\right)x_j + I(r_1) = 0, j = 1, 2. \end{aligned}$$

If x_1 and x_2 are two distinct real roots of any quadratic, then

$$x_2 - x_1 = \frac{\sqrt{\tau}}{a} \Rightarrow (x_2 - x_1)^2 = \frac{\tau}{a^2},$$

where $\tau =$ discriminant and $a =$ leading coefficient. For the specific quadratic in the variable x_j given in (34) we have

$$\begin{aligned} \tau &= \left(D(r_1) + \left(\frac{w+1}{v}\right)E(r_1)\right)^2 - \\ & 4F(r_1)\left(A(r_1) + \left(\frac{w+1}{v}\right)B(r_1) + \left(\frac{w+1}{v}\right)^2 C(r_1)\right) \end{aligned}$$

and

$$a = A(r_1) + \left(\frac{w+1}{v}\right)B(r_1) + \left(\frac{w+1}{v}\right)^2 C(r_1).$$

Now applying Proposition 5.2 and simplifying yields

$$\begin{aligned} \tau &= 16r_1^2 s^3 v^2 (1 - r_1) \zeta, \\ a &= 4r_1 s \beta. \end{aligned}$$

Hence $(x_2 - x_1)^2 = \frac{\tau}{a^2} = \frac{v^2 s (1 - r_1) \zeta}{\beta^2}$. Now $y_2 - y_1 = \frac{w+1}{v}(x_2 - x_1) \Rightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_2 - x_1)^2 \left(1 + \left(\frac{w+1}{v}\right)^2\right)$, which implies that

$$(35) \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = \left(1 + \left(\frac{w+1}{v}\right)^2\right) \frac{v^2 s (1 - r_1) \zeta}{\beta^2}.$$

Similarly, suppose that CD_2 intersects E_I at the two distinct points $P_3 = (x_3, y_3) = \left(x_3, \frac{w-1}{v}x_3 + \frac{s}{\beta}\right)$ and $P_4 = (x_4, y_4) = \left(x_4, \frac{w-1}{v}x_4 + \frac{s}{\beta}\right)$. Since P_3 and P_4 lie on E_I , we have $A(r_1)x_j^2 + B(r_1)x_jy_j + C(r_1)y_j^2 + D(r_1)x_j + E(r_1)y_j + I(r_1) = 0, j = 3, 4$. Substituting $y_j = \frac{w-1}{v}x_j + \frac{s}{\beta}$ and simplifying yields

$$(36) \quad \begin{aligned} & \left(A(r_1) + \frac{w-1}{v}B(r_1) + \left(\frac{w-1}{v}\right)^2 C(r_1)\right)x_j^2 + \\ & \left(\frac{s}{\beta}B(r_1) + \frac{2s}{\beta}\frac{w-1}{v}C(r_1) + D(r_1) + \frac{w-1}{v}E(r_1)\right)x_j + \\ & \left(\frac{s}{\beta}\right)^2 C(r_1) + \left(\frac{s}{\beta}\right)E(r_1) + I(r_1) = 0, j = 3, 4. \end{aligned}$$

For the quadratic in the variable x_j given in (36) we have

$$\begin{aligned} \tau = & \left(\frac{s}{\beta}B(r_1) + \frac{2s}{\beta}\frac{w-1}{v}C(r_1) + D(r_1) + \frac{w-1}{v}E(r_1)\right)^2 - 4\left(A(r_1) + \frac{w-1}{v}B(r_1) + \right. \\ & \left.\left(\frac{w-1}{v}\right)^2 C(r_1)\right) \left(\left(\frac{s}{\beta}\right)^2 C(r_1) + \left(\frac{s}{\beta}\right)E(r_1) + I(r_1)\right). \end{aligned}$$

The leading coefficient is $a = A(r_1) + \frac{w-1}{v}B(r_1) + \left(\frac{w-1}{v}\right)^2 C(r_1)$.

Applying Proposition 5.2 again and simplifying yields

$$\tau = \frac{16r_1s^3v^2(r_1-1)^2\zeta^2}{\beta}, \quad a = 4s(1-r_1)\zeta.$$

Thus

$$(x_4 - x_3)^2 = \frac{\tau}{a^2} = \frac{16r_1s^3v^2(r_1-1)^2\zeta^2}{\beta(4s(1-r_1)\zeta)^2} = \frac{r_1sv^2}{\beta}.$$

Now $(y_4 - y_3)^2 = \left(\frac{w-1}{v}\right)^2 (x_4 - x_3)^2 \Rightarrow (x_4 - x_3)^2 + (y_4 - y_3)^2 = \left(1 + \left(\frac{w-1}{v}\right)^2\right) (x_4 - x_3)^2$, which implies that

$$(37) \quad (x_4 - x_3)^2 + (y_4 - y_3)^2 = \left(1 + \left(\frac{w-1}{v}\right)^2\right) \frac{r_1sv^2}{\beta}.$$

L_1 and L_2 are **equal** conjugate diameters if and only if $|P_1P_2| = |P_3P_4| \iff (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2$. Using (35) and (37), $|P_1P_2| = |P_3P_4| \iff$

$$\begin{aligned} & \left(1 + \left(\frac{w+1}{v}\right)^2\right) \frac{v^2s(1-r_1)\zeta}{\beta^2} = \left(1 + \left(\frac{w-1}{v}\right)^2\right) \frac{r_1sv^2}{\beta} \iff \\ & \left(1 + \left(\frac{w+1}{v}\right)^2\right) (v^2s(1-r_1)\zeta)\beta - \left(1 + \left(\frac{w-1}{v}\right)^2\right) r_1sv^2\beta^2 = 0 \iff \end{aligned}$$

$$\left(1 + \left(\frac{w+1}{v}\right)^2\right) ((1-r_1)\zeta) - \left(1 + \left(\frac{w-1}{v}\right)^2\right) r_1\beta = 0 \iff$$

$$\frac{2(s-v)(v^2+w^2+1)r_1^2 + 2v(v^2+w^2+1)r_1 - s(v^2+(w+1)^2)}{v^2} = 0$$

$\iff \alpha(r_1) = 0$, where α is given by (33). Since r_1 is a root of α by definition, that completes the proof of Theorem 5.1(ii).

7. EXAMPLE

Consider the quadrilateral, Q , with vertices $(0, 0)$, $(0, 1)$, $(8, 4)$, and $(6, 2)$. Thus $Q = Q_{s,t,v,w}$ with $s = 8$, $t = 4$, $v = 6$, and $w = 2$, which satisfy (17), (19), and (20).

Q is a type 1 midpoint diagonal quadrilateral since $vt = (w + 1)s$. The diagonal lines of Q are $\overleftrightarrow{D_1}: y = \frac{1}{2}x$, which is also the equation of L , and $\overleftrightarrow{D_2}: y = 1 + \frac{1}{6}x$.

First, let E_0 be the ellipse inscribed in Q corresponding to $r = \frac{3}{7}$. By Proposition 5.2(i) the equation of E_0 is $33x^2 - 148xy + 196y^2 + 28x - 168y = -36$.

By (3), E_0 has center $\left(\frac{7}{2}, \frac{7}{4}\right)$ and the points of tangency of E_0 with Q are given by $q_1 = \left(0, \frac{3}{7}\right)$, $q_2 = \left(\frac{32}{9}, \frac{7}{3}\right)$, $q_3 = \left(\frac{62}{9}, \frac{26}{9}\right)$, and $q_4 = \left(\frac{18}{7}, \frac{6}{7}\right)$.

• As guaranteed by Theorem 3.1, the slope of $\overleftrightarrow{q_2q_3} = \text{slope of } \overleftrightarrow{q_1q_4} = \frac{1}{6} = \text{slope of } D_2$.

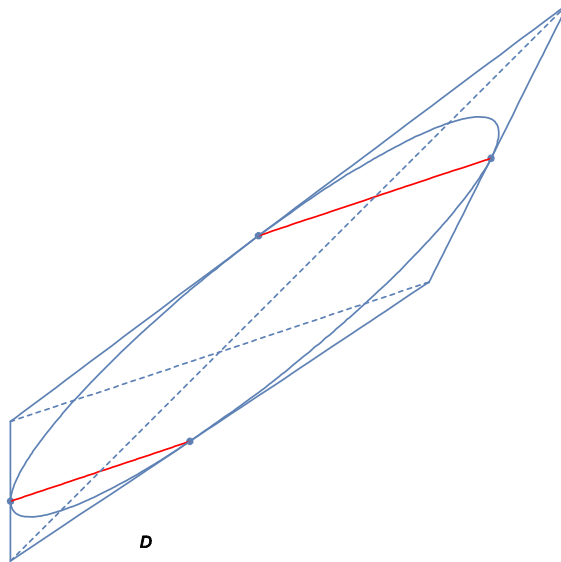


Figure 9: Tangency chords parallel to D_2

• Suppose that DI_1 and DI_2 are diameters of E_0 which are parallel to the diagonals D_1 and D_2 , respectively, and suppose that DI_1 intersects E_0 at the two distinct points P_1 and P_2 , while DI_2 intersects E_0 at the

two distinct points P_3 and P_4 . The equations of $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ are thus $y - \frac{7}{4} = \frac{1}{2} \left(x - \frac{7}{2} \right)$ and $y - \frac{7}{4} = \frac{1}{6} \left(x - \frac{7}{2} \right)$, respectively. To determine the coordinates of P_1 and P_2 , substitute the equation of DI_1 into the equation of E_0 . That yields $33x^2 - 148x \left(\frac{7}{4} + \frac{1}{2} \left(x - \frac{7}{2} \right) \right) + 196 \left(\frac{7}{4} + \frac{1}{2} \left(x - \frac{7}{2} \right) \right)^2 + 28x - 168 \left(\frac{7}{4} + \frac{1}{2} \left(x - \frac{7}{2} \right) \right) = -36$, which has solutions $x = \frac{7}{2} \pm \frac{1}{2} \sqrt{31}$. Hence $P_1 = \frac{1}{4}(7 - \sqrt{31})(2, 1)$ and $P_2 = \frac{1}{4}(7 + \sqrt{31})(2, 1)$. Similarly, $P_3 = \frac{1}{4}(2, 1)(7 - 3\sqrt{2}, 7 - \sqrt{2})$ and $P_4 = \frac{1}{4}(2, 1)(7 + 3\sqrt{2}, 7 + \sqrt{2})$. The family of lines which are parallel to $\overleftrightarrow{P_1P_2}$ have equation $y = \frac{1}{2}x + b$. Substituting into the equation of E_0 yields $33x^2 - 148x \left(\frac{1}{2}x + b \right) + 196 \left(\frac{1}{2}x + b \right)^2 + 28x - 168 \left(\frac{1}{2}x + b \right) = -36$. Solving for x gives $x = \frac{7}{2} - 3b \pm \frac{1}{2} \sqrt{31 - 62b^2}$, and thus the chords of E_0 which are parallel to $\overleftrightarrow{P_1P_2}$ have midpoints $\left(\frac{7}{2} - 3b, \frac{7}{4} - \frac{1}{2}b \right)$, which each lie on $\overleftrightarrow{P_3P_4}$. Hence DI_1 and DI_2 are the pair of conjugate diameters of E_0 which are parallel to the diagonals of Q , as guaranteed by Theorem 4.1.

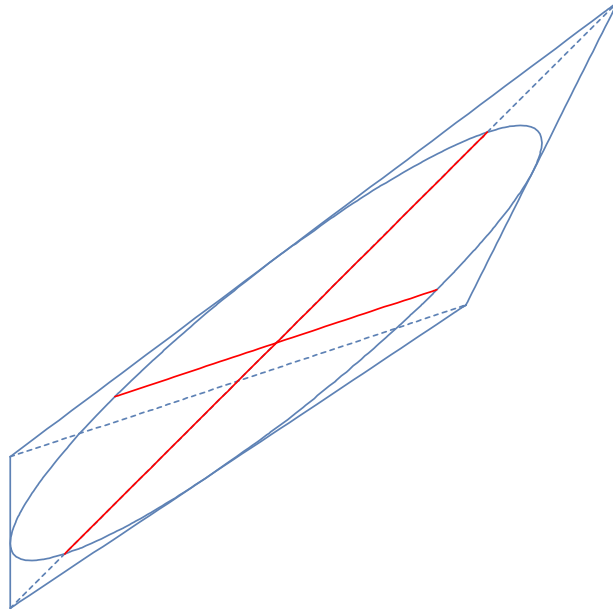


Figure 10: Conjugate diameters parallel to the diagonals

Second, let E_I be the unique ellipse of minimal eccentricity inscribed in Q . By (33), $\alpha(r) = 164r^2 + 492r - 360$ and the unique root of α in J is $r_1 = -\frac{3}{2} + \frac{27}{82} \sqrt{41}$. By Proposition 5.2(i), after some simplification, the equation of E_I is $3(427 - 63\sqrt{41})x^2 + 8(-793 + 117\sqrt{41})xy + 164(61 - 9\sqrt{41})y^2 + 4(-2911 + 459\sqrt{41})x + 24(2911 - 459\sqrt{41})y = 36(-3521 + 549\sqrt{41})$. By

Theorem 4.1, there are conjugate diameters DI_1 and DI_2 of E_I which are parallel to the diagonals D_1 and D_2 , respectively. Suppose that DI_1 intersects E_I at the two distinct points P_1 and P_2 , while DI_2 intersects E_I at the two distinct points P_3 and P_4 . As we did above, one can show that $P_1 = \frac{-82+18\sqrt{41}-9\sqrt{74}+\sqrt{3034}}{20}(2, 1)$, $P_2 = \frac{-82+18\sqrt{41}+9\sqrt{74}-\sqrt{3034}}{20}(2, 1)$, $P_3 = \left(\frac{-82+18\sqrt{41}-27\sqrt{10}+3\sqrt{410}}{10}, \frac{-82+18\sqrt{41}-9\sqrt{10}+\sqrt{410}}{20}\right)$, and $P_4 = \left(\frac{-82+18\sqrt{41}+27\sqrt{10}-3\sqrt{410}}{10}, \frac{-82+18\sqrt{41}+9\sqrt{10}-\sqrt{410}}{20}\right)$. It then follows that $|\overline{P_1P_2}|^2 = |\overline{P_3P_4}|^2 = \frac{37}{5}(61 - 9\sqrt{41})$. As guaranteed by Theorem 5.1, $\overline{P_1P_2}$ and $\overline{P_3P_4}$ are **equal** conjugate diameters of E_I which are parallel to the diagonals of Q .

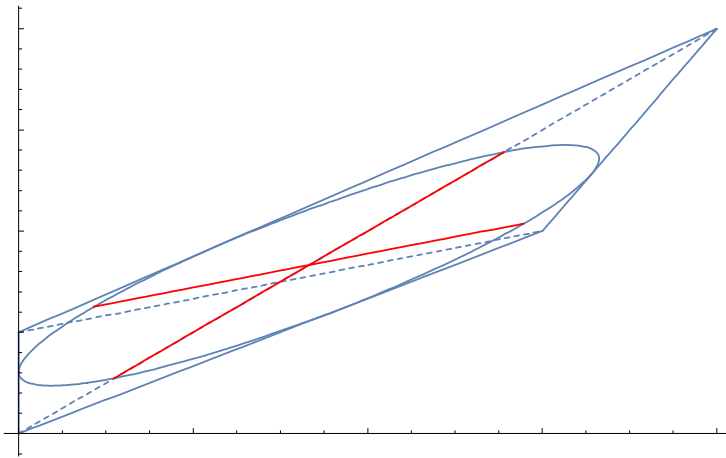


Figure 11: Equal conjugate diameters parallel to the diagonal

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