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TANGENTS TO A CUBIC CURVE

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Abstract. Six tangents may be drawn from a point X to a planar cubic curve U (a.k.a. a curve of the third degree, triangle cubic or 3-curve). It is shown that their six points of contact determine another cubic curve V which passes through eleven easily determined points. Furthermore there is quartic curve which connects no fewer than 41 points related to these two cubic curves.

1. Introduction.

1.1. **Algebraic Projective Geometry.** When studying planar figures it is often the case that areal coordinates are more convenient than Cartesian. Such coordinates come with various names: trilinear, homogeneous and barycentric being popular but they are all very similar and the basic idea is the same.

Choose any three points in the plane which are not collinear and label them A, B and C. These form the *triangle of reference*. If P is any point in the plane then the sum of the (signed) areas of the triangles BCP, CAP and ABP (written as $\Delta_1, \Delta_2, \Delta_3$ respectively) will be Δ, the area of ABC. Thus

$$r_1 = \Delta_1/\Delta,\ r_2 = \Delta_2/\Delta,\ r_3 = \Delta_3/\Delta$$

may used as the (absolute) coordinates of P. Since Δ acts simply as a scale factor it is usual to call $(\mu\Delta_1, \mu\Delta_2, \mu\Delta_3)$ the coordinates of P for any value of μ. Clearly the coordinate of A, B and C are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Since only ratios of the coordinates are involved, the equation of any curve will be a homogeneous polynomial in (r_1, r_2, r_3). In particular, any linear expression in r_1, r_2, r_3 represents a line. However, the one with equation

$$x + y + z = 0$$

is clearly special; no finite point satisfies this condition and it is referred to as *the line at infinity*. The elegance of the subject is greatly added to by allowing the coefficients in the polynomials and the coordinates of points to be complex numbers. Thus a line and a conic always meet in two points.

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Only properties of curves which are preserved under a linear transformation are considered important in projective geometry. However, the formalism just introduced may be used to discuss metrical properties although this is generally not so elegant. For example, if the lengths of the sides of the triangle ABC are a, b, c then the equation of the circumcircle of ABC is

$$a^2 r_2 r_3 + b^2 r_3 r_1 + c^2 r_1 r_2 = 0$$

(the coefficients vary according to exactly which flavour of coordinates is used). Also the equation of any circle may be written as

$$a^2 r_2 r_3 + b^2 r_3 r_1 + c^2 r_1 r_2 + (r_1 + r_2 + r_3)(\alpha r_1 + \beta r_2 + \gamma r_3) = 0$$

for some set of constants α, β, γ. This shows that any circle in the plane passes through the two common points of the line at infinity and the circumcircle of ABC. These are the *circle points* I and J. Also any pair of circles have four common points.

Because only projectively invariant properties will be considered in what follows, it is permissible to choose any point (not on the lines forming the triangle of reference) as a distinguished point with coordinates $(1, 1, 1)$. This is referred to as the *unit point*.

Let S be a homogeneous polynomial of degree n ($n \geq 1$) in $\mathbf{r} = (r_1, r_2, r_3)$. The curve represented by $S(r_1, r_2, r_3) = 0$, where now (r_1, r_2, r_3) are considered to be homogeneous coordinates, is called an n-curve or curve of the nth degree. If $X(\boldsymbol{\alpha})$ is any point, the *polar curve* of X in S is the $(n-1)$-curve S_X defined by

$$nS_X \equiv \alpha_1 \frac{\partial S}{\partial r_1} + \alpha_2 \frac{\partial S}{\partial r_2} + \alpha_3 \frac{\partial S}{\partial r_3} \equiv \boldsymbol{\alpha} \cdot \nabla S = 0 \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. Note that the same symbol will be used for the polynomial (such as S) and the curve with equation $S = 0$. Thus if C is a homogeneous quadratic one may refer to the conic C and line C_X. Much information about cubic curves is to be found in Gibert's website [3] and in Salmon's book [5]. The latter reference also contains an extensive treatment of quartic curves. The texts [4] and [6] provide a more modern, abstract approach to projective geometry while [2], which was the author's school text, gives an excellent introduction to the subject but only considers lines and conics. The methods of projective geometry are also used in the more recent articles [1] and [7] which shows that the methods may be used in unusual contexts.

1.2. **The Methodology of this Paper.** Here is explained the genesis of this work and the organization of the paper.

In the context of cubic curves, the following results are well known. Let $U(\mathbf{r})$ be a homogeneous polynomial of degree three and $X(\boldsymbol{\alpha})$ a given point.

a) If two cubic curves U and U' have eight points in common then they also have a ninth common point. Furthermore, any cubic curve through this set of eight points also passes through this ninth point and may be expressed as $U + \lambda U'$.

b) If X is not on U then there are six tangents from X to U and their six points of contact satisfy

$$U = 0 = U_X. \tag{2}$$

c) (Attributed to Maclaurin.) If X is on U then let the four points of contact of the tangents from X to U (ignoring the tangent at X) be P_i $(1 \le i \le 4)$.
   i) The three diagonal points of the quadrangle $P_1P_2P_3P_4$ lie on U.
   ii) The tangents at these diagonal points are concurrent at a point on U and the fourth tangent from this point touches U at X.

d) Let the cubic curve U meet a 2-curve (conic) C at the points P_i $(1 \le i \le 6)$ and let the line P_iP_j meet U again at Q_{ij}. Then the points Q_{12}, Q_{34}, Q_{56} are collinear. (This is a generalization of *Pascal's mystic hexagram* which in turn is a generalization of Pappus' theorem.)

The motivation for this investigation was to find results similar in style to c) when X is not on U and the author's exploration was in several distinct phases. The six points P_i are now where the tangents from X touch U.

- Firstly it was realised that the lines

$$P_1Q_{23},\ P_2Q_{31},\ P_3Q_{12}$$

  were concurrent (this is repeated in c) in the theorem below) and since there are twenty such triangles formed by the six P-points there are twenty points of concurrency.
- A numerical investigation of these twenty points revealed that only ten of them are distinct; complementary triangles (such as $P_1P_2P_3$ and $P_4P_5P_6$) produce the same point.
- Nine points are rather special in the context of cubic curves (see a) above) and so the cubic curve through nine of the ten points was found numerically. Not only did this cubic pass through the tenth point, it also passed through X. Could this be called 'the eleven-point cubic'?
- From the numerical results for this cubic it was eventually established that the coefficients could be expressed as polynomials in the parameters of the problem. This is how (8) below was established.
- The challenge now was to prove that this equation did indeed meet the properties just referred to. This was an algebraic exercise of some length but did not involve any novel methods.
- Having established the existence of V it was relatively straightforward to extend the exploration to the quartic W and the other properties listed below of these two curves.
- The chief results of this paper are the geometric properties already alluded to (especially the eleven-point property), together with the algebraic identities of section 5 and the coordinate-free expression for W (and hence V) given by (4).

It is hoped that this puts the work into context and not only makes the paper more accessible but might induce some to take an active interest in projective geometry. In particular it has to be admitted that the emphasis here is very much on algebra. It would be most interesting if a geometrical proof could be found for the existence of the cubic through the eleven points.

2. Notation.

It will be very convenient to use vector notation (including the scalar and vector products) throughout this article even though no vectors are involved. This notation has already been used in equation (1). Any solution to

$$\nabla U = \mathbf{r} \times \boldsymbol{\alpha}. \tag{3}$$

will automatically satisfy equations (2) because of Euler's theorem on homogeneous functions. However, this last equation is not homogeneous and so imposes a particular scaling upon the solutions to (2). In what follows

$$\mathbf{p}_i = (x_i, y_i, z_i) \ (1 \le i \le 6)$$

will always refer to these solutions for the six points of contact of the tangents from $X(\boldsymbol{\alpha})$ to the cubic curve U. These are not homogeneous coordinates; they satisfy (3) and cannot be re-scaled.

The polar curve of the point $P_1(\mathbf{p}_1)$ in U should be written, following equation (1), as U_{P_1}. However, this is more neatly written as simply U_1. Since $(U_1)_2$ is equal to $(U_2)_1$, either may be written as U_{12}: it is the polar line of P_2 in the conic U_1 (and the polar line of P_1 in the conic U_2). The presence of the factor n in (1) ensures that, for example,

$$U_{112} = U_{11}(\mathbf{p}_2) = U_{12}(\mathbf{p}_1) = U_2(\mathbf{p}_1) \text{ and } U_{111} = U(\mathbf{p}_1).$$

It will be convenient to use $[\ldots]_X$ to indicate that the enclosed expression is to be evaluated at X. Also $[\ldots]_i$ will mean that it is evaluated at P_i.

3. The Main Theorem.

Theorem 3.1. *Let the six tangents from a point $X(\boldsymbol{\alpha})$ to the cubic curve U (where X does not lie on U or its Hessian and U is not factorizable) touch U at the points $P_i(\mathbf{p}_i)$ $(1 \le i \le 6)$ where the coordinates $\mathbf{p}_i$ satisfy equation (3). Let the line P_iP_j $(i \ne j)$ meet U again at Q_{ij}. Then the following are claimed:*

a) *The coordinates of Q_{ij} are $(\mathbf{p}_i + \mathbf{p}_j)$.*

b) $\displaystyle\sum_{i=1}^{6} \mathbf{p}_i = \mathbf{0}$.

c) *The lines*

$$P_1Q_{23},\ P_2Q_{31},\ P_3Q_{12}$$

are concurrent, their common point will be denoted by R_{123} and has coordinates $(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$.

d) *There are 15 Q-points and 20 proper triangles $P_iP_jP_k$ (i.e. with i, j, k different) but only 10 distinct R-points.*

e) *If $1, 2, i, j$ are all different then the lines*

$$P_1P_2,\ Q_{1i}Q_{2i},\ R_{1ij}R_{2ij}$$

(a total of 8 distinct lines) have a common point and this point is labelled S_{12}. There are 15 such S-points. The points Q_{12} and S_{12} divide the line segment P_1P_2 harmonically.

f) *There exists a cubic curve V which passes through the ten R-points.*

g) *V passes through X.*

h) *The six values $V(\mathbf{p}_i)$ are equal.*

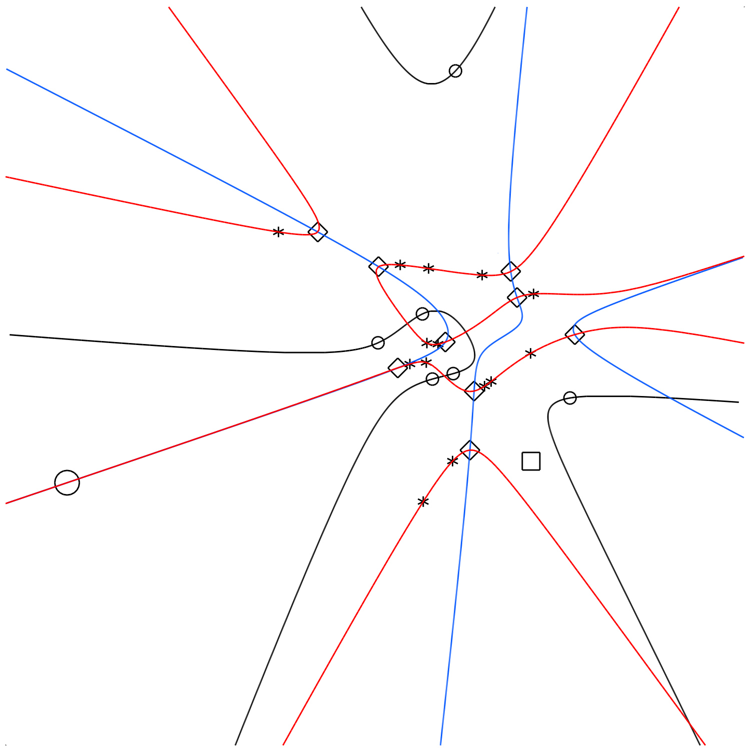

FIGURE 1. The curves U, V and W are shown as black, blue and red lines respectively. P, R and S-points are shown by small circles, diamonds and stars, respectively. Also X is at the large circle and Y the square.

i) *Denote the Hessian of U by H. Let the lines U_{XX} and H_{XX} (i.e. the polar lines of X in U and H) meet at the point Y. The conics U_Y and V_X coincide.*

j) *There exists a quartic curve W with the following properties:*
  - *The equation of W is*

$$3W \equiv 2(\nabla H \times \nabla U) \cdot [\nabla U)]_X = 0. \tag{4}$$

  - *The polar curve of X in W is V.*
  - *W passes through the 15 S-points.*
  - *W touches V at X.*
  - *The R-points are the other 10 common points of V and W so that they are where the tangents from X touch W.*

Figure 1 shows a typical configuration. When X lies on the Hessian of U the theorem is still correct but its direct proof requires some modification.

Proof.

It has already been explained that $P_i(\mathbf{p}_i)$ $(1 \le i \le 6)$ are the points of contact of the tangents from $X(\boldsymbol{\alpha})$ to the cubic curve U. [Note that the condition of U being not factorizable is necessary here, e.g. if $U \equiv LC$ where L is a line and C a conic then the only non-trivial solutions of (3) are the two solutions of $C = 0 = C_X$.]

a) With
$$\mathbf{s}_i = [\nabla U]_i \ (1 \le i \le 6)$$
we have
$$\mathbf{p}_i \cdot \mathbf{s}_j = \mathbf{p}_i \cdot (\mathbf{p}_j \times \boldsymbol{\alpha}) = -\mathbf{p}_j \cdot (\mathbf{p}_i \times \boldsymbol{\alpha}) = -\mathbf{p}_j \cdot \mathbf{s}_i.$$
Also
$$3U_i \equiv \mathbf{p}_i \cdot \nabla U \ \Rightarrow \ 3U_{122} = \mathbf{p}_1 \cdot \mathbf{s}_2$$
and so $U_{112} = -U_{122}$. But $U_{111} = 0 = U_{222}$ since P_1 and P_2 lie on U, hence
$$U(\mathbf{p}_1 + \mathbf{p}_2) = U_{111} + 3U_{112} + 3U_{122} + U_{222} = 0.$$
Thus the point with coordinates $(\mathbf{p}_1 + \mathbf{p}_2)$ lies on U and is collinear with P_1 and P_2, so it is the point Q_{12}.

b) From d) in the Introduction, the points Q_{12}, Q_{34}, Q_{56} are collinear and so
$$\mathbf{p}_1 + \mathbf{p}_2, \ \mathbf{p}_3 + \mathbf{p}_4, \ \mathbf{p}_5 + \mathbf{p}_6$$
are linearly dependent. But so are each of the sets
$$\mathbf{p}_1 + \mathbf{p}_2, \ \mathbf{p}_3 + \mathbf{p}_5, \ \mathbf{p}_4 + \mathbf{p}_6 \text{ and } \mathbf{p}_1 + \mathbf{p}_2, \ \mathbf{p}_3 + \mathbf{p}_6, \ \mathbf{p}_4 + \mathbf{p}_5.$$
Since no set of three of the $\mathbf{p}_i$ is linearly dependent (this would imply that P_1, P_2, P_3 were collinear and so X lies on the Hessian of U) it follows that the sum of all of them is zero.

c) This is an immediate consequence of a). The converse is also true: *The lines $P_1Q_{23}, P_2Q_{31}, P_3Q_{12}$ are concurrent if and only if the tangents to U at P_1, P_2, P_3 are concurrent.* A coordinate-free proof of this result is included in [8].

d) Since $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = -(\mathbf{p}_4 + \mathbf{p}_5 + \mathbf{p}_6)$ it follows that the points R_{123} and R_{456} coincide.

e) The point with coordinates $(\mathbf{r}_1 - \mathbf{r}_2)$ clearly lies on each of the lines listed and so is the point S_{12}. The points $(\mathbf{r}_1 \pm \mathbf{r}_2)$ are the harmonic separators of $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$.

f) When using homogeneous coordinates, the coordinates of any four points may be specified. Let X be the point $(1,1,1)$ and let three of the points of contact of the tangents from X to U be the vertices of the triangle of reference. Then U will take the form

(5) $$U \equiv ar_1^2(r_2 - r_3) + br_2^2(r_3 - r_1) + cr_3^2(r_1 - r_2) + 2dr_1r_2r_3 = 0.$$

The substitution $r_1 = 0 = r_2$ into (3) gives $r_3 = -1/c$ and so the six $P(\mathbf{p}_i)$-points are

(6) $$\mathbf{p}_1 = (-1/a, 0, 0), \ \mathbf{p}_2 = (0, -1/b, 0), \ \mathbf{p}_3 = (0, 0, -1/c)$$

together with three points $\mathbf{p}_i = (x_i, y_i, z_i)$ $(4 \le i \le 6)$ where we know that
$$\sum_{i=4}^{6} x_i = 1/a, \ \sum_{i=4}^{6} y_i = 1/b, \ \sum_{i=4}^{6} z_i = 1/c.$$
Thus the R-points are $(1/a, 1/b, 1/c)$ and the nine points

(7) $$(x_i, y_i - 1/b, z_i - 1/c), \ (x_i - 1/a, y_i, z_i - 1/c), \ (x_i - 1/a, y_i - 1/b, z_i) \ (4 \le i \le 6).$$

Consider the cubic curve V defined by

$$(8)\qquad \begin{aligned} V/24 \equiv -4d^2(a^3r_1^3 + b^3r_2^3 + c^3r_3^3) + r_1^2(f_1r_2 + g_1r_3) \\ + r_2^2(f_2r_3 + g_2r_1) + r_3^2(f_3r_1 + g_3r_2) + Dr_1r_2r_3 = 0 \end{aligned}$$

where

$$\begin{aligned} f_1 &= 2ad^2[a(a+b-c) + 2b^2] + 2ad(c-b)[a^2 + b^2 - c(a+b)] \\ &\qquad + a(a+b-c)(b-c)(c-a)(a-b) + 4a^2d^3 \\ g_1 &= 2ad^2[a(a-b+c) + 2c^2] + 2ad(c-b)[a^2 + c^2 - b(a+c)] \\ &\qquad - a(a-b+c)(b-c)(c-a)(a-b) - 4a^2d^3, \\ D &= -2d(a+b+c)(b-c)(c-a)(a-b) \\ &\qquad - 4d^2[a^2(b+c) + b^2(c+a) + c^2(a+b)], \end{aligned}$$

(f_2, f_3, g_2, g_3 may be found by cyclic permutations of a, b, c).

It is easily verified that $V(1/a, 1/b, 1/c) = 0$. The equation (3) is equivalent to

$$[\nabla U]_i = \mathbf{p}_i \times \boldsymbol{\alpha}.$$

With $i = 1, 2$ or 3, this equation is already satisfied but for $i = 4, 5$ or 6 each gives three scalar equations which are linear in a, b, c. When solved, they give

$$(9)\qquad a = \frac{1}{x_i} - \frac{dy_iz_i(x_iz_i + x_iy_i - y_iz_i)}{x_i(y_i - z_i)(z_i - x_i)(x_i - y_i)}$$

with corresponding formulae for b and c. The substitution of these into

$$V(x_i, y_i - 1/b, z_i - 1/c)$$

gives zero. Likewise for the other two forms required by (7). Hence the ten R-points have been verified as lying on V.

g) It is readily found that $V(1,1,1) = 0$.

h) The substitution (9) into $V(\mathbf{r})$ yields $96d^2$ which is also the value of V at each of the points given by (6).

i) The Hessian of U is

$$(10)\qquad H \equiv \left| \frac{\partial^2 U}{\partial r_i \partial r_j} \right| = 0$$

and the lines U_{XX} and H_{XX} meet at the point $Y(\boldsymbol{\beta}) \equiv Y(\beta_1, \beta_2, \beta_3)$ where

$$(11)\qquad \boldsymbol{\beta} = [\nabla U \times \nabla H]_X.$$

When U is given by equation (5) this gives

$$\beta_1 = 24[(c-b)(a-b-c) + 2d(b+c)][(a-b)(a-c) + 2d(b-c)]$$

and β_2, β_3 are obtained by cyclic permutations of a, b, c. It may be verified that $U_Y \equiv V_X$.

j) When U is given by (5) and W by (4) it may be verified that $W_X = V$ and that V is as given by (8). The algebraic details for W involve yet more sesquipedalian expressions; in the next section a more convenient form of U is used and then W is explicitly displayed.

The result that W passes through the 15 S-points is not easily proved from the current form of W; it is S_{45}, S_{56}, S_{64} which are troublesome. So the proof of this is deferred to Section 4.2. However, there is no such difficulty with the 10 R-points. The technique used to show that they lie on V may be applied to show that they also lie on W.

Since X lies on W it follows that W and $W_X = V$ touch at X.

4. V when U is in Normal Form.

Let U be any cubic curve and A any point. The conic U_A (the polar curve of A in U) and the line U_{AA} (the polar line of A) will meet in two points, say B and C. Then the polar lines of both B and C pass through A. Now impose the two new conditions that the polar line of B shall pass through C and the polar line of C shall pass through B. Since choosing the position of A also involves two parameters, it follows that *for any cubic curve there exist three points such that the polar line of any one is the join of the other two.* There are four such sets of points, see [5], and a triangle formed by such a set is sometimes called a *polar triangle* of the cubic curve. When a polar triangle is taken to be the triangle of reference, the equation for U assumes its *normal form*

$$U(\mathbf{r}) \equiv fr_1^3 + gr_2^3 + hr_3^3 + 6kr_1r_2r_3 = 0. \tag{12}$$

The Hessian, $H(\mathbf{r})$, of U is now

$$H(\mathbf{r}) \equiv 216[-k^2U + (fgh + 8k^3)r_1r_2r_3] = 0 \tag{13}$$

and, using (11),

$$\boldsymbol{\beta} = 648(fgh + 8k^3)(\alpha_1(g\alpha_2^3 - h\alpha_3^3),\ \alpha_2(h\alpha_3^3 - f\alpha_1^3),\ \alpha_3(f\alpha_1^3 - g\alpha_2^3)).$$

The requirement that U shall not be factorizable implies that $(fgh + 8k^3)$ is different from zero.

Equation (4) for W produces

$$\begin{aligned} W \equiv\ & 1296(fgh + 8k^3)[\ fr_1^3\{r_2(g\alpha_2^2 + 2k\alpha_3\alpha_1) - r_3(h\alpha_3^2 + 2k\alpha_1\alpha_2)\} \\ & + gr_2^3\{r_3(h\alpha_3^2 + 2k\alpha_1\alpha_2) - r_1(f\alpha_1^2 + 2k\alpha_2\alpha_3)\} \\ & + hr_3^3\{r_1(f\alpha_1^2 + 2k\alpha_2\alpha_3) - r_2(g\alpha_2^2 + 2k\alpha_3\alpha_1)\}\ ] \end{aligned} \tag{14}$$

and, from $V = W_X$,

$$\begin{aligned} V \equiv\ & 324(fgh + 8k^3)[\\ & fr_1^2\{r_1(g\alpha_2^3 - h\alpha_3^3) + 3\alpha_1 r_2(g\alpha_2^2 + 2k\alpha_1\alpha_3) - 3\alpha_1 r_3(h\alpha_3^2 + 2k\alpha_1\alpha_2)\} \\ & + gr_2^2\{r_2(h\alpha_3^3 - f\alpha_1^3) + 3\alpha_2 r_3(h\alpha_3^2 + 2k\alpha_2\alpha_1) - 3\alpha_2 r_1(f\alpha_1^2 + 2k\alpha_2\alpha_3)\} \\ & + hr_3^2\{r_3(f\alpha_1^3 - g\alpha_2^3) + 3\alpha_3 r_1(f\alpha_1^2 + 2k\alpha_3\alpha_2) - 3\alpha_3 r_2(g\alpha_2^2 + 2k\alpha_3\alpha_1)\}\ ]. \end{aligned} \tag{15}$$

4.1. Points on W.

It is apparent that any point which satisfies any one of

$$\nabla U = \mu_1 \nabla H,\ \nabla U = \mu_2 [\nabla U]_X,\ \nabla H = \mu_3 [\nabla U]_X$$

(where the μ_i are constants) lies on W. The first of these conditions implies that W passes through each vertex of the four polar triangles of U (twelve points). The second implies that W passes through the three points (other than X) which have the same polar line in U as X. Likewise the third gives the four points which have the same polar line in H as X does in U (but these last ones were deemed too esoteric to be included in the total of 'related points').

It has now been established that W connects

- X,
- 10 R-points,
- 15 S-points,
- 12 vertices of the polar triangles,
- 3 points which have the same polar line as X in U,

giving 41 points in total.

4.2. **Completion of the Proof.** Only an outline of the proof is given. The form of U given in (12) will be employed. A preliminary result is first established.

Since $P_1,\ P_2,\ Q_{12}$ lie on U it follows that

$$U(\mathbf{p}_1) = 0 = U(\mathbf{p}_2) = U(\mathbf{p}_1 + \mathbf{p}_2).$$

Each of these three equations is linear in f, g, h and, assuming that the determinant of the matrix of coefficients is not zero, these may be solved for f, g, h. However, when this is carried out, it is found that the solutions satisfy $fgh + 8k^3 = 0$ which is unreasonable. Thus the determinant must be zero and this gives, using $\mathbf{p}_i = (x_i, y_i, z_i)$,

$$\theta \equiv (x_1 + x_2)(y_1 + y_2)(z_1 + z_2) - (x_1y_1z_1 + x_2y_2z_2) = 0. \tag{16}$$

The equations $U(\mathbf{p}_1) = 0 = U(\mathbf{p}_2)$ may be solved for f, g. Now find the point X at which the tangents at P_1 and P_2 meet, i.e. solve $U_{11} = 0 = U_{22}$. Write the solution as (a_1, a_2, a_3) so that $\boldsymbol{\alpha} = \mu(a_1, a_2, a_3)$ for some constant μ, substitute into any one of the three components of $[\nabla U]_1 = \mathbf{p}_1 \times \boldsymbol{\alpha}$ and solve for μ. The other two components will be automatically satisfied. Furthermore, it is found that the three components of $[\nabla U]_2 = \mathbf{p}_2 \times \boldsymbol{\alpha}$ are also satisfied by virtue of $\theta = 0$. Also the substitution of the results for $f, g, \boldsymbol{\alpha}$ into $W(\mathbf{p}_1 - \mathbf{p}_2)$ gives an expression with a factor of θ. Thus the proof is complete.

As an unexpected bonus, it is also found that

$$W(\mathbf{p}_1) + W(\mathbf{p}_1 + \mathbf{p}_2) + W(\mathbf{p}_2) = 0 \tag{17}$$

(because this also has a factor of θ) but the geometrical implication of this eludes the writer.

5. Algebraic Properties of W and V.

Recall that

$$3W = 2[\nabla U]_X \cdot (\nabla H \times \nabla U),\ \boldsymbol{\beta} = -[\nabla H \times \nabla U]_X \ \text{ and } \ V = W_X.$$

Alternatively,

$$9W \equiv \sum_{i,j,l} r_i r_j \frac{\partial^2 \beta_l}{\partial \alpha_i \partial \alpha_j} \frac{\partial U}{\partial r_l} \ \text{ and } \ 6V \equiv \sum_{i,j} r_j \frac{\partial \beta_i}{\partial \alpha_j} \frac{\partial U}{\partial r_i}. \tag{18}$$

All summations are from 1 to 3.

That $V \equiv V(\boldsymbol{\alpha}, \mathbf{r})$ is homogeneous of degree three in both $\boldsymbol{\alpha}$ and $\mathbf{r}$ is immediate from (15) and also $W \equiv W(\boldsymbol{\alpha}, \mathbf{r})$ has degree four in $\mathbf{r}$ and two in $\boldsymbol{\alpha}$. But it is found, considering V as a function of $\boldsymbol{\alpha}$ and $\mathbf{r}$, that

$$V(\boldsymbol{\alpha}, \mathbf{r}) \equiv -V(\mathbf{r}, \boldsymbol{\alpha}).$$

Several identities follow from (18):

$$2W \equiv \sum_{i,j} r_j \alpha_i \frac{\partial^2 W}{\partial \alpha_j \partial r_i}, \quad 3V \equiv \sum_{i,j} r_j \alpha_i \frac{\partial^2 V}{\partial \alpha_j \partial r_i}$$

$$\sum_{i,j} \alpha_i \alpha_j \frac{\partial^2 W}{\partial r_i \partial r_j} \equiv -6W(\mathbf{r}, \boldsymbol{\alpha}) \text{ and } 3W \equiv 2\sum_i r_i \frac{\partial V}{\partial \alpha_i} \text{ or } W(\mathbf{r}, \boldsymbol{\alpha}) \equiv -2V_X.$$

6. Concluding Remarks.

The formulae (18) are quite remarkable, not least because they involve the Jacobian matrix $[\partial \beta_i / \partial \alpha_j]$. It has been demonstrated algebraically that

- at each of the six P-points
  - * $V(\mathbf{p}_i) = 24[U(\boldsymbol{\alpha})]^2$,
  - * $W(\mathbf{p}_i) = -4H(\mathbf{p}_i)U(\boldsymbol{\alpha})$,
  - * $H_X(\mathbf{p}_i) = -8U(\boldsymbol{\alpha})$.
- $H(\mathbf{p}_1 + \mathbf{p}_2) = H(\mathbf{p}_1) + H(\mathbf{p}_2)$ c.f. (17).
- With $J = [\partial \beta_i / \partial \alpha_j]$,

$$\text{trace}\,(J) = 0,$$
$$\det(J)U(\boldsymbol{\alpha}) + 8U(\boldsymbol{\beta}) = 0,$$
$$48V(\boldsymbol{\beta}) + U(\boldsymbol{\alpha}) \sum_i \beta_i \frac{\partial \det(J)}{\partial \alpha_i} = 0.$$

- When $U \equiv LC$, where L is a line and C a proper conic, the geometrical constructions of W and V are problematic. However, equation (4) may still be used and it is found that

$$W = \frac{4L^3 C_Y}{3L_X^2} \quad \text{and} \quad V = \frac{L^2 C_Y}{L_X}$$

  where now Y is the common point of the lines L and C_X.

The proofs given here rely heavily upon algebra and a geometric demonstration of the existence of W and V would be welcome.

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