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# NEW CHARACTERIZATIONS OF EXTANGENTIAL QUADRILATERALS

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**Abstract.** We prove 24 new necessary and sufficient conditions for when a convex quadrilateral can have an excircle.

## 1. Introduction

One of the lesser known classes of convex quadrilaterals is the one that admits an exterior circle that is tangent to the *extensions* of all four sides. These were named *extangential quadrilaterals* in [9] and the circle is called the *excircle* of the quadrilateral (see Figure 1). The first author has previously written the two papers [5, 6] on properties that are characteristic for such quadrilaterals, where 15 necessary and sufficient conditions were proved. Here we shall continue that exploration and prove another 24 such conditions. First let us briefly review four old ones, two of which will be used in the majority of the proofs.

In 1846 Jakob Steiner proved in [11] that a convex quadrilateral $ABCD$ can have an excircle outside one of the vertices $A$ or $C$ if and only if

$$AB + BC = CD + DA. \tag{1}$$

By symmetry, it can have an excircle outside $B$ or $D$ if and only if

$$DA + AB = BC + CD. \tag{2}$$

There is at most one excircle belonging to a convex quadrilateral and it must be located outside the one of the two vertices in each case that has the biggest vertex angle. (In contrast, all quadrilaterals have four escribed circles which are each tangent to one side of the quadrilateral and the extensions of the two adjacent sides.) Except in the next section, we will only discuss characterizations for an excircle outside of vertex $A$ or $C$, but the other case has symmetric conditions that are easy to realize.

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Two other important characterizations are expressed in terms of the points of intersection $J = AB \cap CD$ and $K = BC \cap DA$. Here only non-trapezoids are considered so that these points exist. A convex quadrilateral $ABCD$ is extangential with the capacity of having an excircle outside of vertex $A$ or $C$ if and only if

$$AJ + JC = CK + KA. \tag{3}$$

The second similar condition, also regarding an excircle outside of $A$ or $C$ (see Figure 1), is

$$BJ + JD = DK + KB. \tag{4}$$

The first of these was proved by Jakob Steiner in 1846 and the second was added in 1973 by Howard Grossman in [2]. For the reader's convenience, we prove the two most useful characterizations (1) and (4) in an appendix to this paper.

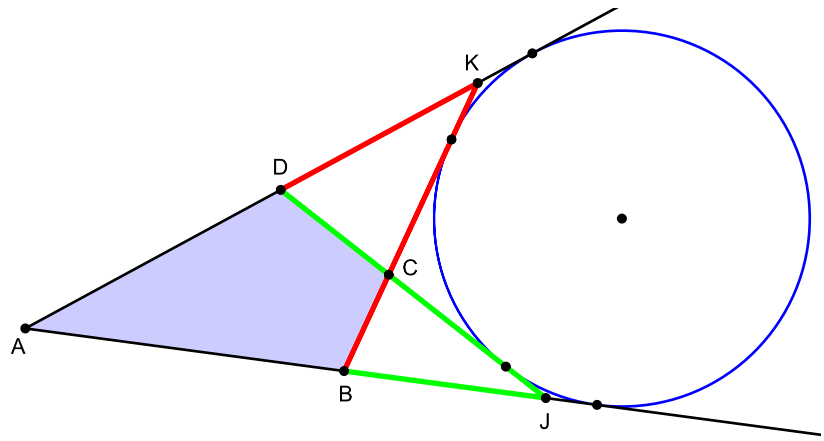


FIGURE 1. A quadrilateral with an excircle outside of $C$

In 1976 Daniel Sokolowsky contributed in [10] with one more characterization related to the configuration in Figure 1 (but with other notations). He proved that a convex quadrilateral $ABCD$ can have an excircle outside of $A$ or $C$ if and only if triangles $ABK$ and $ADJ$ have equal perimeters.

## 2. Basic theorems

In this section we study three basic characterizations of extangential quadrilaterals. The first was stated in another form in [5], but since we cannot find a published proof of it, we include that here.

**Theorem 2.1.** *In a convex quadrilateral that is not a trapezoid, consider the internal angle bisectors at a pair of opposite vertices, the external angle bisectors at the other two vertices, and the external angle bisectors at the angles formed where the extensions of opposite sides intersect. Any four of these six angle bisectors are concurrent if and only if it is an extangential quadrilateral, where the point of intersection is the center of the excircle.*

**Proof.** There are several different cases to consider, but they are all proved in the same way using congruent triangles, so we only study one of them.

($\Rightarrow$) If the quadrilateral is extangential, then its center lies on all six angle bisectors by RHS congruence ($\triangle AEF \cong \triangle AEI$ and so on in Figure 2).

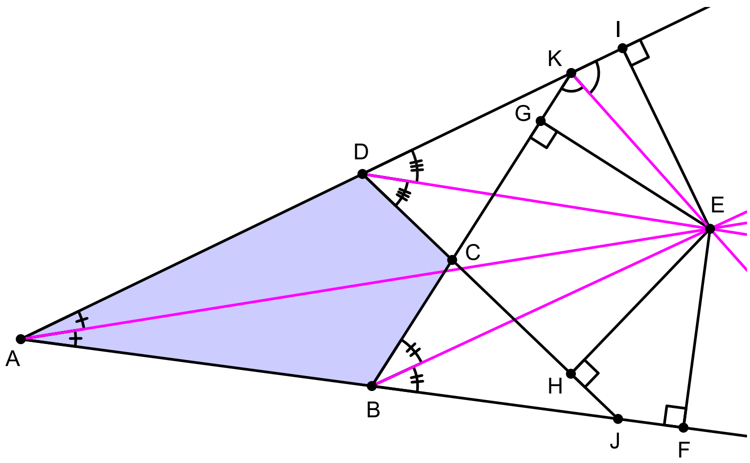


FIGURE 2. Four concurrent angle bisectors at $E$

($\Leftarrow$) Conversely, let (for instance) the angle bisectors of $A$, $B$, $K$ and $D$ intersect at a point $E$ and let $F$, $G$, $H$, $I$ be the projections of $E$ on the extensions of $AB$, $BC$, $CD$ and $DA$ respectively (see Figure 2). Then, by AAS congruence, four pairs of triangles are congruent, so $EF = EG$, $EF = EI$, $EG = EI$ and $EH = EI$. Hence $E$ is equidistant to the extensions of all four sides, making $ABCD$ an extangential quadrilateral. $\square$

We note that there are cases when it is enough with three concurrent angle bisectors instead of four. This happens when the points where these angle bisectors are drawn from are collinear, as for example $A$, $D$, $K$. But there are more cases, as for example with bisectors at $A$, $B$, $K$, when it cannot be proved that the extensions of all four sides are at equal distance from the intersection $E$ of these (internal or external) bisectors. Therefor the theorem is formulated to require four concurrent angle bisectors.

**Theorem 2.2.** *In a convex quadrilateral where two opposite external angle bisectors intersect at an exterior point, this point is equidistant to the extensions of any two opposite sides if and only if it is an extangential quadrilateral.*

**Proof.** ($\Rightarrow$) By the previous theorem, the point of intersection of the two opposite external angle bisectors is equidistant to the extensions of all four sides in an extangential quadrilateral.

($\Leftarrow$) Conversely we have that any point on an angle bisector is equidistant to the two legs of the angle where it is drawn, so the point of intersection of the two opposite external angle bisectors is equidistant to the extensions of two adjacent sides of the quadrilateral (see Figure 3). If the point of intersection is also equidistant to the extensions of any two opposite sides,

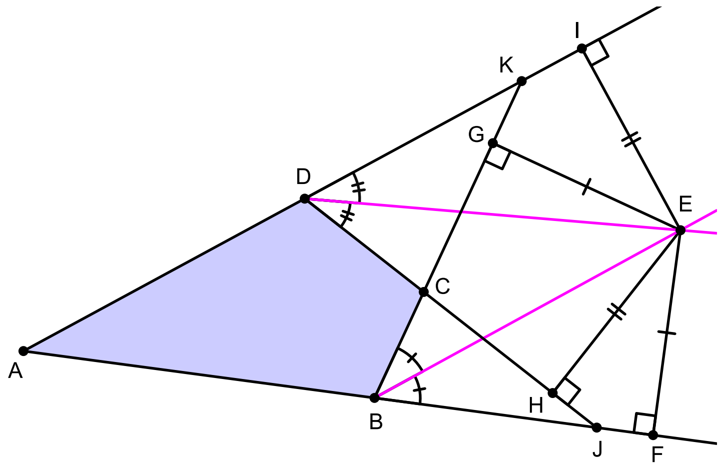


FIGURE 3. $ABCD$ is extangential $\quad\Leftrightarrow\quad$ $EF = EH$

then this point is equidistant to the extensions of all four sides making the quadrilateral extangential by definition. □

A variant of (4) appears in the next characterization.

**Theorem 2.3.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose the external angle bisectors at $B$ and $D$ intersect at $E$, and that $F$, $G$, $H$, $I$ are the projections of $E$ on the extensions of $AB$, $BC$, $CD$, $DA$ respectively. Then*

$$FJ + KG = JH + IK$$

*if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $A$ or $C$.*

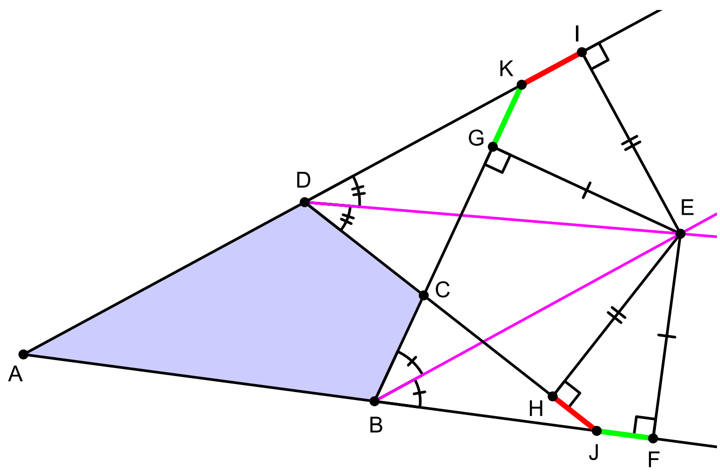


FIGURE 4. $ABCD$ is extangential $\quad\Leftrightarrow\quad$ $FJ + KG = JH + IK$

**Proof.** By congruent triangles (AAS), we have $BF = GB$ and $DH = ID$ (see Figure 4). We have that (4) is equivalent to

$$BF - FJ + JH + HD = DI - IK + KG + GB,$$

which is equivalent to the equality stated in the theorem. □

## 3. Equal line segments

In this section we will need the following well-known triangle formulas, which are illustrated in Figure 5. If they are unknown to the reader, we offer proving them as an exercise before reading further.

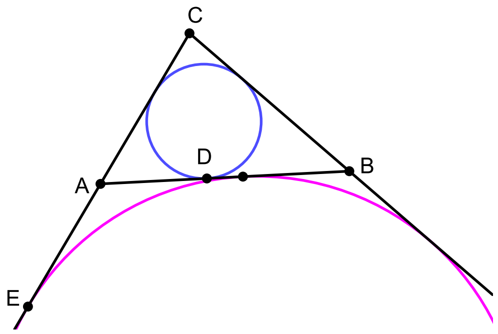


Figure 5. Tangent points for the incircle and one excircle to a triangle

**Lemma 3.1.** *In triangle $ABC$, suppose the incircle is tangent to $AB$ at $D$, and that the excircle tangent to $AB$ is tangent to the extension of $CA$ at $E$. Then*

$$AD = \tfrac{1}{2}(AB - BC + CA), \qquad AE = \tfrac{1}{2}(AB + BC - CA),$$
$$CE = \tfrac{1}{2}(AB + BC + CA).$$

The theorems in this section has the common theme that they deal with configurations where a convex quadrilateral is extangential if and only if two line segments have equal lengths. As we shall see, there are a lot of different such characterizations.

**Theorem 3.1.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $AJK$ ($CJK$) is completely outside of $ABCD$ and that the incircles in triangles $ADJ$ ($BCJ$) and $ABK$ ($CDK$) are tangent to $AJ$ ($CJ$) and $AK$ ($CK$) at $Y_1$ and $Y_2$ respectively. Then $JY_1 = KY_2$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $A$ ($C$).*

**Proof.** The first case is illustrated in Figure 6 and the second (in parenthesis) in Figure 7. Since the proofs are identical, we only prove the theorem in the second case and let the reader write the proof in the first case. We apply the first formula in Lemma 3.1 twice to get

$$JY_1 = \tfrac{1}{2}(-BC + BJ + CJ), \qquad KY_2 = \tfrac{1}{2}(-CD + CK + DK)$$

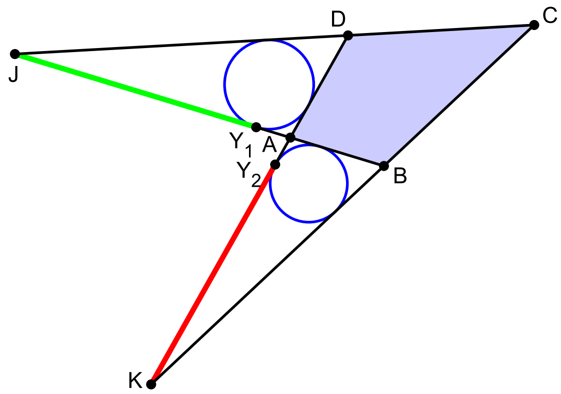


FIGURE 6. $ABCD$ is extangential $\quad\Leftrightarrow\quad JY_1 = KY_2$

so

$$2(JY_1 - KY_2) = -BC + BJ + CJ + CD - CK - DK$$
$$= -BK + BJ + JD - DK.$$

Hence we have

$$JY_1 = KY_2 \quad\Leftrightarrow\quad BJ + JD = DK + KB$$

which according to (4) proves the claim. □

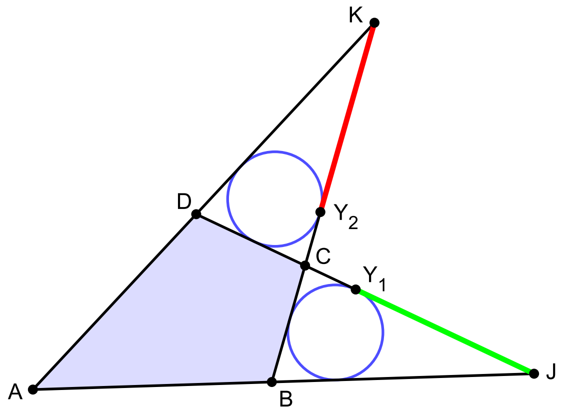


FIGURE 7. $ABCD$ is extangential $\quad\Leftrightarrow\quad JY_1 = KY_2$

As we saw in the previous theorem, the formulation of the theorem is a bit different in the case with an excircle outside of $A$ instead of $C$. This holds in several theorems in this paper regarding the points $J$ and $K$, and in order to avoid double formulations, from now on we only state theorems for an excircle outside of vertex $C$ in those cases.

**Theorem 3.2.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$ and that the excircles to triangles $ABK$ and $DAJ$ are tangent to $AB$ and $DA$ at $E'$ and $H'$ respectively. Then $JE' = KH'$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

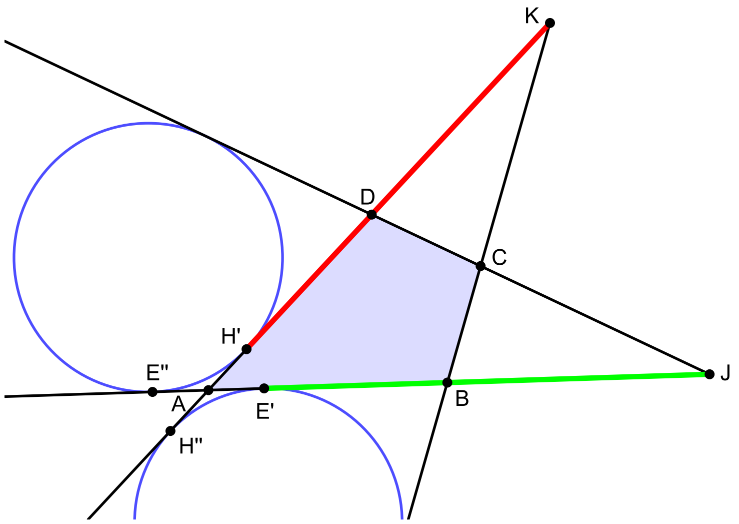


FIGURE 8. $ABCD$ is extangential $\Leftrightarrow$ $JE' = KH'$

**Proof.** With notations as in Figure 8, we have $JE' = JE'' - E''E'$ and

$$KH' = KH'' - H''H' = KH'' - E''E'.$$

Applying the third formula in Lemma 3.1 yields

$$\begin{aligned} 2(JE' - KH') = 2(JE'' - KH'') &= AJ + JD + DA - AB - BK - KA \\ &= BJ + JD - DK - KB. \end{aligned}$$

Hence

$$JE' = KH' \quad \Leftrightarrow \quad BJ + JD = DK + KB$$

which proves the theorem according to (4). □

Next we have a characterization about two other triangle incircles.

**Theorem 3.3.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$ and that the incircles in triangles $AJK$ and $CJK$ are tangent to $JK$ at $V$ and $U$ respectively. Then $JU = KV$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

**Proof.** We apply the first formula in Lemma 3.1 twice to get

$$KV = \tfrac{1}{2}(-AJ + JK + KA), \qquad JU = \tfrac{1}{2}(-CK + CJ + JK)$$

(see Figure 9), so

$$2(KV - JU) = -AJ + KA + CK - CJ.$$

Thus

$$KV = JU \quad \Leftrightarrow \quad AJ + CJ = AK + CK$$

which by (3) completes the proof. □

There is the following excircle version of the previous theorem.

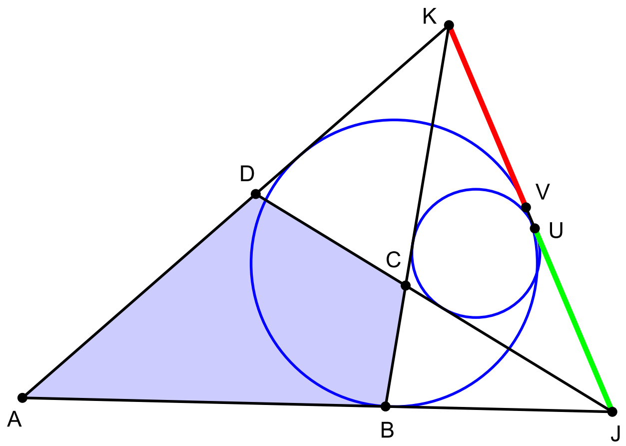


FIGURE 9. $ABCD$ is extangential $\Leftrightarrow$ $JU = KV$

**Theorem 3.4.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$ and that the excircles to triangles $AJK$ and $CJK$ that are outside of $JK$ are tangent to $JK$ at $S$ and $T$ respectively. Then $JS = KT$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

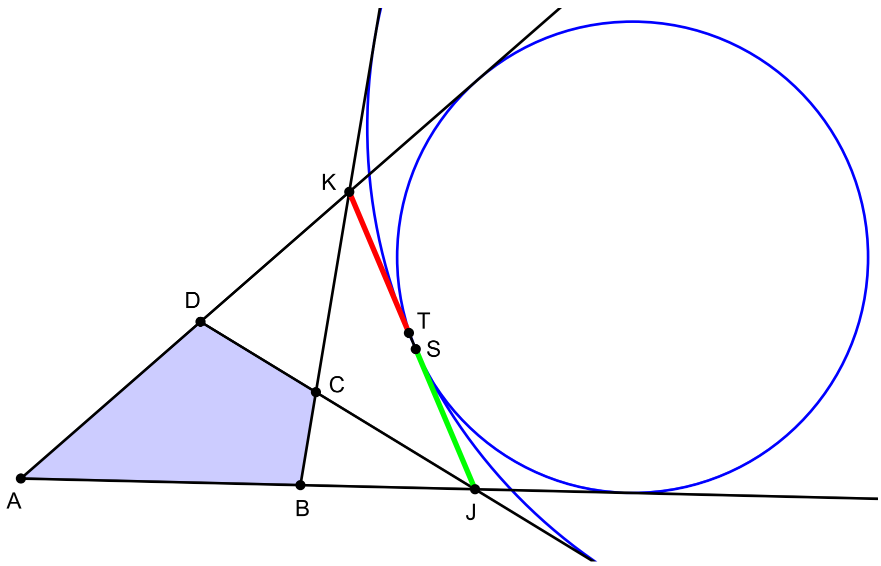


FIGURE 10. $ABCD$ is extangential $\Leftrightarrow$ $JS = KT$

**Proof.** According to Lemma 3.1, we have (see Figure 10)

$$JS = \tfrac{1}{2}(-AJ + JK + AK), \qquad KT = \tfrac{1}{2}(-CK + JK + CJ).$$

Then

$$2(JS - KT) = -AJ + AK + CK - CJ$$

which means that

$$JS = KT \quad \Leftrightarrow \quad AJ + CJ = AK + CK.$$

Thus the theorem is true according to (3). $\square$

Now we prove three characterization with four triangle excircles.

**Theorem 3.5.** *In a convex quadrilateral $ABCD$, suppose a pair of opposite excircles to triangles $ABC$ and $CDA$ are tangent to the sides $BC$ and $DA$ at $B'$ and $D'$ respectively. Then $AD' = CB'$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $A$ or $C$.*

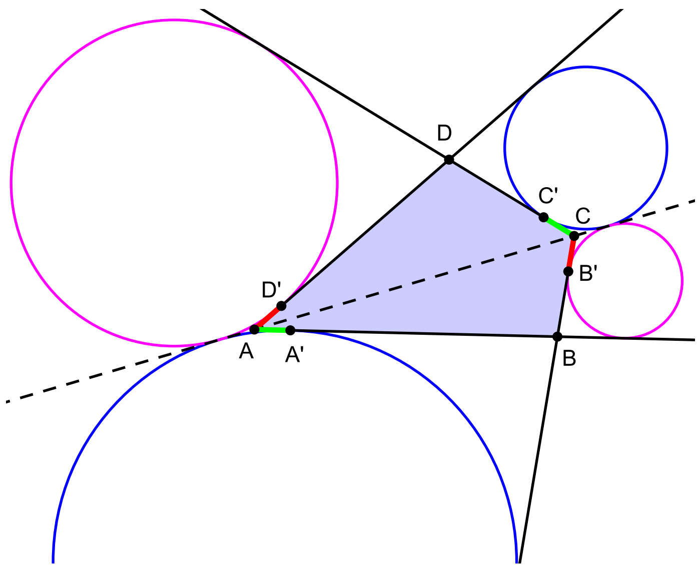


FIGURE 11. $ABCD$ is extangential $\quad\Leftrightarrow\quad AD' = CB'$

**Proof.** We have (see Figure 11)

$$AD' = \tfrac{1}{2}(-AC + CD + DA), \qquad CB' = \tfrac{1}{2}(-AC + AB + BC).$$

Then

$$2(AD' - CB') = CD + DA - AB - BC$$

so that

$$AD' = CB' \quad \Leftrightarrow \quad AB + BC = CD + DA$$

which is characterization (1) for an excircle outside of $A$ or $C$. $\square$

There is of course the similar characterization $AA' = CC'$ for the other pair of opposite excircles. We further have that $AA' = CB' = CC' = AD'$ in an extangential quadrilateral, which is a consequence of Lemma 3.1.

**Theorem 3.6.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. If two of the excircles to triangle $CJK$ are tangent to $CJ$ and $CK$ at $T_1$ and $U_1$ respectively, and two of the excircles to triangles $ABK$ and $ADJ$ are tangent to the extensions of $BC$ and $CD$ at $S_1$ and $V_1$ respectively, then $S_1U_1 = T_1V_1$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

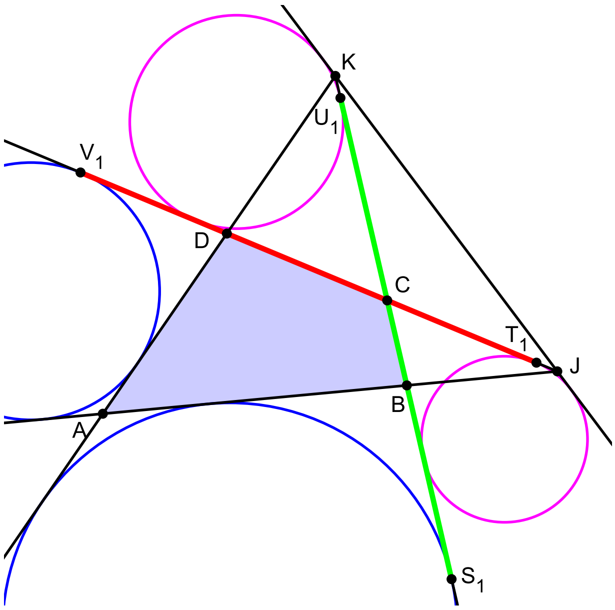


FIGURE 12. $ABCD$ is extangential $\quad\Leftrightarrow\quad S_1U_1 = T_1V_1$

**Proof.** Applying Lemma 3.1, we get (see Figure 12)

$$\begin{aligned} T_1V_1 = JV_1 - JT_1 &= \tfrac{1}{2}(JD + DA + AJ) - \tfrac{1}{2}(KC + CJ - JK) \\ &= \tfrac{1}{2}(CD + DA + AJ - KC + JK). \end{aligned}$$

In the same way

$$S_1U_1 = \tfrac{1}{2}(KA + AB + BC - CJ + JK)$$

so

$$\begin{aligned} 2(T_1V_1 - S_1U_1) &= CD + DA + AJ - KC - KA - AB - BC + CJ \\ &= BJ - KD + JD - KB. \end{aligned}$$

Hence

$$S_1U_1 = T_1V_1 \quad\Leftrightarrow\quad BJ + JD = DK + KB$$

completing the proof according to (4). □

**Theorem 3.7.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. Let an excircle to each of triangles $ABK$ and $AJK$ be tangent to $AJ$ at $S_2$ and $T_2$ respectively and an excircle to each of triangles $AKJ$ and $ADJ$ be tangent to $AK$ at $U_2$ and $V_2$ respectively. Then $S_2T_2 = U_2V_2$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

**Proof.** We have (see Figure 13)

$$AS_2 = \tfrac{1}{2}(AB + BK - KA), \qquad AT_2 = \tfrac{1}{2}(AJ + JK - KA)$$

so

$$2S_2T_2 = 2(AT_2 - AS_2) = AJ + JK - AB - BK = BJ + JK - BK > 0$$

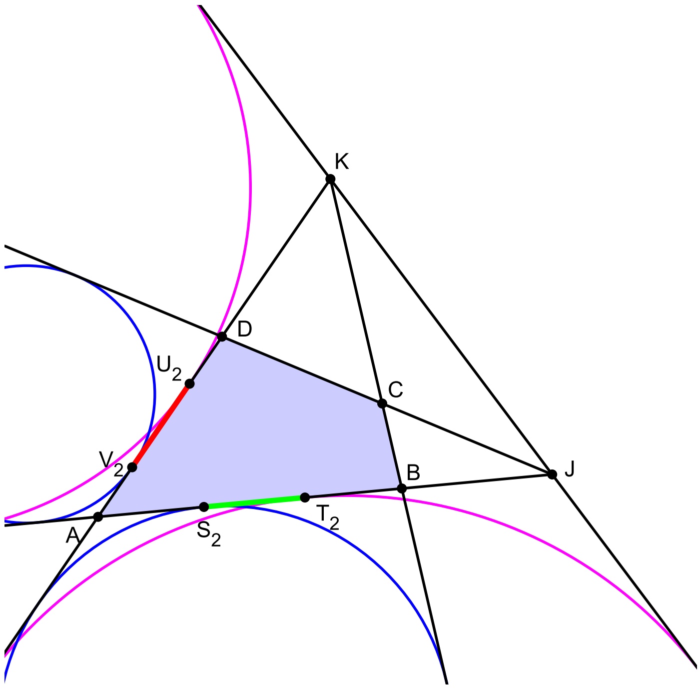


FIGURE 13. $ABCD$ is extangential $\quad\Leftrightarrow\quad$ $S_2T_2 = U_2V_2$

which is positive according to the triangle inequality applied in triangle $BJK$. Thus $AT_2 > AS_2$ always holds. In the same way

$$2U_2V_2 = JK + KD - JD > 0.$$

Hence we get

$$2(S_2T_2 - U_2V_2) = BJ - BK - KD + JD$$

and so

$$S_2T_2 = U_2V_2 \quad\Leftrightarrow\quad BJ + JD = DK + KB$$

which proves the theorem. □

The next four theorems include two triangle excircles and two incircles.

**Theorem 3.8.** *In a convex quadrilateral $ABCD$ where the diagonals intersect at $P$, let $B_1$ and $C_1$ be the points where the excircle to triangle $ADP$ outside of $AD$ and the incircle in triangle $BCP$ are tangent to diagonal $AC$ or its extension, and let $D_1$ and $E_1$ be the points where the incircle in triangle $CDP$ and the excircle to triangle $ABP$ outside of $AB$ are tangent to diagonal $BD$ or its extension. Then $B_1C_1 = D_1E_1$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $A$ or $C$.*

**Proof.** We have that (see Figure 14)

$$PB_1 = \tfrac{1}{2}(AP + PD + AD), \qquad PC_1 = \tfrac{1}{2}(CP + PB - BC)$$

so

$$B_1C_1 = PB_1 + PC_1 = \tfrac{1}{2}(AC + BD + AD - BC).$$

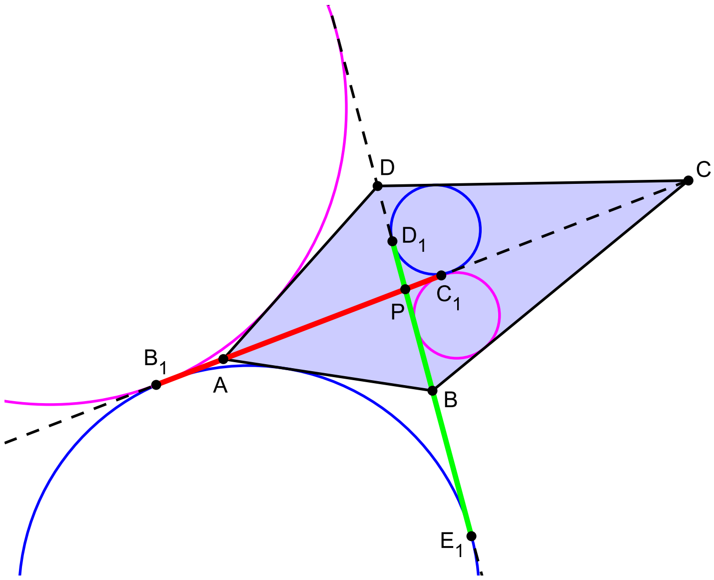


FIGURE 14. $ABCD$ is extangential $\Leftrightarrow$ $B_1C_1 = D_1E_1$

In the same way

$$D_1E_1 = \tfrac{1}{2}(AB - CD + AC + BD).$$

Hence

$$2(B_1C_1 - D_1E_1) = AD - BC - AB + CD$$

and we get that

$$B_1C_1 = D_1E_1 \quad \Leftrightarrow \quad AB + BC = CD + DA$$

which proves that $B_1C_1 = D_1E_1$ a characterization for an excircle outside of $A$ or $C$ in quadrilateral $ABCD$. □

**Theorem 3.9.** *In a convex quadrilateral $ABCD$ where the diagonals intersect at $P$, let $E_1$, $F_1$, $D_1$, $G_1$ be the points where the excircle to triangle $ABP$ outside of $AB$, the incircle in triangle $BCP$, the incircle in triangle $CDP$, and the excircle to triangle $DAP$ outside of $DA$ are tangent to diagonal $AC$ or its extension respectively. Then $E_1F_1 = D_1G_1$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $A$ or $C$.*

**Proof.** We have (see Figure 15)

$$\begin{aligned} D_1G_1 = DD_1 + DG_1 &= \tfrac{1}{2}(-CP + CD + DP) + \tfrac{1}{2}(-DP + DA + AP) \\ &= \tfrac{1}{2}(-CP + CD + DA + AP). \end{aligned}$$

An identical calculation yields

$$E_1F_1 = \tfrac{1}{2}(-CP + BC + AB + AP).$$

Hence

$$2(D_1G_1 - E_1F_1) = CD + DA - BC - AB$$

so

$$D_1G_1 = E_1F_1 \quad \Leftrightarrow \quad AB + BC = CD + DA$$

completing the proof. □

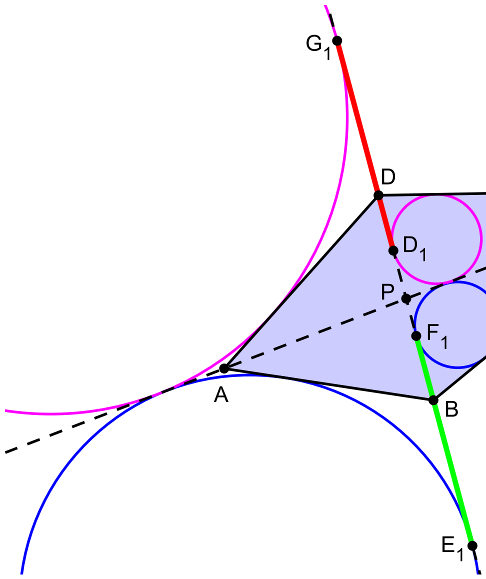


FIGURE 15. $ABCD$ is extangential $\quad\Leftrightarrow\quad E_1F_1 = D_1G_1$

**Theorem 3.10.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. If the incircles in triangles $ACJ$ and $ACK$ are tangent to $CJ$ and $CK$ at $B_3$ and $C_3$ respectively, and the excircles to triangles $ABK$ and $ADJ$ outside of $AB$ and $AD$ are tangent to the extensions of $BC$ and $CD$ at $A_3$ and $D_3$ respectively, then $A_3C_3 = B_3D_3$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

**Proof.** Since (see Figure 16)

$$\begin{aligned} A_3C_3 = KA_3 - KC_3 &= \tfrac{1}{2}(AB + BK + KA) - \tfrac{1}{2}(KA - AC + CK) \\ &= \tfrac{1}{2}(AB + BC + AC) \end{aligned}$$

and, in the same way,

$$B_3D_3 = \tfrac{1}{2}(DA + CD + AC),$$

then we have

$$2(A_3C_3 - B_3D_3) = AB + BC - CD - DA.$$

Hence

$$A_3C_3 = B_3D_3 \quad\Leftrightarrow\quad AB + BC = CD + DA$$

and the theorem follows from (1). □

**Theorem 3.11.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. If the incircles in triangles $ACJ$ and $ACK$ are tangent to $AJ$ and $AK$ at $B_2$ and $C_2$ respectively, and the excircles to triangles $ABK$ and $ADJ$ are tangent to $AB$ and $AD$ at $A_2$ and $D_2$ respectively, then $A_2B_2 = C_2D_2$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

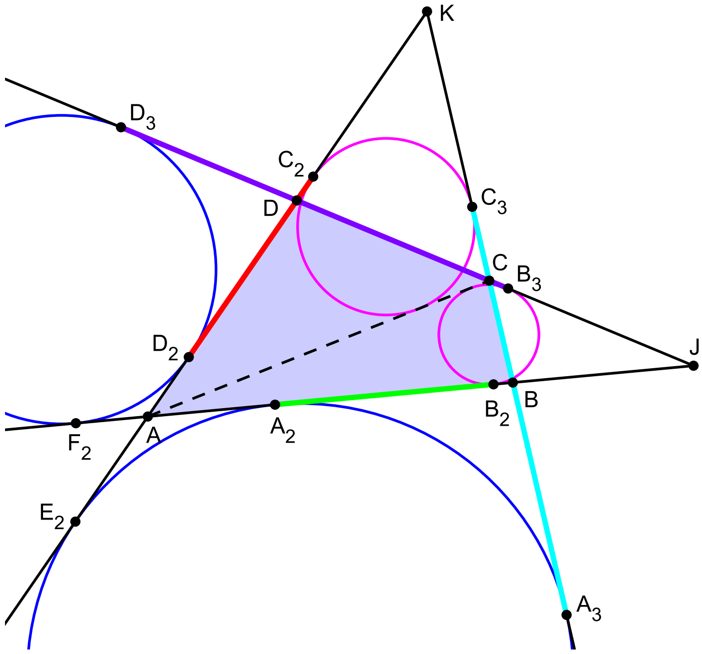


FIGURE 16. $A_3C_3 = B_3D_3 \quad \Leftrightarrow \quad ABCD$ is extangential $\quad \Leftrightarrow \quad A_2B_2 = C_2D_2$

**Proof.** As a consequence of the property that two tangents to a circle from an external point have equal lengths, we have that $A_3C_3 = C_2E_2$, $B_2F_2 = B_3D_3$ and $A_2F_2 = D_2E_2$ (see Figure 16). Then we get

$$\begin{aligned} A_2B_2 - C_2D_2 &= (B_2F_2 - A_2F_2) - (C_2E_2 - D_2E_2) \\ &= B_2F_2 - C_2E_2 = B_3D_3 - A_3C_3 \end{aligned}$$

so

$$A_2B_2 = C_2D_2 \quad \Leftrightarrow \quad A_3C_3 = B_3D_3$$

which according to the previous theorem proves this theorem. □

Now we have two characterization with four subtriangle incircles.

**Theorem 3.12.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. If the incircles in triangles $ACJ$ and $BCJ$ are tangent to $AJ$ at $P_3$ and $Q_3$ respectively, and the incircles in triangles $ACK$ and $DCK$ are tangent to $AK$ at $W_3$ and $V_3$ respectively, then $P_3Q_3 = W_3V_3$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

**Proof.** According to Lemma 3.1, we have (see Figure 17)

$$JQ_3 = \tfrac{1}{2}(BJ - BC + CJ), \qquad JP_3 = \tfrac{1}{2}(AJ - AC + CJ)$$

so

$$2P_3Q_3 = 2(JP_3 - JQ_3) = AJ - AC - BJ + BC = AB - AC + BC > 0$$

according to the triangle inequality. In the same way it holds that

$$2W_3V_3 = 2(KW_3 - KV_3) = CD - AC + AD > 0.$$

Thus we get

$$2(P_3Q_3 - W_3V_3) = AB + BC - CD - AD.$$

Hence

$$P_3Q_3 = W_3V_3 \quad \Leftrightarrow \quad AB + BC = CD + DA$$

so the quadrilateral has an excircle outside of $A$ or $C$ if and only if $P_3Q_3 = W_3V_3$. $\square$

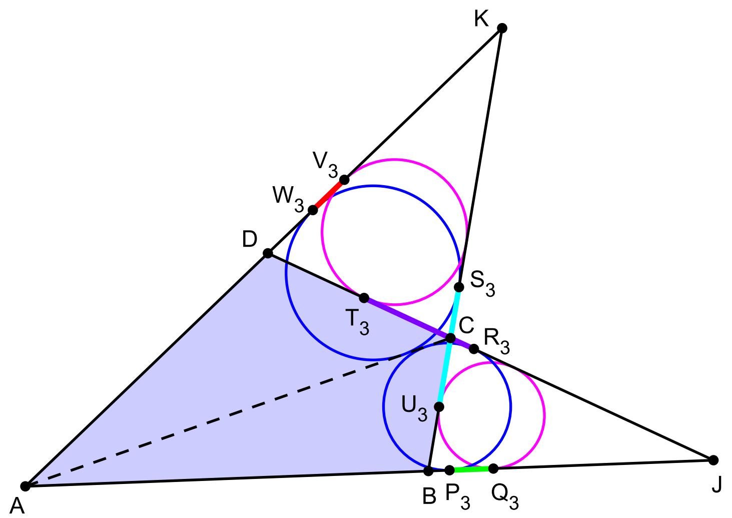


FIGURE 17. $P_3Q_3 = W_3V_3 \quad \Leftrightarrow \quad ABCD$ is extangential $\quad \Leftrightarrow \quad R_3T_3 = S_3U_3$

**Theorem 3.13.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. If the incircles in triangles $ACJ$ and $CDK$ are tangent to $DJ$ at $R_3$ and $T_3$ respectively, and the incircles in triangles $ACK$ and $BCJ$ are tangent to $BK$ at $S_3$ and $U_3$ respectively, then $R_3T_3 = S_3U_3$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

**Proof.** Using Lemma 3.1, we have (see Figure 17)

$$CU_3 = \tfrac{1}{2}(-BJ + BC + CJ), \qquad CS_3 = \tfrac{1}{2}(-AK + AC + CK)$$

and

$$CR_3 = \tfrac{1}{2}(-AJ + AC + CJ), \qquad CT_3 = \tfrac{1}{2}(-DK + CD + CK).$$

Then

$$\begin{aligned} 2(S_3U_3 - R_3T_3) &= 2(CS_3 + CU_3 - CR_3 - CT_3) \\ &= -BJ + BC - AK + AJ + DK - CD \\ &= AB + BC - AD - CD \end{aligned}$$

so

$$S_3U_3 = R_3T_3 \quad \Leftrightarrow \quad AB + BC = AD + CD$$

which proves that $ABCD$ has an excircle outside of $A$ or $C$ if and only if $S_3U_3 = R_3T_3$. $\square$

The following two characterizations are related to the previous two theorems and concern four triangle excircles.

**Theorem 3.14.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. If one of the excircles to triangles $ACJ$ and $ACB$ are tangent to $AJ$ at $P'$ and $Q'$ respectively, and one of the excircles to triangles $ACK$ and $ACD$ are tangent to $AK$ at $W'$ and $V'$ respectively, then $P'Q' = V'W'$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

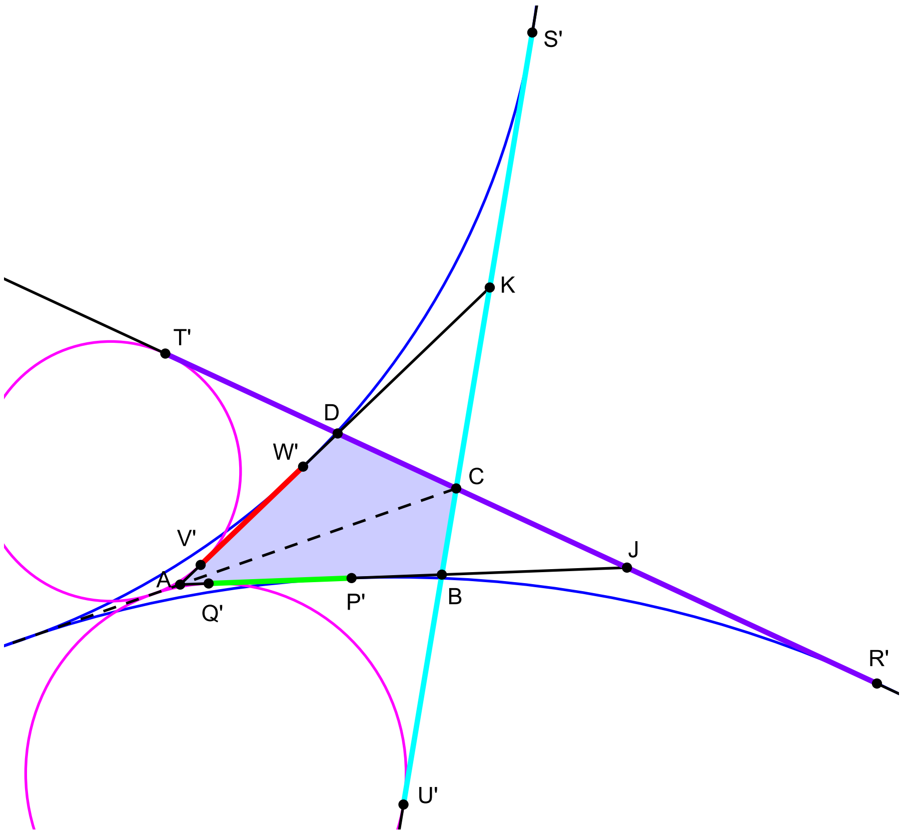


FIGURE 18. $P'Q' = V'W' \quad \Leftrightarrow \quad ABCD$ is extangential $\quad \Leftrightarrow \quad S'U' = R'T'$

**Proof.** Using Lemma 3.1, we have (see Figure 18)

$$AQ' = \tfrac{1}{2}(-AC + AB + BC), \qquad AP' = \tfrac{1}{2}(-AC + AJ + JC)$$

so

$$Q'P' = AP' - AQ' = \tfrac{1}{2}(AJ + JC - AB - BC) = \tfrac{1}{2}(BJ + JC - BC) > 0$$

according to the triangle inequality. In the same way we get

$$V'W' = \tfrac{1}{2}(CK + KD - CD) > 0.$$

Hence

$$\begin{aligned}2(Q'P'-V'W') &= BJ+JC-BC-CK-KD+CD\\ &= BJ+JD-DK-KB\end{aligned}$$

and therefore

$$Q'P'=V'W' \quad\Leftrightarrow\quad BJ+JD=DK+KB$$

which proves this theorem according to (4). □

Writing the proof of the last characterization in this section is left as an exercise for the reader. It is similar to previous proofs and the theorem is illustrated in Figure 18.

**Theorem 3.15.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let $J = AB \cap CD$ and $K = BC \cap DA$. Suppose triangle $CJK$ is completely outside of $ABCD$. If one of the excircles to triangles $ACK$ and $ACB$ are tangent to the extension of $BC$ at $S'$ and $U'$ respectively, and one of the excircles to triangles $ACJ$ and $ACD$ are tangent to the extension of $CD$ at $R'$ and $T'$ respectively, then $S'U' = R'T'$ if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $C$.*

## 4. Radii and areas

In [5] we proved several metric characterizations of extangential quadrilaterals. One that we missed was the following:

**Theorem 4.1.** *In a convex quadrilateral $ABCD$ where the diagonals intersect at $P$, let $r_b$ and $r_c$ be the inradii in triangles $BCP$ and $CDP$ respectively, and $\rho_a$ and $\rho_d$ be the radii in the excircles to triangles $ABP$ and $DAP$ that are tangent to $AB$ and $DA$ respectively. Then*

$$\frac{1}{\rho_a}+\frac{1}{r_c}=\frac{1}{r_b}+\frac{1}{\rho_d}$$

*if and only if $ABCD$ is an extangential quadrilateral with an excircle outside of $A$ or $C$.*

**Proof.** Two quite well-known formulas in triangle geometry are that the inradius $r$ and the exradius to side $c$, which we label $\rho_c$, can be expressed in terms of the heights $h_a$, $h_b$, $h_c$ to sides $a$, $b$, $c$ respectively as

$$\frac{1}{r}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}, \qquad \frac{1}{\rho_c}=\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}.$$

They were for instance stated in [4, p. 189], but with other notations.

In a convex quadrilateral $ABCD$ with diagonals $p = AC$ and $q = BD$, let $h_{pB}$ be the height from vertex $B$ to diagonal $p$, and similar for the other vertices. Then

$$\frac{1}{\rho_a}=\frac{1}{h_{pB}}+\frac{1}{h_{qA}}-\frac{1}{h_1}, \qquad \frac{1}{\rho_d}=\frac{1}{h_{qA}}+\frac{1}{h_{pD}}-\frac{1}{h_4}$$

and (see Figure 19, where only four of the eight used heights are drawn)

$$\frac{1}{r_b}=\frac{1}{h_{pB}}+\frac{1}{h_{qC}}+\frac{1}{h_2}, \qquad \frac{1}{r_c}=\frac{1}{h_{qC}}+\frac{1}{h_{pD}}+\frac{1}{h_3}$$

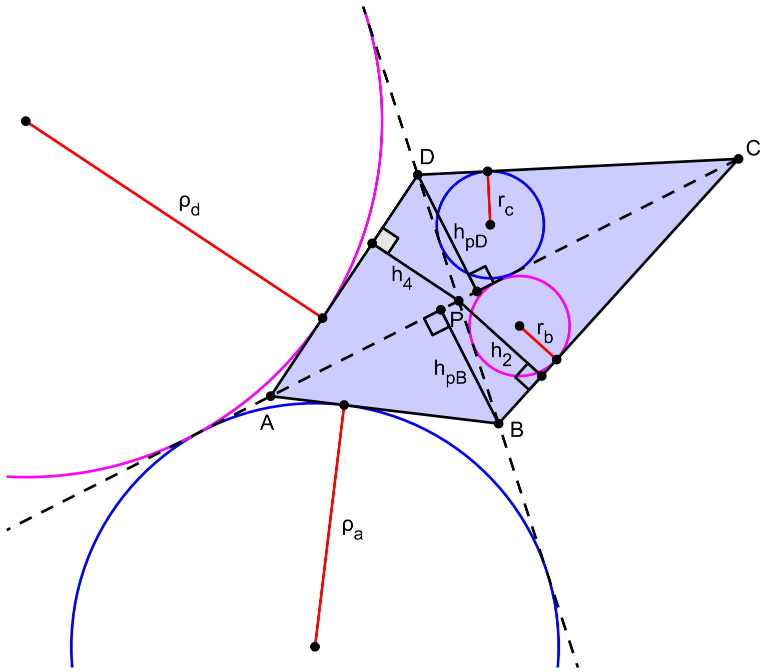


FIGURE 19. $ABCD$ is extangential $\Leftrightarrow$ $\frac{1}{\rho_a}+\frac{1}{r_c}=\frac{1}{r_b}+\frac{1}{\rho_d}$

where $h_1$, $h_2$, $h_3$, $h_4$ are heights from $P$ to $AB$, $BC$, $CD$, $DA$ respectively. We get

$$\frac{1}{\rho_a}+\frac{1}{r_c}-\frac{1}{r_b}-\frac{1}{\rho_d}=-\frac{1}{h_1}+\frac{1}{h_3}-\frac{1}{h_2}+\frac{1}{h_4}$$

so

$$\frac{1}{\rho_a}+\frac{1}{r_c}=\frac{1}{r_b}+\frac{1}{\rho_d}\quad\Leftrightarrow\quad\frac{1}{h_1}+\frac{1}{h_2}=\frac{1}{h_3}+\frac{1}{h_4}.$$

Theorem 3 in [5] states that the right hand side equality is a necessary and sufficient condition for $ABCD$ to have an excircle outside of $A$ or $C$. Hence so is the left hand side equality, completing the proof. □

The next characterization includes the radii of two pairs of subtriangle excircles.

**Theorem 4.2.** *In a convex quadrilateral, the product of the radii in adjacent excircles to those triangles created by a diagonal which are on the same side of that diagonal and also tangent to a side of the quadrilateral, are equal if and only if it is an extangential quadrilateral.*

**Proof.** It is well-known that the radius of the excircle that is tangent to side $z$ of a triangle with sides $x$, $y$, $z$ is given by the formula

$$R_z=\frac{1}{2}\sqrt{\frac{(x+y+z)(-x+y+z)(x-y+z)}{x+y-z}}.$$

Let the excircles tangent to sides $AB=a$, $BC=b$, $CD=c$, $DA=d$ of the quadrilateral have radii $R_a$, $R_b$, $R_c$, $R_d$ respectively, and let diagonal

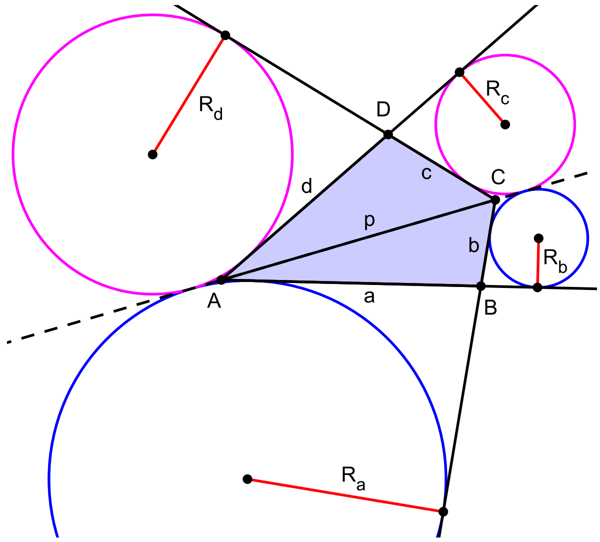


FIGURE 20. $ABCD$ is extangential $\quad\Leftrightarrow\quad R_aR_b = R_cR_d$

$AC = p$ (see Figure 20). Then $4R_aR_b$ is equal to

$$\sqrt{\frac{(a+b+p)(a-b+p)(a+b-p)}{(-a+b+p)}\cdot\frac{(a+b+p)(-a+b+p)(a+b-p)}{(a-b+p)}}$$

which is simplified into

$$4R_aR_b = (a+b+p)(a+b-p).$$

In the same way we get

$$4R_cR_d = (c+d+p)(c+d-p).$$

Then

$$4(R_aR_b-R_cR_d) = (a+b)^2-p^2-((c+d)^2-p^2) = (a+b+c+d)(a+b-c-d).$$

Hence

$$R_aR_b = R_cR_d \quad\Leftrightarrow\quad a+b = c+d$$

which is equivalent to that the quadrilateral $ABCD$ can have an excircle outside of $A$ or $C$. □

The three points where an excircle is tangent to a side of a triangle and the extensions of the adjacent two sides determine a new triangle that is called the *contact triangle*. If the original triangle has area $S$, circumradius $R$ and the considered excircle has radius $R_a$, then this contact triangle has area

$$S_a = \frac{R_a}{2R}S. \tag{5}$$

Several different proofs of this formula can be found at [1].

**Theorem 4.3.** *In a convex quadrilateral, the product of the areas of adjacent triangles determined by the points of tangency of the excircles to the two triangles created by a diagonal and are on opposite sides of that diagonal* *and each tangent to a side of the quadrilateral, are equal if and only if it is an extangential quadrilateral.*

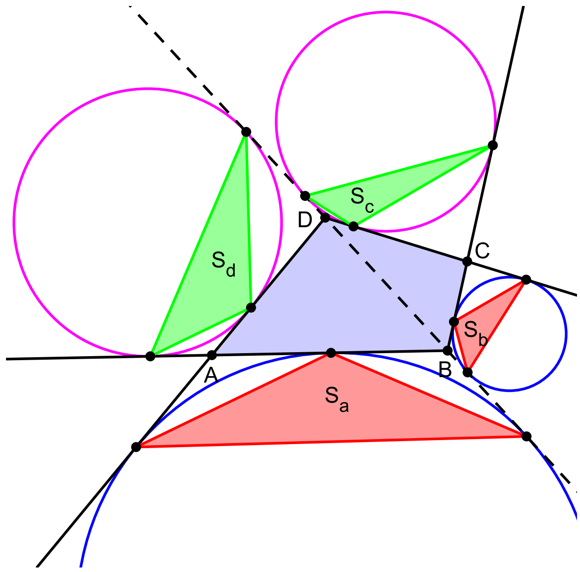


FIGURE 21. $ABCD$ is extangential $\quad\Leftrightarrow\quad S_aS_b = S_cS_d$

**Proof.** In a convex quadrilateral $ABCD$, let triangles $ABD$, $BCD$ have areas $S_1$, $S_2$ and circumradii $R_1$, $R_2$ respectively. Let the excircles tangent to sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ have radii $R'_a$, $R'_b$, $R'_c$, $R'_d$ and their corresponding contact triangles have areas $S_a$, $S_b$, $S_c$, $S_d$ respectively (see Figure 21). Applying (5) yields

$$S_aS_b - S_cS_d = \frac{R'_aS_1}{2R_1}\cdot\frac{R'_bS_2}{2R_2} - \frac{R'_cS_2}{2R_2}\cdot\frac{R'_dS_1}{2R_1} = \frac{S_1S_2}{4R_1R_2}\left(R'_aR'_b - R'_cR'_d\right).$$

Now we use the well-known formula for the radius of an excircle to a triangle ($R_a = \frac{S}{s-a}$ where $S$ is the triangle area and $s$ is the semiperimeter) to get

$$R'_aR'_b - R'_cR'_d = \frac{2S_1}{-a+q+d}\cdot\frac{2S_2}{-b+q+c} - \frac{2S_2}{-c+q+b}\cdot\frac{2S_1}{a+q-d}$$

where $q = BD$. The right hand side is factorized as

$$\frac{8qS_1S_2(a+b-c-d)}{(-a+q+d)(-b+q+c)(-c+q+b)(a+q-d)}.$$

Thus we have derived that

$$S_aS_b - S_cS_d = \frac{2qS_1^2S_2^2(a+b-c-d)}{R_1R_2(-a+q+d)(-b+q+c)(-c+q+b)(a+q-d)}$$

where the denominator is never zero according to the triangle inequality. Hence

$$S_aS_b = S_cS_d \quad\Leftrightarrow\quad a+b = c+d$$

which proves this condition according to (1). □

Directly from the previous proof, we get the following characterization, which is illustrated in Figure 22.

**Theorem 4.4.** *In a convex quadrilateral, the product of the radii in adjacent excircles to those triangles created by a diagonal which are on opposite sides of that diagonal and also tangent to a side of the quadrilateral, are equal if and only if it is an extangential quadrilateral.*

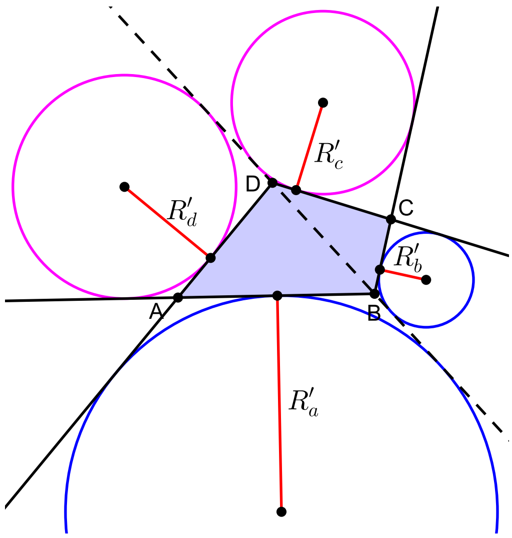


FIGURE 22. $ABCD$ is extangential $\Leftrightarrow$ $R'_aR'_b = R'_cR'_d$

## 5. The Newton line

In [7] we defined the *height* of a convex quadrilateral to be the perpendicular distance from one side or its extension to the point on the opposite side or its extension where the Newton line intersects it. The *Newton line* ($MN$ in Figure 23) is defined as the line that goes through the two diagonal midpoints. (This way the height is undefined in quadrilaterals having one or two pairs of opposite parallel sides, but in those quadrilaterals the height can be interpreted in the old fashioned way.) We proved in [7] that the area $Q$ of a convex quadrilateral is given by the simple formulas

$$Q = ah_a = bh_b = ch_c = dh_d$$

where $h_a$, $h_b$, $h_c$, $h_d$ are the heights to side $a = AB$, $b = BC$, $c = CD$, $d = DA$ respectively. These formulas were discovered by Milton Favio Donaire Peña. (Usually the area of a quadrilateral is denoted $K$, but since we use that label for the intersection of two opposite sides we use another label for the area here.) It is now easy to prove the following characterization.

**Theorem 5.1.** *In a convex quadrilateral $ABCD$ with consecutive heights $h_a$, $h_b$, $h_c$ and $h_d$, it holds that*

$$\frac{1}{h_a} + \frac{1}{h_b} = \frac{1}{h_c} + \frac{1}{h_d}$$

*if and only if $ABCD$ is an extangential quadrilateral with the excircle outside of $A$ or $C$.*

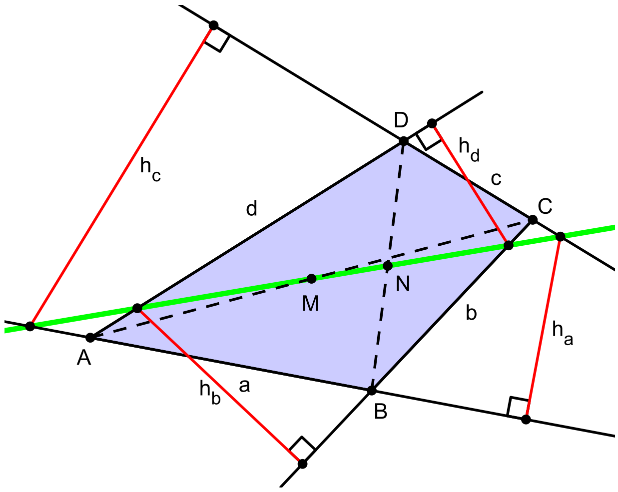


FIGURE 23. $ABCD$ is extangential $\Leftrightarrow$ $\frac{1}{h_a}+\frac{1}{h_b}=\frac{1}{h_c}+\frac{1}{h_d}$

**Proof.** Inserting Donaire's formulas into (1), we get that the quadrilateral is extangential with the excircle outside of $A$ or $C$ if and only if

$$\frac{Q}{h_a}+\frac{Q}{h_b}=\frac{Q}{h_c}+\frac{Q}{h_d}$$

which directly simplifies to the stated necessary and sufficient condition. □

Now we prove an excircle version of Theorem 7.3 in [8] (that theorem gave a characterization of tangential quadrilaterals).

**Theorem 5.2.** *If two opposite external angle bisectors to a convex quadrilateral intersect at an exterior point, then this point lies on Newton's line if and only if it is an extangential quadrilateral.*

**Proof.** (⇒) Let the external angle bisectors at $B$ and $D$ intersect at a point $E$ outside of $C$ in an extangential quadrilateral with exradius $\rho$. Then we have

$$AB+BC=CD+DA \quad \Rightarrow \quad AB-CD=DA-BC$$

which we rewrite as

$$\tfrac{1}{2}\rho\cdot AB-\tfrac{1}{2}\rho\cdot CD=\tfrac{1}{2}\rho\cdot DA-\tfrac{1}{2}\rho\cdot BC.$$

This is an alternative way of expressing the equality

$$T_{ABE}-T_{CDE}=T_{DAE}-T_{BCE}$$

where $T_{ABE}$ denotes the area of triangle $ABE$. According to the converse of Leon-Anne's theorem, this confirms that $E$ lies on Newton's line (see Theorem 2 and its proof in [7]; the reason two terms are negative is that the triangles they represent are completely outside of $ABCD$).

(⇐) If the intersection $E$ of the external angle bisectors at $B$ and $D$ lies on Newton's line $MN$ in a convex quadrilateral, then $E$ has the same distance to the pairs of sides $AB$, $BC$ and $CD$, $DA$; let these distances be denoted by $x$ and $y$ respectively. Applying Leon-Anne's theorem, we get that

$$T_{ABE}-T_{CDE}=T_{DAE}-T_{BCE}$$

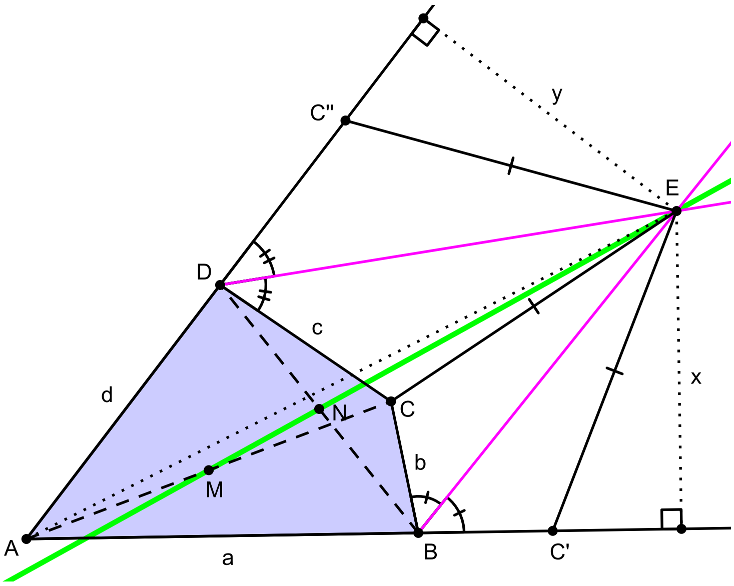


FIGURE 24. $ABCD$ is extangential $\Leftrightarrow$ $E \in MN$

implies

$$\tfrac{1}{2}x \cdot AB - \tfrac{1}{2}y \cdot CD = \tfrac{1}{2}y \cdot DA - \tfrac{1}{2}x \cdot BC$$

which is factorized as

$$\tfrac{1}{2}x(AB + BC) = \tfrac{1}{2}y(CD + DA). \tag{6}$$

Now we reflect triangles $BCE$ and $CDE$ in the external angle bisectors, which yields $C'$ on the extension of $AB$ and $C''$ on the extension of $AD$ (see Figure 24). Then (6) can be expressed as

$$\tfrac{1}{2}x \cdot AC' = \tfrac{1}{2}y \cdot AC''$$

or equivalently

$$\tfrac{1}{2}AE \cdot EC' \sin\angle AEC' = \tfrac{1}{2}AE \cdot EC'' \sin\angle AEC''. \tag{7}$$

Since $\triangle BCE \cong \triangle BC'E$ and $\triangle DCE \cong \triangle DC''E$ (SAS), we get $CE = EC'$ and $CE = EC''$ respectively; thus $EC' = EC''$. Then it follows from (7) that

$$\sin\angle AEC' = \sin\angle AEC''$$

with the two possible solutions

$$\angle AEC' = \angle AEC'' \qquad \vee \qquad \angle AEC' = \pi - \angle AEC''.$$

In the first case we get $\triangle AEC' \cong \triangle AEC''$ (SAS), so $AC' = AC''$ and thus

$$AB + BC' = AD + DC'' \quad \Rightarrow \quad AB + BC = AD + DC.$$

Hence $ABCD$ is an extangential quadrilateral.

In the second case, since

$$2\pi - \angle C = \angle ECB + \angle ECD = \angle EC'B + \angle EC''D$$

we get in quadrilateral $AC'EC''$ that

$$\begin{aligned}\angle A &= 2\pi - (\angle BC'E + \angle DC''E) - (\angle AEC' + \angle AEC'') \\ &= 2\pi - (2\pi - \angle C) - \pi \\ &= \angle C - \pi\end{aligned}$$

so we would have

$$\angle C = \angle A + \pi,$$

which is impossible in a convex quadrilateral. This concludes the proof of the converse. □

We note that the last angle equality is possible in a concave quadrilateral, so the converse is not valid in such quadrilaterals.

## 6. Concluding remarks

As in our previous collaborations, the majority of the theorems in this paper were discovered by the second author, while the first author wrote most of their proofs.

There are certainly more characterizations of extangential quadrilaterals to be discovered than the ones published so far. We know of 43 such conditions which we have given references for in this paper, including the 24 it contains, so with this paper we have more than doubled the number of known characterizations. In a follow-up paper we will study more than a dozen of new necessary and sufficient conditions for when a convex quadrilateral can have an excircle regarding tangent circles, concurrent lines, and cyclic quadrilaterals.

## Appendix A. Proofs of two old characterizations

The characterizations (1) and (4) that were reviewed in the introduction have been used in about 80 % of the proofs, so they are certainly important in the study of new characterizations of extangential quadrilaterals. Since it is very rare to find proofs of them in modern literature, we conclude by proving them here.

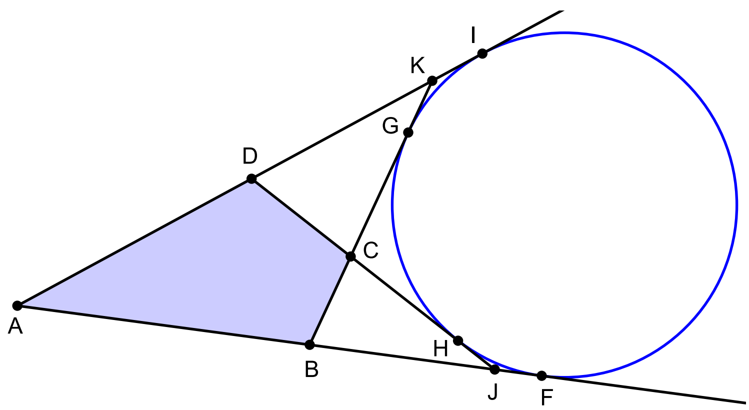


Figure 25. Tangent points for the excircle

First we prove that both these equalities hold in an extangential quadrilateral. With notations as in Figure 25, we get

$$AB + BC = AF - BF + BG - CG = AF - CG = AI - CH$$
$$= AD + DI - (DH - CD) = AD + CD$$

where we repeatedly used that the two tangents to a circle from an external point have equal lengths ($BF = BG$ and so on). In the same way

$$BJ + JD = BF - FJ + JH + HD = BG + DI$$
$$= BK - KG + KI + DK = DK + KB$$

completing the proofs of the two direct theorems.

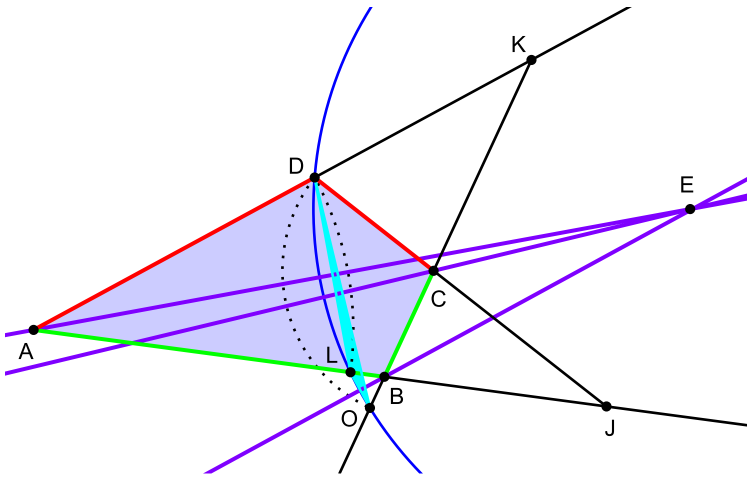


FIGURE 26. $AB + BC = CD + DA \quad \Rightarrow \quad ABCD$ is extangential

Now for the first converse. In a convex quadrilateral $ABCD$ where $AB + BC = CD + DA$, suppose without loss of generality that $AB > DA$, and thus $BC < CD$. (In the case $AB = DA$, the quadrilateral is a kite and it is trivial that there exists a point equidistant to the extensions of all four sides.). Construct $L$ on $AB$ such that $AL = DA$ and $O$ on the extension of $BC$ so that $CD = CO$ (see Figure 26). Then the given equality implies that

$$AL + LB + CO - BO = CD + DA$$

from which we get $LB = BO$. This means that all three triangles $ADL$, $CDO$ and $BLO$ are isosceles, which in turn implies that the perpendicular bisectors to the sides $DL$, $DO$ and $LO$ are also the angle bisectors of the vertex angles at $A$, $C$ and the exterior angle at $B$. Since the three perpendicular bisectors of the sides of any triangle (such as $DLO$) intersect in a point, which we call $E$, so does the three angle bisectors just mentioned. As a consequence, $E$ is equidistant from the three pairs of extended sides $\{AD, AB\}$, $\{BC, CD\}$ and $\{AB, BC\}$. Hence $E$ is equidistant from all four extended sides of $ABCD$, making it the center of a circle tangent to those four extended sides (the excircle, not drawn in Figure 26).

In order to prove the converse of (4), consider a convex quadrilateral $ABCD$ where $BJ + JD = DK + KB$. Constructing $Q$ and $R$ on the

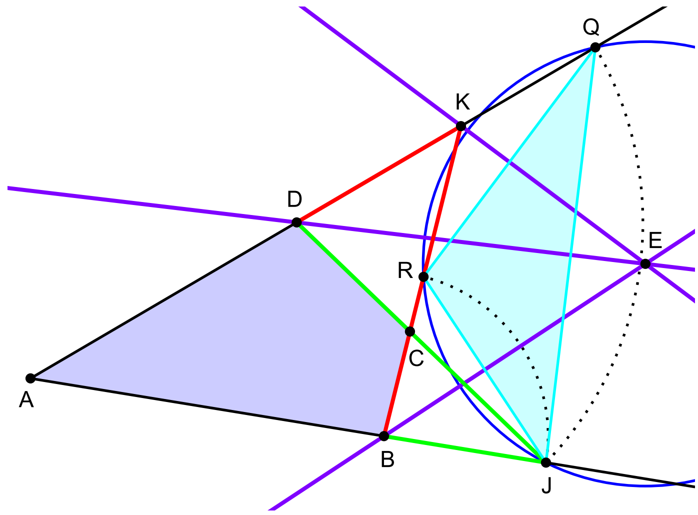


FIGURE 27. $BJ + JD = DK + KB \quad \Rightarrow \quad ABCD$ is extangential

extensions of $AD$ and $BC$ respectively such that $JD = DQ$ and $BJ = BR$ (see Figure 27), we get that the given equality implies

$$BJ + JD = DQ - QK + KR + RB,$$

so $KR = KQ$. Then there are three new isosceles triangles (do you see them?), and the perpendicular bisectors of their bases are also the exterior angle bisectors at $B$, $D$ and $K$. These are concurrent at a point $E$, which is the center of a circle tangent to the extensions of all four sides of $ABCD$, making it an extangential quadrilateral. This completes the proof.

As an exercise, we invite the reader to prove the old characterization (3).

Another way to interpret the three necessary and sufficient conditions (1), (3) and (4) is in the following way: *Any quadrilateral can have an excircle if and only if the sum of two adjacent sides equals the sum of the other two sides.* Then (1) is the version for a *convex* quadrilateral $ABCD$, (3) is for a *concave* quadrilateral $AJCK$, and (4) is for a *complex* quadrilateral $BJDK$ (see Figure 25). But more generally, all three of them are valid in all three cases (since they are equivalent). A short trigonometric proof of the equivalence of (3) and (4) was given in [3].

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