INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 11 (2022), No. 3, 41–47

A GENERALIZATION OF THE SIMSON LINE USING TWO ISOGONAL CONJUGATE POINTS

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Abstract. We shall give an extension to the famous Simson line theorem using two isogonal conjugate points instead of the pair of isogonal conjugate orthocenter and circumcenter in a triangle. Along with that, we also introduce a pure synthetic proof for this extension.

1. Introduction

The line through three projections on the sidelines of an arbitrary point on the circumcircle of a triangle named Simson line [12, 16]. However, this concept is sometimes known as the Wallace–Simson line because it was first published by William Wallace in 1799 [11]. Historically, it is a classical result of elementary geometry and has many applications; see [2, 3, 16]. There have been a lot of extensions to Simson's classical theorem, notably the Dao's expansion [4]. Many solutions to this extension have been proposed, including the one to the converse theorem; see [5, 6, 10, 13]. However, these extensions still only revolve around the orthocenter and the orthogonal projections. Orthocenter and circumcenter of a triangle are considered as a particular case of two isogonal conjugate points. In this paper, we have further generalized the Simson line using a pair of isogonal conjugates in the triangle.

We recall some concepts of elementary geometry; see [7, 14, 15].

Definition 1 (See [7], §§2)**.** If A and B are any two points, $\overline{AB}$ means the distance from A to B, and $\overline{BA}$ the distance from B to A. One of these will be represented by a positive number, the other by the same number with the negative sign. The notations $\overline{AB}$ and $\overline{BA}$ are called by signed lengths of segments.

Definition 2 (Isogonal conjugate points, [15])**.** Let ABC be a triangle and X is a point in the plane of the triangle ABC. Reflecting the lines AX, BX, CX about the angle bisectors at A, B, C. The three reflected lines then concur at the isogonal conjugate X^{-1} of X with respect to triangle ABC.

Date: March 27, 2022.
2010 *Mathematics Subject Classification.* 51M04, 51-03.
Key words and phrases. Simson line, general theorem, isogonal conjugate points, triangle, parallel lines, directed angle, signed length.
Received: 18.09.2021. In revised form: 16.02.2022. Accepted: 24.01.2022.

Definition 3 (See [7], §§16–19)**.** The directed angle from a line ℓ to a line ℓ' denoted by (ℓ, ℓ') is that angle through which ℓ must be rotated in the positive direction to become parallel to ℓ' or to coincide with ℓ'.

Throughout this paper, the symbol (ℓ, ℓ') always mean the directed angle of two lines ℓ and ℓ', taken modulo 180°.

Using the above concepts, we establish the following general theorem.

Theorem 1 (Further generalization of the Simson line)**.** *Let ABC be a triangle inscribed in a circle Ω. Let P and Q be two isogonal conjugate points with respect to the triangle ABC. Let A', B', and C' be the second intersections of QA, QB, and QC with the circle Ω, respectively. Let R be an arbitrary point lying on the circle Ω. Let X, Y, and Z be three points lying on line PR such that*

$$\frac{\overline{XP}}{\overline{XR}} = \frac{\overline{QA}}{\overline{QA'}}, \frac{\overline{YP}}{\overline{YR}} = \frac{\overline{QB}}{\overline{QB'}}, \frac{\overline{ZP}}{\overline{ZR}} = \frac{\overline{QC}}{\overline{QC'}}. \tag{1}$$

Let Δ be a line passing through Q. Let U, V, and W be the intersections of Δ with the lines RA, RB, and RC, respectively. Choose the points D, E, and F on the lines BC, CA, and AB, respectively, such that

$$UD \parallel PA, \ VE \parallel PB, \ WF \parallel PC. \tag{2}$$

Then, three lines XD, YE, and ZF are parallel.

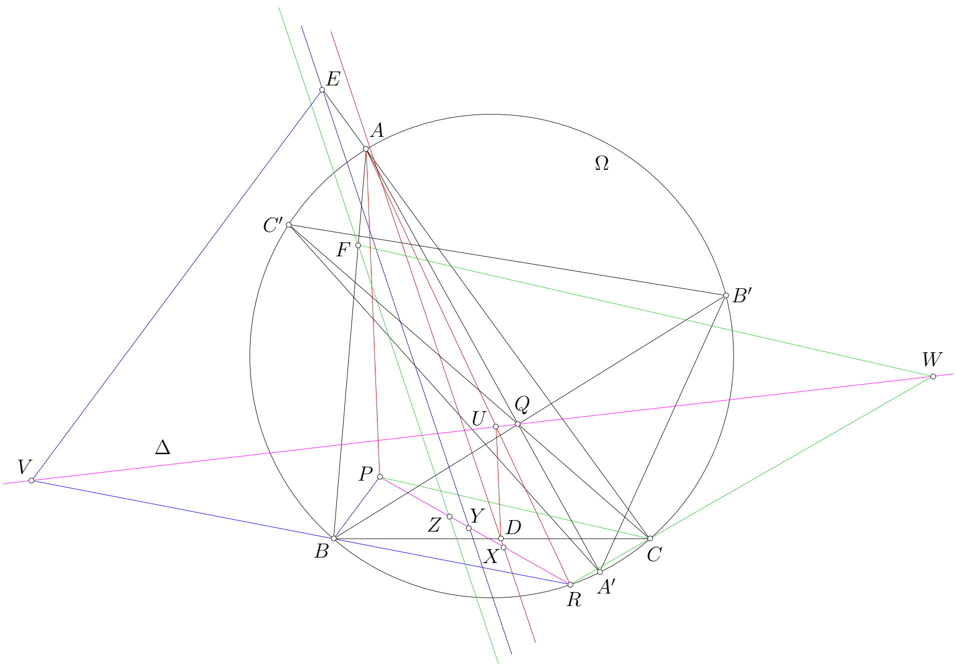

FIGURE 1. A further generalization of the Simson line.

Remark. When P is the orthocenter of triangle ABC then Q is the circumcenter of triangle ABC. It is easily seen that Q is the midpoints of AA', BB', and CC'. Thus X, Y, and Z coincide with the midpoint of PR.

Since the conclusion of Theorem 1, three lines XD, YE, and ZF are parallel. Hence, D, E, F, and the midpoint of RP are collinear. We get the generalization of the Simson line proposed by Dao Thanh Oai [4, 5].

2. Proof of Theorem 1

In this section, we give a proof to Theorem 1. We first introduce a some lemmas as follows:

Lemma 1 (K. L. Nguyen, 2005 [8]). *Let ABC be a triangle inscribed in a circle Ω. Let P and Q be two isogonal conjugate points with respect to the triangle ABC. Let Y be the point on line BC such that $QY \parallel AP$. Then, two lines PY and AQ intersect at point X lying on Ω.*

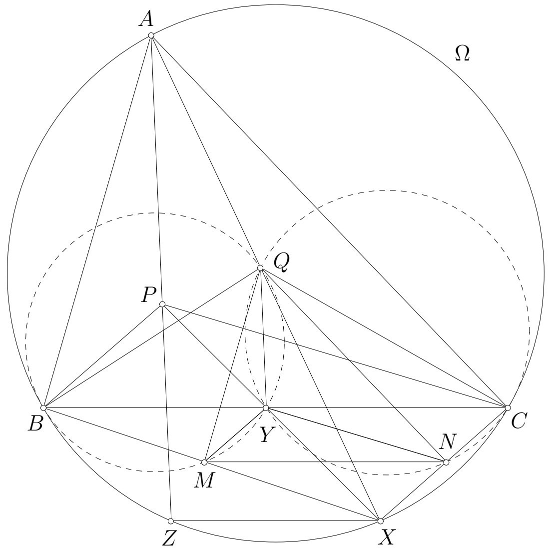

Figure 2. Proof of Lemma 1.

The proof of this lemma is based on the idea of the nickname Leonhard Euler [9], which is strictly based on the directed angle and Thales's theorem using signed lengths.

Proof. Let X be the second intersection of AQ and Ω, we need to prove that P, Y, and X are collinear. Indeed, let M, N be the points lying on the lines XB, XC, respectively, such that $QM \parallel AB$, $QN \parallel AC$. We have

$$\frac{\overline{MX}}{\overline{MB}} = \frac{\overline{QX}}{\overline{QA}} = \frac{\overline{NX}}{\overline{NC}}.$$

So that

$$MN \parallel BC. \tag{3}$$

Since $QM \parallel AB$ and $QY \parallel AP$ combined with P and Q are two isogonal conjugate points with respect to the triangle ABC, we have the following directed angle chasing

$$(QM, QY) = (AB, AP) = (AQ, AC) = (BX, BC) = (BM, BY).$$

This means B, Q, Y, and M are four concyclic points. Similarly, we have C, Q, Y, and N are four concyclic points. From these, combining with P and Q are two isogonal conjugate points with respect to triangle ABC and $QY \parallel AP$, we have

$$\begin{aligned}(MY, BP) &= (MY, YQ) + (AP, AB) + (AB, BP)\\ &= (BM, BQ) + (AC, AQ) + (BQ, BC)\\ &= (BM, BC) + (BC, BX)\\ &= 0.\end{aligned}$$

This deduces that $MY \parallel PB$. Similarly, we have $NY \parallel PC$. Combining with (3), we get triangles PBC and YMN are homothetic, so PY passes through X. This completes the proof of Lemma 1. □

Lemma 2. *Let ABC be a triangle inscribed in a circle Ω. Let P and Q be two isogonal conjugate points with respect to the triangle ABC. Lines PA and BC meet at A_1. Let R be an arbitrary point on Ω. Let A_2 be an arbitrary point lying on line AR. Let A_3 be a point lying on line BC such that $A_2A_3 \parallel PA$. Choose a point S on Ω such that*

$$(AS, AC) = (AB, QA_2). \tag{4}$$

Let A_4 be a point lying on line PR such that

$$\frac{\overline{A_4P}}{\overline{A_4R}} = \frac{\overline{QA}}{\overline{QA_6}}. \tag{5}$$

Then, two lines A_3A_4 and SR are parallel.

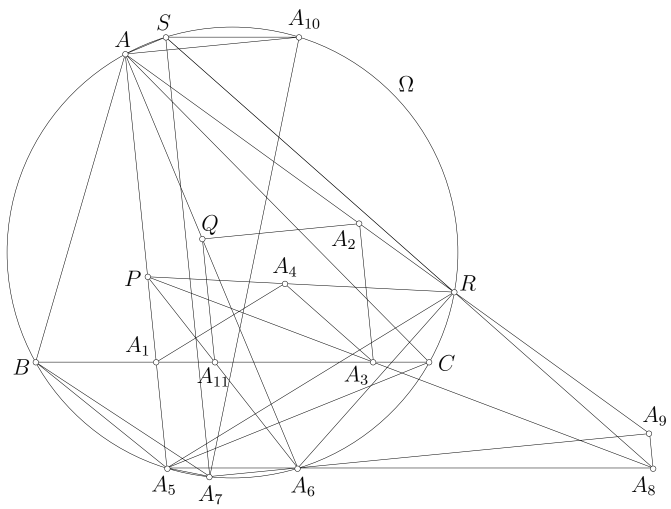

FIGURE 3. Proof of Lemma 2.

Proof. Let A_5 and A_6 be the second intersection points of lines AP and AQ with circle Ω. Since P and Q are two isogonal conjugate points with respect to the triangle ABC, two lines A_5A_6 and BC are parallel.

Let A_7 be a point lying on Ω such that $SA_7 \parallel AP$. Let A_8 be the intersection of two lines PA_3 and A_5A_6. Let A_9 be the intersection of two lines A_6A_7 and AR. Choose point A_{10} on Ω such that $AA_{10} \parallel QA_2$. Combining with (4), we have $(AS, AC) = (AB, AA_{10})$. From this, we easily see that

$$SA_{10} \parallel BC \parallel A_5A_6. \tag{6}$$

Thus, $A_6A_{10} = SA_5 = AA_7$, so

$$A_7A_6 \parallel AA_{10} \parallel QA_2. \tag{7}$$

Let A_{11} be the intersection of two lines PA_6 and BC. It follows from Lemma 1, $QA_{11} \parallel PA$. Now using Thales's theorem, we have

$$\frac{\overline{A_2A}}{\overline{A_2A_9}} = \frac{\overline{QA}}{\overline{QA_6}} = \frac{\overline{A_{11}P}}{\overline{A_{11}A_6}} = \frac{\overline{A_1P}}{\overline{A_1A_5}} = \frac{\overline{A_3P}}{\overline{A_3A_8}}.$$

This deduces that $A_8A_9 \parallel A_2A_3 \parallel AP$. Therefore,

$$(A_8A_6, A_8A_9) = (A_5A_6, A_5A) = (RA_6, RA) = (RA_6, RA_9),$$

we get that four points A_6, A_8, A_9, and R are concyclic. Using (6) and (7), we have the following directed angle chasing

$$\begin{aligned}(RA_8, RS) &= (RA_8, RA_9) + (RA_9, RS)\\ &= (A_6A_8, A_6A_9) + (RA, RS)\\ &= (A_6A_5, A_6A_7) + (A_{10}A, A_{10}S)\\ &= (A_6A_5, A_6A_7) + (A_6A_7, A_6A_5)\\ &= 0.\end{aligned}$$

This means S, R, and A_8 are collinear. Now using Thales's theorem again, we obtain

$$\frac{\overline{A_4P}}{\overline{A_4R}} = \frac{\overline{QA}}{\overline{QA_6}} = \frac{\overline{A_{11}P}}{\overline{A_{11}A_6}} = \frac{\overline{A_3P}}{\overline{A_3A_8}}$$

or $A_3A_4 \parallel SR$. This completes the proof of Lemma 2. $\square$

Coming back to the main theorem.

Proof of Theorem 1. Let S be the point on Ω such that

$$(AS, AC) = (AB, \Delta) = (AB, QU). \tag{8}$$

Using (8) and the conditions of Theorem 1, Lemma 2, we get that

$$UD \parallel RS. \tag{9}$$

From (8), we have

$$(BS, BC) = (AS, AC) = (AB, \Delta) = (BA, QV). \tag{10}$$

From (10) and the conditions of Theorem 1, Lemma 2, we have

$$VE \parallel RS. \tag{11}$$

From (8), we have

$$(CS, CB) = (AS, AB) = (AC, \Delta) = (CA, QW). \tag{12}$$

Using (12) and the conditions of Theorem 1, Lemma 2, we get

$$WF \parallel RS. \tag{13}$$

Thus, from (9), (11), and (13), we reach the conclusion three lines XD, YE, and ZF are parallel. This completes the proof of Theorem 1. $\square$

3. Conclusion

If we take P and Q coinciding different pairs of isogonal in the triangle, we can get some new theorems. On the other hand, from the general theorem, we can easily solve the following converse of generalization of the Simson line:

Theorem 2 (Converse of generalization of the Simson line)**.** *Let ABC be a triangle inscribed in a circle Ω. Let P be an arbitrary point. Let R be an arbitrary point lying on the circle Ω. Let Δ be a line passing through an isogonal conjugate of P. Let U, V, and W be the intersections of Δ with the lines RA, RB, and RC, respectively. Choose the points D, E, and F on the lines BC, CA, and AB, respectively, such that $UD \parallel PA$, $VE \parallel PB$, and $WF \parallel PC$. Prove that if D, E, and F are collinear then P must be the orthocenter of triangle ABC, and the line joining D, E, and F bisects the segment RP.*

Proof. Let Q be the isogonal conjugate of P with respect to the triangle ABC. Let A', B', and C' be the second intersections of QA, QB, and QC with the circle Ω, respectively. Let X, Y, and Z be three points lying on line PR such that

$$\frac{\overline{XP}}{\overline{XR}} = \frac{\overline{QA}}{\overline{QA'}}, \frac{\overline{YP}}{\overline{YR}} = \frac{\overline{QB}}{\overline{QB'}}, \frac{\overline{ZP}}{\overline{ZR}} = \frac{\overline{QC}}{\overline{QC'}}.$$

It follows from Theorem 1, $XD \parallel YE \parallel ZF$. Combining with the assumption D, E, and F are collinear, we deduce that X, Y, and Z must coincide. This means

$$\frac{\overline{QA}}{\overline{QA'}} = \frac{\overline{QB}}{\overline{QB'}} = \frac{\overline{QC}}{\overline{QC'}}.$$

From this, it is not hard to see that Q must be the midpoint of AA', BB', and CC' or Q is the circumcenter of triangle ABC. So P must be the orthocenter of triangle ABC. Obviously the line joining D, E, and F bisects the segment RP. This completes the proof of Theorem 2. $\square$

Acknowledgment. The author is grateful to Professor Catalin Ionel Barbu for his great encouragement over the long period of time.

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