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WHEN NEWTON'S CONIC MEETS APOLLONIUS' PROBLEM

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Abstract. A unified method for a geometric construction of conics which have a prescribed focus, pass through given points, and tangent given lines is presented. The proofs make extensive use of polar reciprocity, which converts these problems into degenerate cases of Apollonius' problem.

1. Introduction

A conic is determined once we specify five of its elements, between points and tangents. A standard use of Pascal's theorem, or a direct algebraic computation easily allows to find a five-point conic. These methods are less friendly when we prescribe tangent conditions, and fail, when a focus and three other conditions between point and tangents are given. In this paper, we give a general method to (geometrically) draw a conic which have a prescribed focus and three other elements between points and tangents.

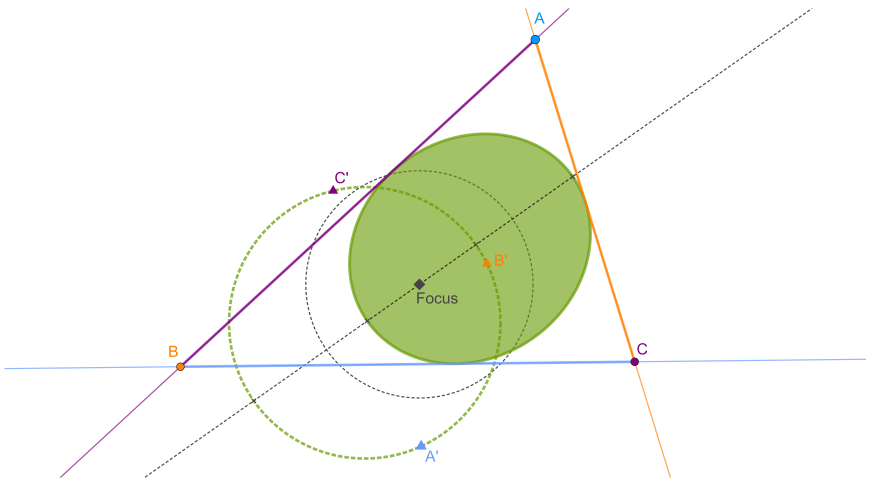

Figure 1. There is a unique conic with a prescribed focus and that tangents $\triangle ABC$'s sides: the polar dual of the circumcircle (dotted green) of the polar triangle $\triangle A'B'C'$.

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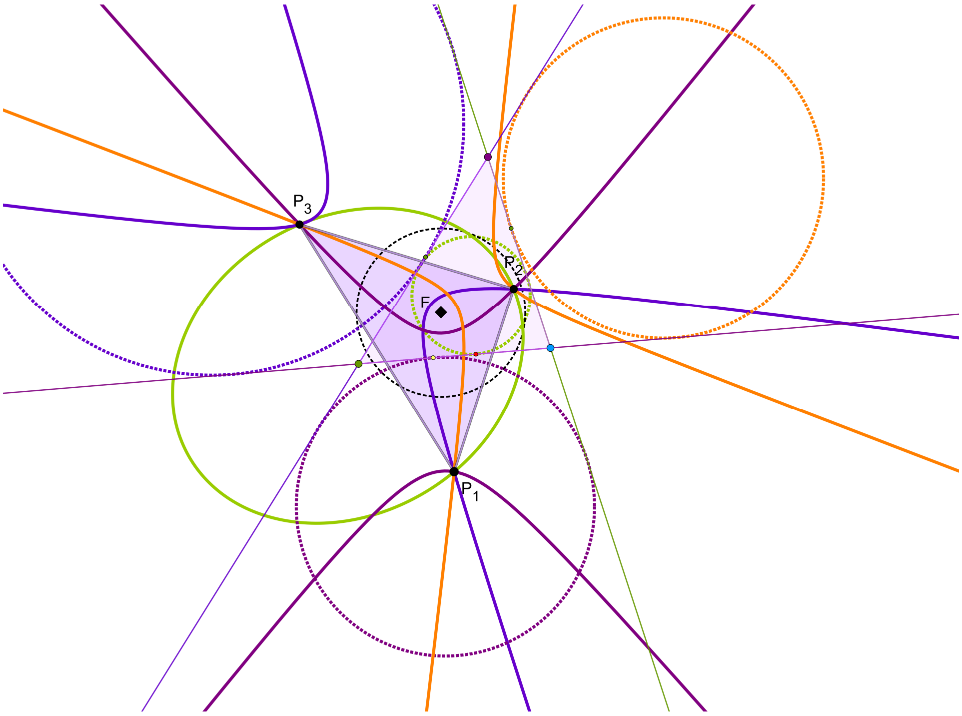

Figure 2. There are four conics that have a common focus, F and that pass through three given points P_1, P_2, P_3: the polar duals of the i-circle and the exinscribed circle of the triangle determined by the polars of these points).

In a broad view, such a problem relates to Kepler's conjecture on elliptical trajectories of planets, which have the Sun as one foci: in order to predict the trajectory of a planet, it suffices to specify three of its positions. Such problems had been extensively studied by Newton, in the first volume of its classic *Principia* [9]; here we solve Proposition XX, Problem XII, Book I, Section IV.

The solution is based on polar reciprocity, a tool developed by V. Poncelet in order to prove his famous Porism. The reader not acquainted with the method may see the Appendix. This approach enables both a natural, as well as unified treatment: there is no significant difference between various cases, as the method adapts to any data.

Some of these problems, as those involving three given tangents, are well posed: the solution exists and is unique; others, are not: there may be four, two, one or no solution. According to the location of the focus, a specific type of conic is obtained. The nature of the inscribed conic is related to a partition of the plane into invariant subsets w.r. to Cremona transform, to which we give a direct, elementary proof.

Related work. Geometric constructions of conics are nowadays a part of so called Descritive geometry; a good text, containing proofs of the constructions, is [5]; see problems 75-79. In [2], foci of circumscribed parabolas are studied; proprieties of conics that tangent triangle's sides at its vertices are in [3],[4]. In-ellipses of convex polygons are studied in [14]. Inscribed conics with prescribed foci are of course a constant presence whenever there's a Poncelet pair for triangles: see [8],[11],[12] for a small sample.

Main result. The first result ensures that a conic that tangents three given lines and having a prescribed focus exists, is unique, and shows how to draw it.

Theorem 1.1. *Let a, b, c three lines that are neither concurrent, nor parallel. Then for any point F in the plane, not on these lines, there exists a unique conic which have focus in F and tangents the lines a, b, c.*

This conic, its tangency points, as well as all its geometric elements (focus, vertices, directrix) admit a construction with line and compass only.

We also solve the dual problem.

Theorem 1.2. *Let A, B, C be three non collinear points. Then for any point F in the plane, that does not belong to the lines of the triangle, there exists four conics which have focus in F and that pass through A, B, C. This conics, as well as its main geometric elements (focus, vertices, directrix) admit a geometric construction. Among those four solutions, there is at most one ellipse.*

2. The proofs

Proof of Theorem 1.1. Refer to figure 1. Perform a dual transform, w.r. to an inversion circle centered in F; the lines a, b, c transforms into their poles A, B, C and the problem converts into: to draw all the circles that pass through three points.

This problem is well posed (always have a solution, and the solution is unique) whenever the three points are not collinear, i.e., whenever the three lines a, b, c are not parallel.

The polar dual of this (unique) circle is the conic we search.

Proof of theorem 1.2. Refer to figure 2. In order to obtain the conic that has one focus in F and pass through the vertices of the triangle, again perform a dual transform w.r. to an inversion circle centered in F. The points converts into their polars, and the problem became the following: to draw all the circles that tangent three distinct lines.

The solution are the i-circle and the three ex-inscribed circles. The polar dual of these circles, w.r. to the inversion circle are the conics that we searched.

All the geometric elements of these inscribed or circumscribed conic (vertices, the other focus, directrix) are obtainable via geometric construction. These details are in Appendix.

3. A partition of the plane into Cremona-invariant regions

As we already seen, it is always possible to draw the i-conic of a triangle, focused at a specific point F: this conic exists and it is unique. Now we show how we can predict its type, according to the location of its focus, F.

Let $\triangle ABC$ and let F be any point that does not belong to either AB, BC, CA; then the isogonal of half-line $[AF$, w.r. to $\angle BAC$ is the half-line $[AF_1$, where F_1 is symmetric of F, with respect to the internal bisector of $\angle BAC$. The isogonal of a line (or point), w.r. to an angle is an involution.

The Cremona transform of a point F w.r.to a triangle $\triangle ABC$ defines as the intersection of Cremona transforms of the half-lines $[AF, [BF, [CF$, w.r.to the angles of said triangle. The definition is good, since these three lines are concurrent.

Lemma 3.1. *Let the triangle $\triangle ABC$ and let $\mathbf{C}$, its circumcircle. Then:*

i) if a point F is on the circumcircle, then the isogonal of $[AF, [BF, [CF$, w.r. to the angles $\angle A$, $\angle B, \angle C$, consist in three parallel lines;

ii) Cremona transform of a point located inside the triangle, is (a point) located inside the triangle;

iii) if a point F is located outside the triangle, but inside its circumcircle, and inside of an angle say $\angle ACB$, then its Cremona transform exists and locates outside the triangle and into opposite by its vertex angle;

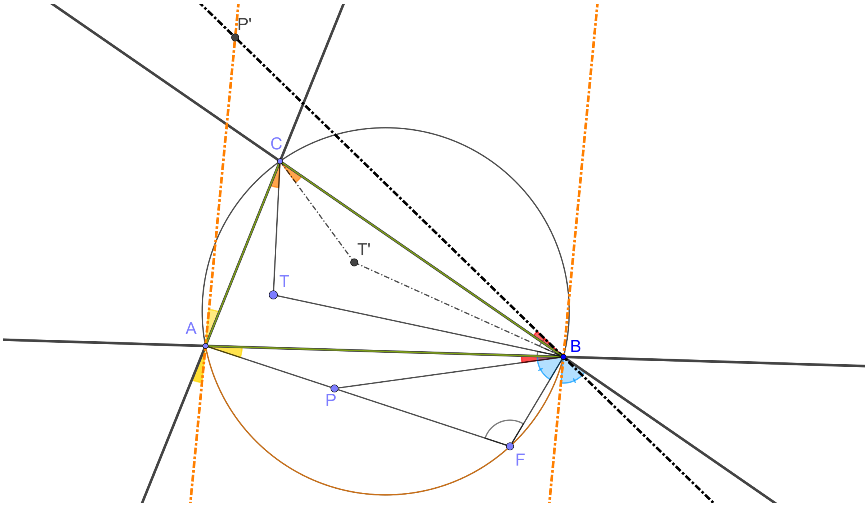

FIGURE 3. i) T', the isogonal conjugate of a point T located inside of a triangle, is also located inside the triangle; ii) the conjugate of point F on the circumcircle is a point at infinity; iii) point P situated on the circular sector delimited by an arc AB and by a triangle's side $[AB]$ are mapped into point P' located into the angle opposed by the vertex of $\angle ACB$.

iv) if F is outside the circumcircle, but inside of some angle of the triangle, then its Cremona transform stays into the same region.

Proof.

i) First assume that F is on the circumcircle; let $[AA_1$ and $[BB_1$ be respectively the isogonal conjugate of $[AF$ and $[BF$ w.r.to the angles $\angle BAC$ and $\angle ABC$ and let Q the interception between AA_1 and BC.

Since the quadrilateral $[AFBC]$ is inscriptible, $\angle FBA = \angle ACQ$; on the other hand, by construction, $\angle FAB = \angle CAQ$. This ensures that the triangles $\triangle ACQ$ and $\triangle AFB$ has a par of respectively equal angles, hence $\angle AQC = \angle ABF$, as well. By construction, $\angle ABF = CBB_1$, which proves that the lines AA_1 and BB_1 are parallel (forms congruent alternate internal angles, w.r. to the secant BC). The same argument proves the other parallelism.

ii) The isogonal of a point inside an angle, stays inside said angle; hence, when P is inside a triangle, it is inside of the three angles of the triangle, hence its Cremona transform stays there, as well.

iii) If P is inside the circle, inside the angle $\angle ACB$ but outside the triangle, then let AP intercept the arc AB in F. Let $\delta = \angle PAB$ and $\epsilon = \angle PBA$. The isogonal conjugate of AP and BP are not parallel (since the point P is not on the circumcircle) and let P' be their their intersection. We shall prove that

$$\angle BAC + \angle\delta + \angle ABC + \angle\epsilon < 180°.$$

In fact $\angle BAC + \angle ABC = 180 - \angle ACB = \angle AFB < \angle APB$, by hypothesis. But $\angle APB = 180 - \epsilon - \delta$, and this proves that the lines BP' and AQ must intercept; denote (also) let P' be their intersection point.

Then P' must be contained into the half-plane limited by BC that does not contain A, and on the half-plane limited by AC, that does not contain B, otherwise the line AQ should intercept the line AC, twice, which is impossible! Hence P' must be located into the opposite angle of $\angle ACB$.

iv) If P is a point inside the angle $\angle ACB$, but outside the circumcircle, then the isogonal conjugate of $[AF$ w.r.to $\angle A$ will necessarily be in the interior of this angle, too; on the other hand, its conjugate cannot be neither inside the circle, nor

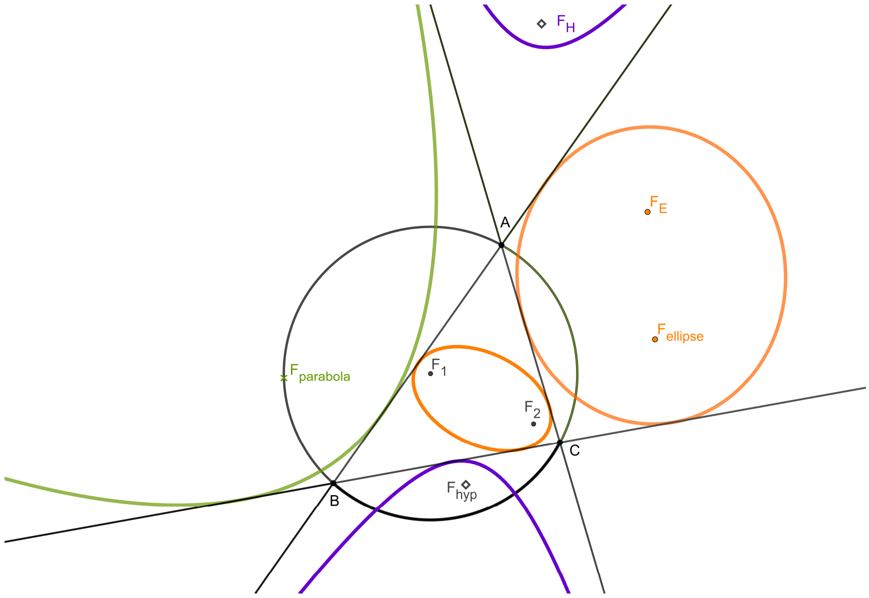

Figure 4. If F is on the circumcircle, the i-conic is a parabola (green); if F is either inside the triangle or inside of an angle but outside the circumcircle, the i-conic is an ellipse (orange); in the reminiscent cases, the i-conic is a hyperbola (violet)

insight the triangle, since otherwise, its conjugate should belong to the opposite angle of $\angle A$, since the isogonal conjugation is an involution. So Cremona transform invarriates this third region, too.

Theorem 3.1. *Let a, b, c and F as above and let $\mathbf{\Gamma}$ a conic that have a focus in F and tangents the lines a, b, c. Then:*

(i) *if the lines a, b, c determines a triangle, then these lines, together with the circumcircle of the triangle induce a partition of the plane such that:*
  (a) *$\mathbf{\Gamma}$ is a parabola if and only if the point F is on the circumcircle;*
  (b) *$\mathbf{\Gamma}$ is an ellipse if and only if F either belong to the interior of the triangle, or is a point that lie outside the circumcircle, and inside some of the angles of the triangle;*
  (c) *$\mathbf{\Gamma}$ is a hyperbola in the reminiscent cases.*

Proof. We first proof the result about the parabola (see e.g. [1] Theorem 1.10 and Lemma 1.3 for a more classic proof). For this, we show it cannot have another (finite) focus. In fact, the focuses of an inscribed conic are isogonal; and since F is on the circumcirlce, its isogonal conjugate (the second focus) is the point at infinity, which proves that the conic is indeed a parabola. Conversely, if certain parabola tangents the sides of a triangle and have a focus in F, then its second focus is the point at infinity, therefore F necessarily belong to the circumcircle. If F is inside the triangle, then the conic is an ellipse whose second focus is also inside the triangle. When one focus is inside of an angle and outside the circumcircle, its Cremona transform stays in that region; this means that the inscribed conic that have one focus in there, will have the other (proper) focus in that region, as well; hence the conic cannot be a parabola. On the other hand, it cannot be a hyperbola, either, since no hyperbola is (entirely) contained into a half-plane, while this region is. So, the conic can only be an ellipse. Finally, in the reminiscent cases, the i-conic is a hyperbola.

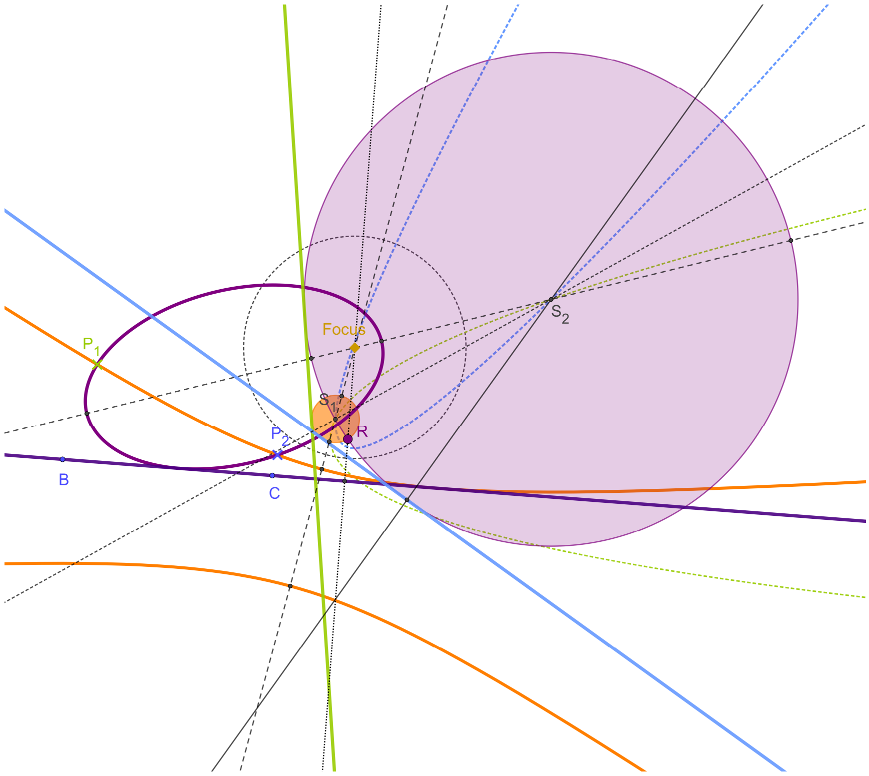

Figure 5. There are precisely two conics (orange hyperbola, purple ellipse) that share a Focus, pass trough P_1, P_2 and tangent one line (purple): these are the duals of the circles (orange and light purple) that tangents the polars of P_1 and P_2 (green and blue lines) and pass through R, the pole of the line. The later circles are the solution of an (LLP) Apollonius' problem.

4. Related problems

For the sake of completeness, we briefly indicate how to solve the other cases, as well. These will led to degenerate cases of Apollonius' problem, that can be tackled with polar duality, as we did in our earlier work [6].

4.1. Point-point-tangent case (PPT).

Proposition 4.1. *To draw all the conics that share a focus, F, two points P_1, P_2 and one tangent, t_3.*

Proof. Refer to figure 5. Perform a polar duality w.r. to a circle centered in F. The conic transforms into a circle; P_1, P_2 transform into their polars, which are two tangents at the dual circle; finally, the tangent t_3, transforms into its pole, P.

The problem converts into the following.

Proposition 4.2. *(line-line-point (LLP)) To find all the circles that pass through a point P and that are tangent to the lines p_1 and p_2.*

This is, of course, the (LLP) case of the Apollonius' problem.

Refer to figure 6. First assume that the lines p_1 and p_2 are concurrent; this happens if and only if the points P_1, P_2, F are not collinear.

- If the point P is neither on p_1 or p_2, and if p_1 and p_2 intercept in a point O, then the center of the tangent circles are the intersection between a parabola with a focus in P and directrix p_1, with the bisector of the

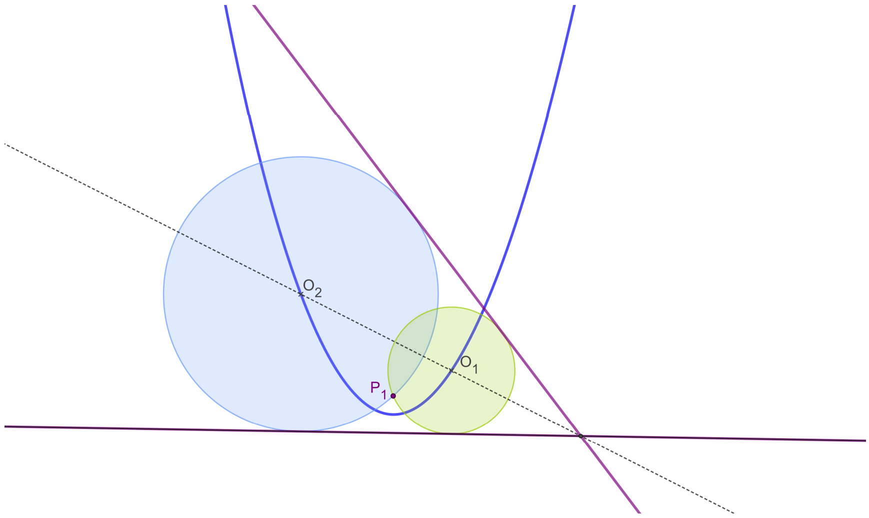

FIGURE 6. (LLP) Apollonius problem. When the two lines are concurrent and the point in not on the lines, there are two circles that tangent two lines and pass through a point, P_1 (not common with the two lines). Their centers are the intersection of a parabola focused in P_1 and directrix one of the lines, with the (internal) bisector of the angle where the point P_1 locates. There are two such parabolas, that produce the same pair of points.

$\angle p_1, p_2$. There are two such solutions. (we get the same solution, when we intercept the parabola with focus in T_3 and directrix p_2).

- if T_3 is on p_1, then the center is simply the intersection between the internal and external bisector of the angle formed by p_1 and p_2 and the perpendicular line in T. This happens when the point P_1 belong to the tangent t_3.
- if T_3 is the intersection point of p_1 and p_2, then there is no solution. This is the case when both P_1 and P_2 are on t_3.

If the lines p_1, p_2 are parallel, then

- if T lie on the strip limited by those two, there are two solutions. This happens when F, P_1, P_2 are collinear and t separates this two points.
- if T belongs to p_1, there is a unique solution.
- Otherwise, t_3 does not separate this points, and there will be no solution.

4.2. **Point-tangent-tangent (PTT).**

Proposition 4.3. *(PTT) To find all the conics that have a given focus, pass through a point P and are tangent to two lines, t_1 and t_2.*

The dual problem is.

Proposition 4.4. *(LPP) To find all the circles that are tangent to a line p and pass through two points T_1, T_2.*

This is the LPP problem of Apollonius.

First assume that the line p does not separate the points T_1 and T_2.

- If the line T_1T_2 is not parallel to p, then we have two distinct solutions, the intersection points of two parabolas: one having focus in T_1 and directrix

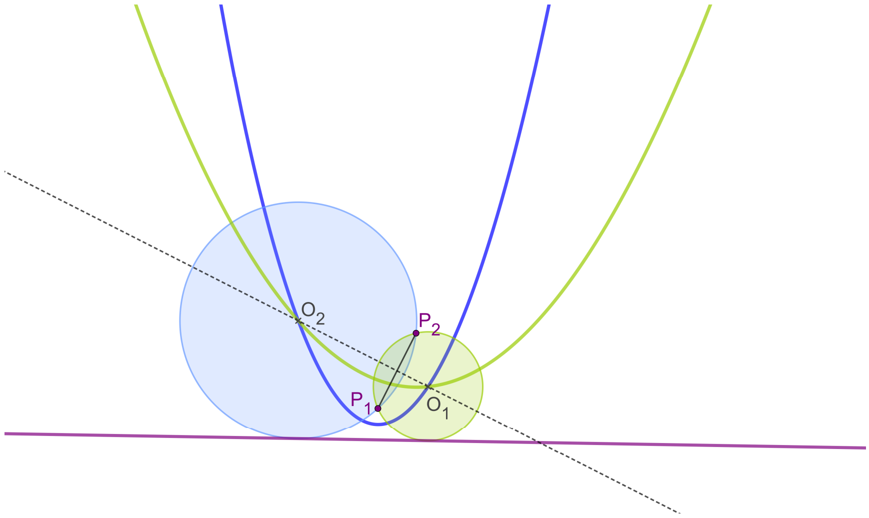

Figure 7. (LPP) Apollonius' problem. The centers of the (circles) solutions of a (LPP) Apollonius' problem (solid blue and green circles), may be obtained as an intersection a parabola and a line (the perpendicular bisector of $[P_1P_2]$. Note one may perform a geometric intersection for the two.

p, and the other with focus in T_2, and directrix p or the intersection of the parabola with focus in T_1 and directrix p, with the perpendicular bisector of T_1T_2.

- If T_1 belong to p, then there is a unique solution, which is the intersection between the normal in T_1 are p, and the mediatrix of T_1T_2.
- If the line T_1T_2 is parallel to p, we get a unique solution: the circumcircle of NT_1T_2, where N is the point where the perpendicular bisector of T_1T_2 intercepts p.

If the line p separates the points T_1 and T_2, then there will be no solution.

Note that all the intersection points between a parabola and a line can be drawn geometrically, again by polar duality. This is because the intersection points between a conic focused in P and a line are the poles of the common tangents from the polar of the line, to the dual circle of the conic (see Appendix).

This finishes the geometric construction of all these conics.

5. Appendix 1. Polar Duality

Let $C(\Omega, R)$ be a circle centered in Ω and of radius R, which we shall call inversion circle. From now on, when we say the inverse or the symmetrical of a point, we subtend symmetry w.r. to this inversion circle.

If p_0 is a line that does not pass through Ω, then its pole is the inverse of the projection of the centre Ω, on the line p_0.

If P_0 is a point ($P_0 \neq \Omega$), the polar of P_0 is the perpendicular line on ΩP_0, that pass through P_1, the inverse of P_0.

The dual of a circle (w.r. to an inversion circle) defines as the curve whose points are the poles of the tangents of the original circle. When we perform the dual of a circle, w.r. to an inversion circle, something quite nice happens.

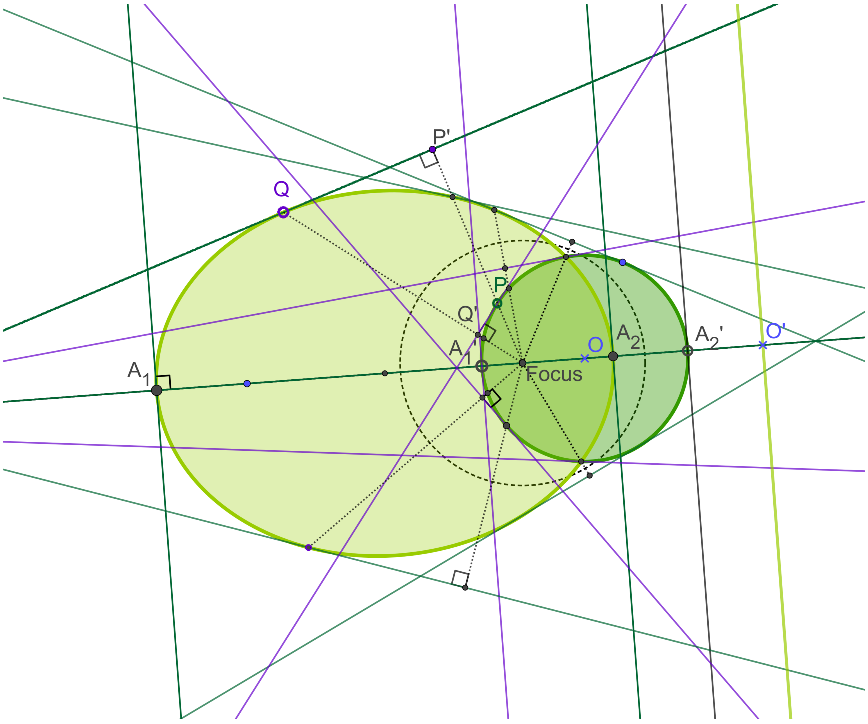

Figure 8. I) the polar dual of a circle (solid green) w.r. to an inversion circle centered in Focus (dotted circle) is an ellipse, if the inversion center, is inside the inverted circle. II) the vertices of the dual ellipse are the inverses of the diameter through O and $Focus$; III) tangents at the reciprocated circle are mapped into points of the dual curve; points on the reciprocated circle are sent into tangents at the dual curve.

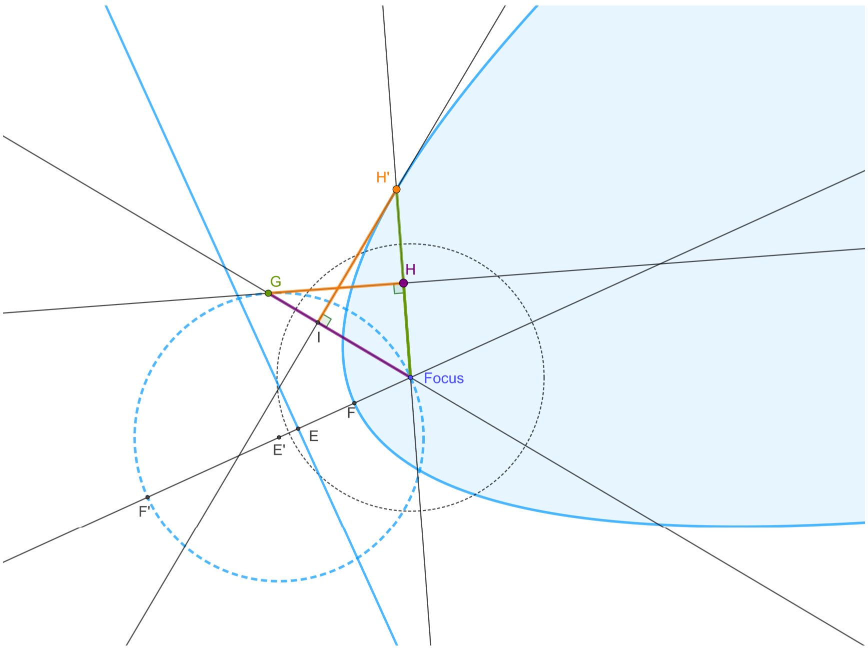

Figure 9. When the center of the inversion circle (dotted black) is on the reciprocated circle (dotted light blue), the polar dual is a parabola (light blue) which have focus in the center of inversion circle (point Focus) and whose directrix (blue) is the polar of E'. the center of the inverted circle.

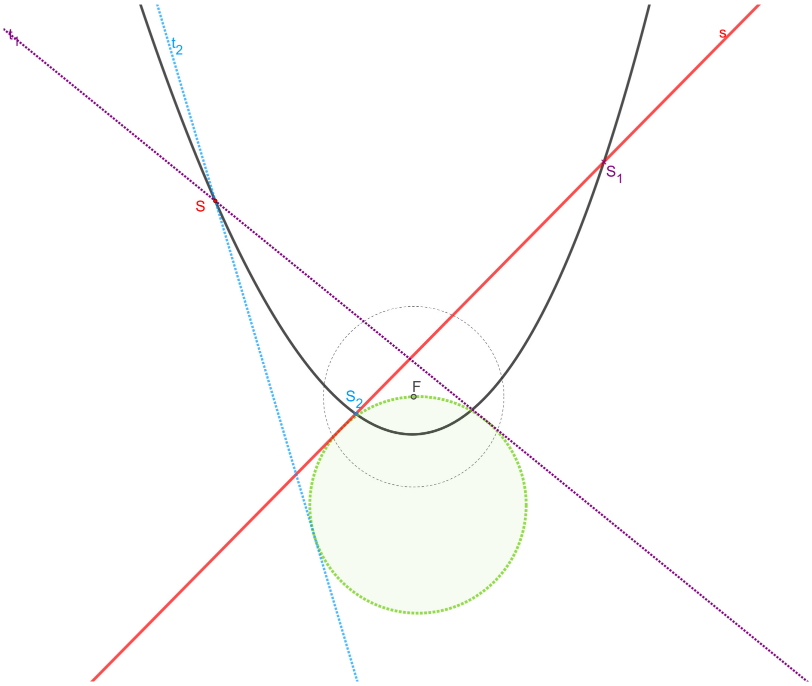

FIGURE 10. The intersection points between a parabola (black) focused in F and a line (red line), are the poles S_1, S_2 of the tangents (blue, purple dotted) from S, the pole of the line, to the dual circle (solid green circle).

Theorem 5.1. *(see e.g. [13], art. 306 and 309) The dual of a circle $\gamma = C(O, r)$, w.r. to an inversion circle $C(\Omega, R)$, is a conic, Γ; if d denotes the distance between the centres of the reciprocated and inversion circles, $d = \Omega O$, then:*

i) Γ is an ellipse, if $r < d$;

ii) Γ is a parabola, if $r = d$;

iii) Γ is a hyperbola, if $r > d$.

Moreover, (one of) the the focus of the dual conic Γ is precisely Ω, the centre of the inversion circle; its directrix is the polar of O, the centre of the reciprocated circle and the eccentricity is $e = \frac{r}{d}$.

This theorem has a very useful corollary.

Corollary 5.1. *The dual of a conic Γ, w.r. to an inversion circle centered into its focus, is a circle, γ.*

The symmetric of the vertices of the conic Γ, are a pair of diametrically opposite points of the dual circle, γ.

The pole of the directrix of Γ, is the center of the circle γ.

The reader may convince himself that all the elements of the dual conic, such as vertices, directrix or the other focus, can be drawn with straight-line and compass, since all the steps involves drawing the symmetric of a point and the pole of a line! For more details on poles, polars and polar reciprocity, see [1], [7], [10], and [13].

6. Appendix 4. A geometric construction of the intersection points between conics and lines

The intersection between a parabola and a line can be performed by polar duality. This is because, according to the fundamental theorem on poles and polars, the intersections of two curves, are (precisely) the poles of their common tangents to their dual curves.

Proposition 6.1. *The intersection of a conic and a line are the poles of the tangents from the pole of the line, to the dual circle of the conic w.r. to any inversion circle, centered into the focus of the conic.*

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