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A HOLISTIC APPROACH TO MORLEY'S GENERAL THEOREM

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Abstract. It is proved in elementary geometry that "the trisectors of a triangle, proximal to a side, intersect on 3 sets of 3 parallel lines forming equilateral triangles". Besides these equilaterals, 27 more are demonstrated with vertices intersections of trisectors revealing that the trisectors meet at the vertices of 54 equilaterals which are on 3 sets of 12 circles passing through two vertices of the triangle.

1. Introduction

The behavior of trisectors of a triangle was ignored across the ages in the shadow of the Ancient Greece legendary problem for trisecting an angle with compass and straightedge. Eventually, in 1837 this old geometrical ambition was confirmed impossible. However, around 1900 Frank Morley, while studying an unrelated problem, among intricate equations surprisingly observed that *in a triangle, the trisectors proximal to a side intersect on three sets of three parallel lines forming equilateral triangles* and then concluded *that the interior trisectors of its angles, proximal to sides respectively, meet at the vertices of an equilateral triangle*, known since then as *Morley's theorem.* Initially the theorem was spread around the world by the word of mouth. In the book *Inversive Geometry,* which was written with his son Frank Vigor Morley, many years later, the general statement was justified as corollary of complicated results on cardioids.[15]

Since Morley's discovery, illustrated in Fig.1, a remarkable number of publications focused on the interior trisectors and produced several simple proofs of the special case. However, the treatment of the original observation remains advanced and highly technical in contrast to its plain phrasing. [17],[13],[7],[9],[8] This note proves it in elementary geometry.

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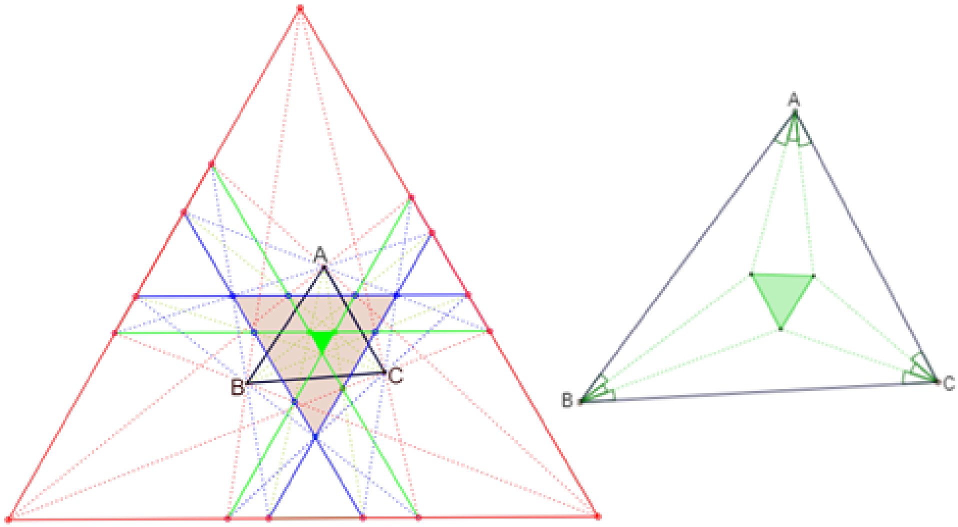

Fig.1 *Morley's observation and Morley's theorem.*

As indicated in Fig.1, there are 27 equilaterals with vertices intersections of trisectors that are either of three angles, proximal to sides respectively, or of two angles, proximal to angles common side. Furthermore, it is shown that besides these equilaterals 27 more have vertices intersections of trisectors of two angles with circumscribed circles through the vertices of angles common side. So, it is deduced that the trisectors meet on 3 sets of 12 circles passing through two vertices of the triangle.

2. SIX TRISECTORS OF AN ANGLE IN A TRIANGLE

Assume that $\angle CAB = 3\alpha$, $0 < \alpha < 60^0$. The interior, exterior and explementary trisectors divide the angle, its exterior angle and its explementary angle (not convex) into three angles of size α, $\frac{1}{3}(180^0 - 3\alpha) = 60^0 - \alpha = \theta$ and $\frac{1}{3}(360^0 - 3\alpha) = 120^0 - \alpha = \phi$, respectively.

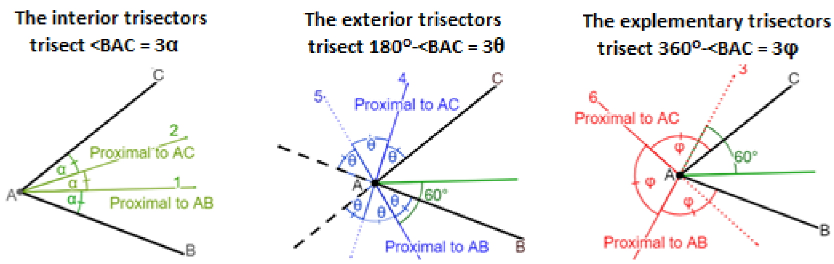

Fig.2 *Three kinds of trisectors of an angle and the corresponding proximal to a side.*

The same kind trisectors are distinguished as proximal and distal with respect to a side of the angle. The trisector **proximal** to a side bisects the angle between the side and the other trisector. The trisector **distal** to a side is proximal to the other side of the angle.

The trisectors proximal to a side make between them 60^0 angles. This is a crucial property for the formation of the considered equilaterals and is utilized extensively in the presented proofs.

For having a uniform notation for all intersections of the trisectors of a $\triangle$ABC suppose that they are indexed sequentially for each angle. If m and n are indices of two trisectors of angles X and Y then their intersection is denoted by Z_{mn} where Z $\in$ {A,B,C}-{X,Y}. The indexing for each angle is designed so that the triangle with the intersections of interior trisectors, proximal to sides respectively, is denoted by $\triangle \mathrm{A}_{12}\mathrm{B}_{12}\mathrm{C}_{12}$. Thus, the indices of trisectors for each angle increase counterclockwise with the interior trisectors having indices 1 and 2. Indices 4 and 5 are for the exterior trisectors while 3 and 6 for the explementary, as the former are always inside the later. Fig.2. The notation Z_{mn} allows identifying proximal trisectors without visual inspection, as trisectors with indices m and n are proximal to side XY precisely for odd m and even n. Also, the indices of trisectors proximal to the same side have the same parity.

In the sequel consider a $\triangle$ABC with trisected angles of sizes 3α, 3β and 3γ that none of them is multiple of 30^0 to avoid degenerated cases.

3. EQUILATERALS WITH VERTICES INTERSECTIONS OF PROXIMAL TRISECTORS

Equilaterals may be formed by intersections of three pairs of trisectors of either two angles, proximal to the same side, or of three angles, proximal to sides respectively. The next theorem identifies 9 equilaterals with vertices intersections of trisectors of two angles. They are called ***Guy Faux (GF) triangles***, a term suggested to Richard Guy by John Conway.[9]

Theorem 1. *In a triangle, trisectors of two angles of one kind with combinations of the remaining kinds, proximal to angles common side, meet at the vertices of an equilateral. The circumcircle of each of these equilaterals passes through the vertices of the angles common side.*

Proof. Consider the GF triangles of trisectors proximal to BC denoted by $\triangle \mathrm{A}_{12}\mathrm{A}_{34}\mathrm{A}_{56}$, $\triangle \mathrm{A}_{54}\mathrm{A}_{16}\mathrm{A}_{32}$ and $\triangle \mathrm{A}_{36}\mathrm{A}_{14}\mathrm{A}_{52}$ having a vertex intersection of interior, exterior or explementary trisectors, respectively.

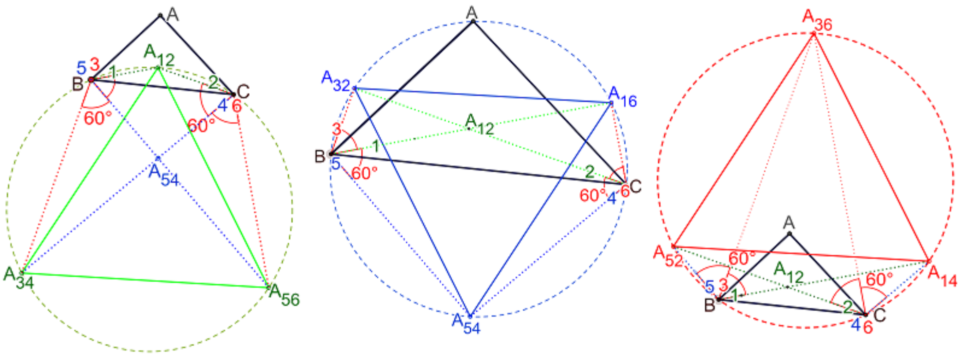

Fig.3 *GF triangles by intersections of trisectors proximal to* BC.

As the trisectors proximal to BC make 60^0 angles, the two quadrangles with vertices B, C, and two intersections of trisectors from which one is of the same kind, are cyclic since they have two opposite angles 120^0 and 60^0. Then the two quadrangles form a cyclic pentagon. Hence the angles of each triangle are 60^0. □

The 9 GF triangles have distinct vertices. There are 3 GF triangles with vertices intersections of trisectors proximal to a side. Notice that they have parallel sides as the angles between the same trisector and their respective sides are equal. Also, there are 3 GF triangles, each having a vertex intersection of same kind trisectors which identifies it. It can be shown as before that they have parallel sides. Thus the 27 intersections of trisectors, proximal to a side, lie on lines of 3 different directions. Morley's observation specifies that these lines are exactly 3 in each direction. Next it is shown that these 9 lines are determined by the parallel sides of three equilaterals with vertices intersections of same kind trisectors, proximal to sides respectively.

Theorem 2. ***(Morley's observation)*** *In a triangle, the trisectors proximal to a side intersect on 3 sets of 3 parallel lines forming equilateral triangles.*

Proof. There are six different intersections according to the kind of intersecting trisectors. The proof establishes gradually that the three intersections of mix kind trisectors are collinear with one side of one triangle having vertices intersections of same kind trisectors. In fact, it is shown that the three triangles are equilateral with sides parallel and the trisectors proximal to a side meet at the intersections of non-parallel sides of these equilaterals.

a. Intersections of interior with explementary trisectors. To establish that these intersections are on a line determined by a side of the triangle with vertices the intersections of interior trisectors, proximal to sides respectively, it is shown that interior trisectors, proximal to a side, meet on a side of a corresponding GF triangle with a vertex intersection of exterior trisectors. Fig.4.

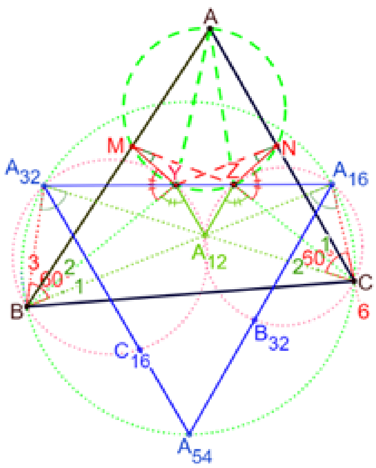

Fig.4 *Interior trisectors, proximal to a side, meet on a side of corresponding GF triangle.*

Let Y and Z be the intersections of interior trisectors proximal to AB and AC respectively, with side $A_{32}A_{16}$, of GF $\triangle A_{54}A_{16}A_{32}$, with vertices intersections of trisectors proximal to BC, where A_{54} is the meeting of exterior trisectors, while A_{16} and A_{32} are the meetings of interior with explementary trisectors. From its corresponding cyclic pentagon $BA_{32}A_{16}CA_{54}$ of Theorem 1,have $\angle A_{32}A_{16}B = \angle A_{32}CB = \gamma$. Then $\angle A_{16}A_{12} = \angle ZCA_{12}$. Hence, the quadrangle $CA_{16}ZA_{12}$ is cyclic. However, $\angle A_{12}CA_{16} = 60^0$, as angle between interior and explementary trisectors, proximal to BC. So $\angle A_{16}ZA_{12} = 120^0$. Similarly, the quadrangle $BA_{32}YA_{12}$ is cyclic and $\angle A_{32}YA_{12} = 120^0$. Thus, $\triangle YA_{12}Z$ is equilateral.

Let M and N be the reflections of A_{12} with respect to bisectors BY and CZ. Then M and N lie on AB and AC respectively, with $YA_{12} = YM$ and $ZA_{12} = ZN$, while $\angle BYM = \angle BYA_{12}$ and $\angle CZN = \angle CZA_{12}$.

Since $\triangle YA_{12}Z$ is equilateral, the first two equations imply that $\triangle MYZ$ and $\triangle NZY$ are isosceles triangles. From the previous cyclic quadrangle $CA_{16}ZA_{12}$, $\angle CZA_{12} = \angle CA_{16}B$. However, the two known angles of $\triangle CA_{16}B$ imply $\angle CA_{16}B = 60^0+\alpha$. So $\angle CZA_{12} = 60^0+\alpha$ and thus $\angle CZN = 60^0+\alpha$. Moreover, $\angle A_{16}ZC = \angle A_{16}A_{12}C = \beta+\gamma = 60^0\text{-}\alpha$. Hence $\angle NZA_{16} = 2\alpha$. Similarly, $\angle MYA_{32} = 2\alpha$. Consequently $\angle YMZ = \alpha$ and $\angle YNZ = \alpha$, as $\triangle MYZ$ and $\triangle NZY$ are isosceles triangles. Then, the quadrangle MYZN is cyclic. As $\angle MYN = (180^0\text{-}2\alpha)\text{-}\alpha = 180^0\text{-}3\alpha$ and $\angle MAN = 3\alpha$, the quadrangle YMAN is cyclic. In turn pentagon YZNAM is cyclic.

But so $\angle MAY = \angle YAZ = \angle ZAN = \alpha$, implying that Y and Z are intersections of the interior trisectors proximal to AB and AC respectively, denoted in the used notation by C_{12} and B_{12} respectively. Therefore, the interior trisectors, proximal to sides, meet at the vertices of an equilateral, as Morley's theorem claims, referred as *inner*. Additionally, interior with explementary trisectors, proximal to a side, intersect on a line determined by a side of inner equilateral.

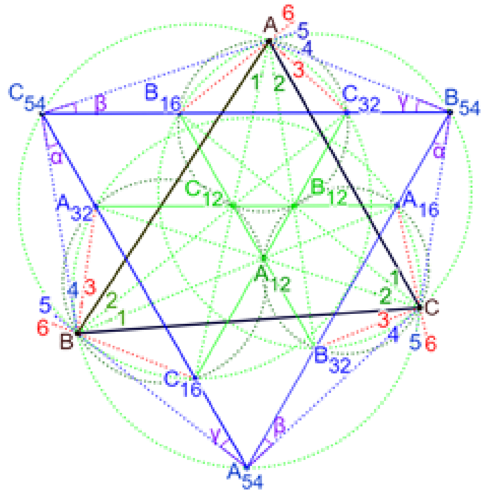

Fig.5 *The interior and exterior trisectors, proximal to sides, meet at the vertices of inner and central equilateral respectively. Interior with explementary, proximal to a side, meet at the intersections of sides or inner and central equilaterals.*

The proved cyclic quadrangle $CA_{12}B_{12}A_{16}$ is determined by vertex C, its corresponding side $A_{12}B_{12}$ of the inner equilateral and the intersection of interior with explementary trisectors proximal to CB. So, the similarly determined quadrangle $CA_{12}B_{12}B_{32}$ by the intersection of the interior with explementary trisectors proximal to CA is also cyclic.Fig.5. Hence, the pentagon $CB_{32}A_{12}B_{12}A_{16}$ is cyclic Thus $\angle B_{12}A_{16}B_{32} = \angle A_{12}B_{32}A_{16} = 60^0$. Then $A_{16}B_{32}$ is collinear with a side of both A_{54} and B_{54} GF triangles and in turn they are collinear with side $A_{54}B_{54}$ of $\triangle A_{54}B_{54}C_{54}$. Similarly, the GF triangles with vertices B_{54} and C_{54}, as well with C_{54} and A_{54}, have a side collinear with a corresponding side of $\triangle A_{54}B_{54}C_{54}$. Therefore, the exterior trisectors, proximal to sides respectively, meet at the vertices of an equilateral, analogously to the interior trisectors in Morley's theorem, referred as *central*. Also, the inner and central equilaterals have corresponding sides parallel, and their non-parallel sides meet at intersections of interior with explementary trisectors, proximal to a side.

b. Intersections of interior with exterior trisectors. Let P and Q be the intersections of sides $C_{54}A_{54}$ and $B_{54}A_{54}$ of central equilateral $\triangle A_{54}B_{54}C_{54}$ with the exterior trisectors, proximal to AC and AB respectively. Note that from the circumscribed circle of each GF triangle having a common vertex with the central equilateral, the size of angles between sides of central equilateral and the exterior trisectors are determined, as indicated in Fig.5. In particular, $\angle QB_{54}P = \angle A_{54}B_{54}C = \alpha$ and $\angle PC_{54}Q = \angle A_{54}C_{54}B = \alpha$. So, PQ is seen from B_{54} and C_{54} with the same angle and so quadrangle $B_{54}PQC_{54}$ is cyclic. Since $\angle AC_{54}B_{54} = \beta$, $\angle A_{54}C_{54}B = \alpha$, and $\angle AB_{54}C_{54} = \gamma$, $\angle AC_{54}Q = \beta + 60^0 + \alpha$ and $\angle AB_{54}Q = \gamma + 60^0$. In turn quadrangle $AC_{54}QB_{54}$ is cyclic. Hence pentagon $AB_{54}PQC_{54}$ is cyclic. Then $\angle C_{54}PQ = \angle C_{54}B_{54}Q = 60^0$ and $\angle B_{54}QP = \angle B_{54}C_{54}P = 60^0$ implying that $\triangle A_{54}PQ$ is equilateral. As $\angle PAB_{54} = \angle PC_{54}B_{54} = 60^0$, AP is the interior trisector proximal to AC, as AB_{54} is its corresponding exterior trisector. Similarly, AQ is the interior trisector proximal to AC. In turn P and Q are denoted in the used notation with B_{52} and C_{14}, respectively. Fig.6.

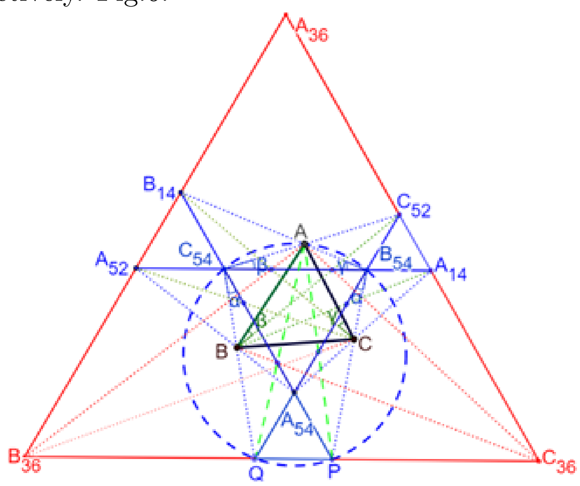

Fig.6 *The explementary trisectors, proximal to sides, meet at the vertices of peripheral equilateral. Interior with exterior trisectors, proximal to a side, meet at the intersections of sides of inner and peripheral equilaterals.*

Consequently, intersections of interior with exterior trisectors, proximal to a side, are collinear with a side of central equilateral. Additionally, the interior trisectors of an angle and the exterior trisectors of the other two intersect at the vertices of an equilateral. But so, each pair of the GF triangles of vertices A_{36}, B_{36}, and C_{36}, intersections of explementary trisectors, has two collinear sides, which are in turn collinear with a side of $\triangle A_{36}B_{36}C_{36}$, implying that it is equilateral. Hence the explementary trisectors, proximal to sides respectively, meet at the vertices of an equilateral, analogously to the interior trisectors in Morley's theorem, referred as peripheral equilateral. Also, the peripheral and central equilaterals have corresponding sides parallel, and their non-parallel sides meet at intersections of interior with exterior trisectors proximal to a side.

c. *Intersections of exterior with explementary trisectors.* Consider A_{56} which is a such intersection. Fig.7. Then $\angle CBA_{56} = 60^0$-β and $\angle BCA_{56} = 120^0$-γ, as angles between the exterior and explementary trisectors proximal to BC respectively, as indicated in Fig.2. Hence from $\triangle BA_{56}C$, $\angle BA_{56}C = \beta+\gamma$. But from the previous cyclic pentagon $AC_{54}C_{14}B_{52}B_{54}$, $\angle A_{54}B_{52}B = \angle A_{54}B_{52}B_{32} = \gamma$ and $\angle B_{32}B_{52}C = \angle AB_{52}B_{54} = \beta$. So $\angle A_{54}B_{52}C = \beta+\gamma$. Thus, quadrangle $A_{54}B_{52}A_{56}C$ is cyclic. Moreover $\angle B_{32}CB_{12} = 60^0$, and $\angle A_{54}CA_{12} = 60^0$, as angles between explementary and interior trisectors and between exterior and interior trisectors, proximal to BC, respectively. These two 60^0 angles have common part $\angle B_{32}CA_{12}$ and so the remaining parts are equal. Then $\angle A_{54}CB_{32} = \angle A_{12}CB_{12} = \gamma$. But from cyclic pentagon $AC_{54}C_{14}B_{52}B_{54}$, $\angle A_{54}B_{52}B_{32} = \gamma$. So, quadrangle $A_{54}B_{52}CB_{32}$ is cyclic. Consequently, pentagon $A_{54}B_{52}A_{56}CB_{32}$ is cyclic. Then $\angle A_{54}B_{32}A_{56} = \angle A_{54}CA_{56} = 60^0$, as angle between the exterior and explementary trisectors proximal to BC, and so $\angle A_{54}B_{52}A_{56} = 120^0$. In turn, $\angle A_{54}B_{32}A_{56}$ is opposite angle of $\angle B_{16}B_{32}B_{54}$ in GF $\triangle B_{16}B_{32}B_{54}$ and so $B_{32}A_{56}$ is collinear with $B_{16}B_{32}$ which has been shown collinear with a side of inner equilateral.

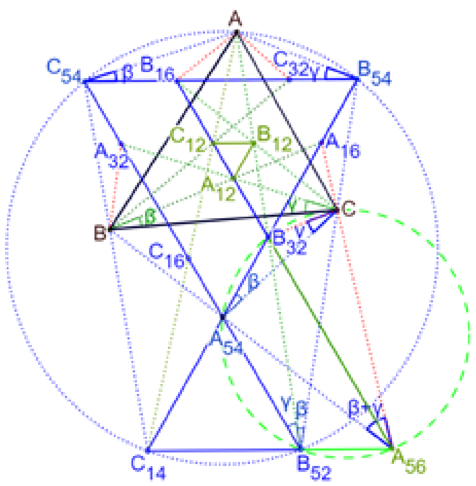

Fig.7 *Exterior with explementary trisectors meet at the intersections of sides of inner and peripheral equilaterals.*

As $\angle A_{54}B_{52}A_{56}= 120^0$ and $\triangle A_{54}B_{52}C_{14}$ is equilateral, $B_{52}A_{56}$ is collinear with $B_{52}C_{14}$ which is collinear to a side of peripheral equilateral. Therefore, A_{52} is intersection of sides of inner and peripheral equilaterals.

Conclude that the trisectors proximal to a side meet on the parallel sides of inner, central, and peripheral equilaterals implying that they are on 3 sets of 3 parallel lines. □

Morley's observation suggests an analog of angle bisectors theorem in a triangle for angle trisectors proved previously. The following statement generalizes Morley's theorem to the full observation.

Theorem 3. *In a triangle, the same kind trisectors, proximal to sides respectively, meet at the vertices of a corresponding equilateral.*

Furthermore, the three equilaterals have sides parallel and the trisectors, proximal to a side, meet at the intersections of their non-parallel sides.

The non-parallel sides of the above three equilaterals, with vertices intersections exclusively of one kind trisectors for all angles, meet at 27 points. They are vertices of 18 equilaterals which are intersections of trisectors, proximal to sides respectively, proper combinations of kinds. Specifically, besides the initial three, 9 more equilaterals involve one kind trisectors for an angle and another for the other two, while 6 extra equilaterals are formed by intersections of a different kind trisectors for each angle.[12]

4. EQUILATERALS WITH VERTICES INTERSECTIONS OF TRISECTORS OF TWO ANGLES

Are there more equilaterals with vertices intersections of angle trisectors?

Among triangles with vertices intersections of trisectors of two angles, proximal to a side, Theorem 1 identifies 9 equilaterals which have a vertex intersection of same kind trisectors and two vertices that are intersections of trisectors combination of the remaining kinds. Fig.1 and Fig.3.

The next theorem reveals more equilaterals among triangles with vertices intersections of trisectors of two angles.

Theorem 4. *In a triangle, the trisectors of two angles meet at the vertices of an equilateral provided*:

(a) Two are of same kind and the other combinations of the remaining kinds, distal to the angles common side.

(b) All of them are of same kind, one proximal and the other distal to the angles common side.

(c) All of them are combinations of different kinds, one proximal and the other distal to the angles common side.

The circumcircle of each of the above equilaterals passes through the vertices of the angles common side.

Proof. It is similar to the proof of Theorem 1 using the fact that the angle between interior and exterior, or between exterior and explementary trisectors, proximal to a side, is 60^0. Fig.8.

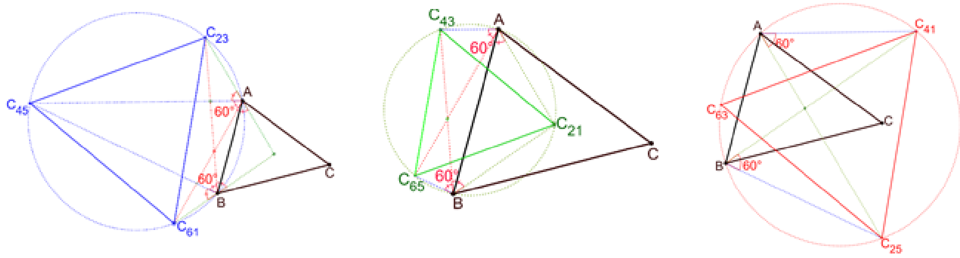

Fig.8a *Equilaterals with vertices intersections of trisectors, one of same kind and two combinations of remaining kinds, distal to* AB.

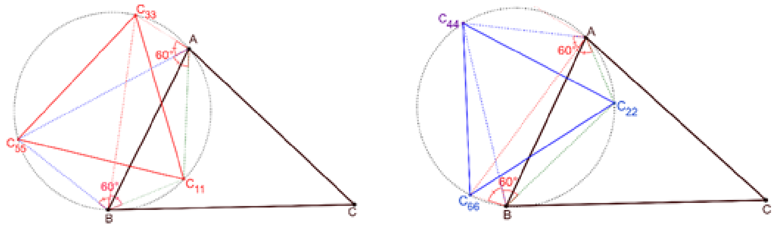

Fig.8b *Equilaterals with vertices intersections of same kind trisectors, one proximal and one distal to* AB.

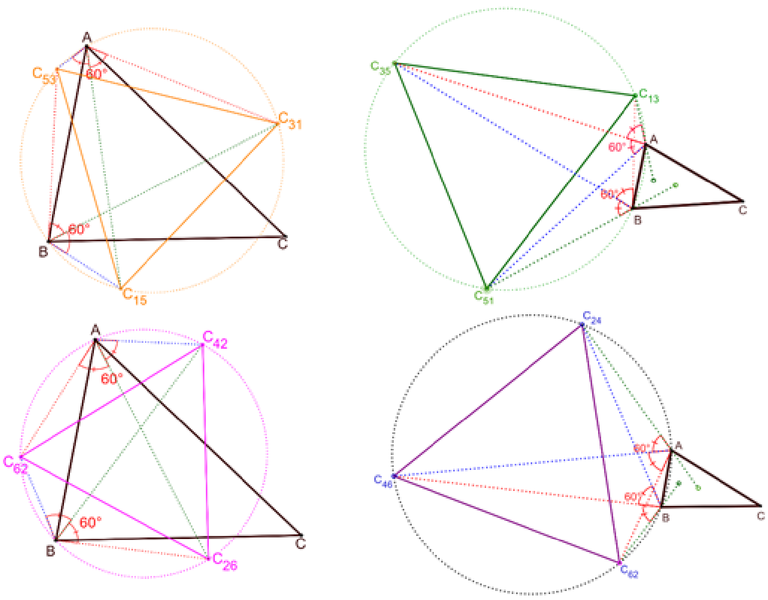

Fig.8c *Equilaterals with vertices intersections of trisectors of different combinations of kinds, one proximal and one distal to* AB.

□

The 9 equilaterals of Theorem 3 and the corresponding 3 GF triangles of Theorem 1 have 36 distinct vertices, covering all 36 intersections of trisectors of two angles. Therefore:

Theorem 5. *In a triangle the intersections of trisectors of just two angles lie in triples on twelve circles, each passing through two vertices of the triangle.*

5. Conclusion

The trisectors of a triangle meet at 108 points that are on 3 sets of 12 circles passing through two vertices of the triangle, forming 54 equilaterals - 36 have vertices intersections of trisectors proximal to a triangle side, and 18 have intersections of trisectors proximal to sides respectively. So, the trisectors proximal to a side meet on 3 sets of 3 parallel lines, as Morley observed.

So far Morley's theorem has absorbed almost all attention in investigating the behavior of angle trisectors in a triangle and thus results have been discovered regarding mainly interior trisectors. However, most of them must be true for exterior and explementary trisectors too, begging for validation.

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