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CIRCLES IN THE COMPLETE QUADRILATERAL[1]

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Abstract. The book of nature is written in the language of mathematics, and the symbols are triangles, circles and other geometrical figures: That is how Galileo Galilei said it in his important book Il Saggiatore 1623. The present text adds a few new and interesting characters to this alphabet. Cascades of circles are generated, starting from the incircle and the excircles of the four triangles contained in a complete quadrilateral. Their centres lie on certain circles whose centers lie on certain circles whose centers......finally lie on a last and very special circle.

1 Introduction

In [4] we were concerned with statements about circles containing the in- and excentres of a triangle and two of its sub-triangles, which arise when one connects a vertex with a point on the opposite side. We call such a configuration a triangle situation. Starting from the idea that such a situation can be conceived as a special case or limit situation of a complete quadrilateral, we study in the following the same questions for such a quadrilateral. Since the quadrilateral is more complex, the relations are even more numerous and varied.
We denote the elements of the complete quadrilateral by lowercase letters. If we imagine the triangle situation created by a boundary transition from the quadrilateral, they either merge into the elements labelled accordingly in capital letters in [4], coincide with other elements or reduce to a point. The statements a) to e) in Theorem 1 are completely analogous to those found for the triangle situation in [4]. They can also be proved with the same arguments. We therefore do not give the proofs of Theorem 1b),d) here but show only the existence of the most important circles - Theorem 1a),c),e) - with which we deal further. On the other hand, in Theorem 1i) to iv) we prove the statements for the quadrilateral, which are analogous to those only mentioned in [4] without proof.

The starting point of our considerations are results of Auguste Miquel [2] and Jakob Steiner [3], who have studied the quadrilateral very thoroughly. We summarise only those of their numerous interesting results (see, for example, [1], [5]) which we will need later (Figure 1).

Proposition 1. *a) A complete quadrilateral contains four triangles. Their circumcentres lie on a circle, the Steiner circle of the quadrilateral, and the circumcircles intersect at a point - the Miquel or Clifford point C of the quadrilateral - on the Steiner circle.*

b) Each of the four triangles in a quadrilateral has an incircle and three excircles. There exist eight circles - we call them the e-circles $e_j, j = 1, 2, ..., 8$ - which pass each through four of the centres of these sixteen circles - through one from each of the four triangles. There exist two lines d_1, d_2 intersecting vertically at the Clifford point C and carrying each four e-circle centers and four e-circles intersect at two points a_1, a_2 on one of these two lines.

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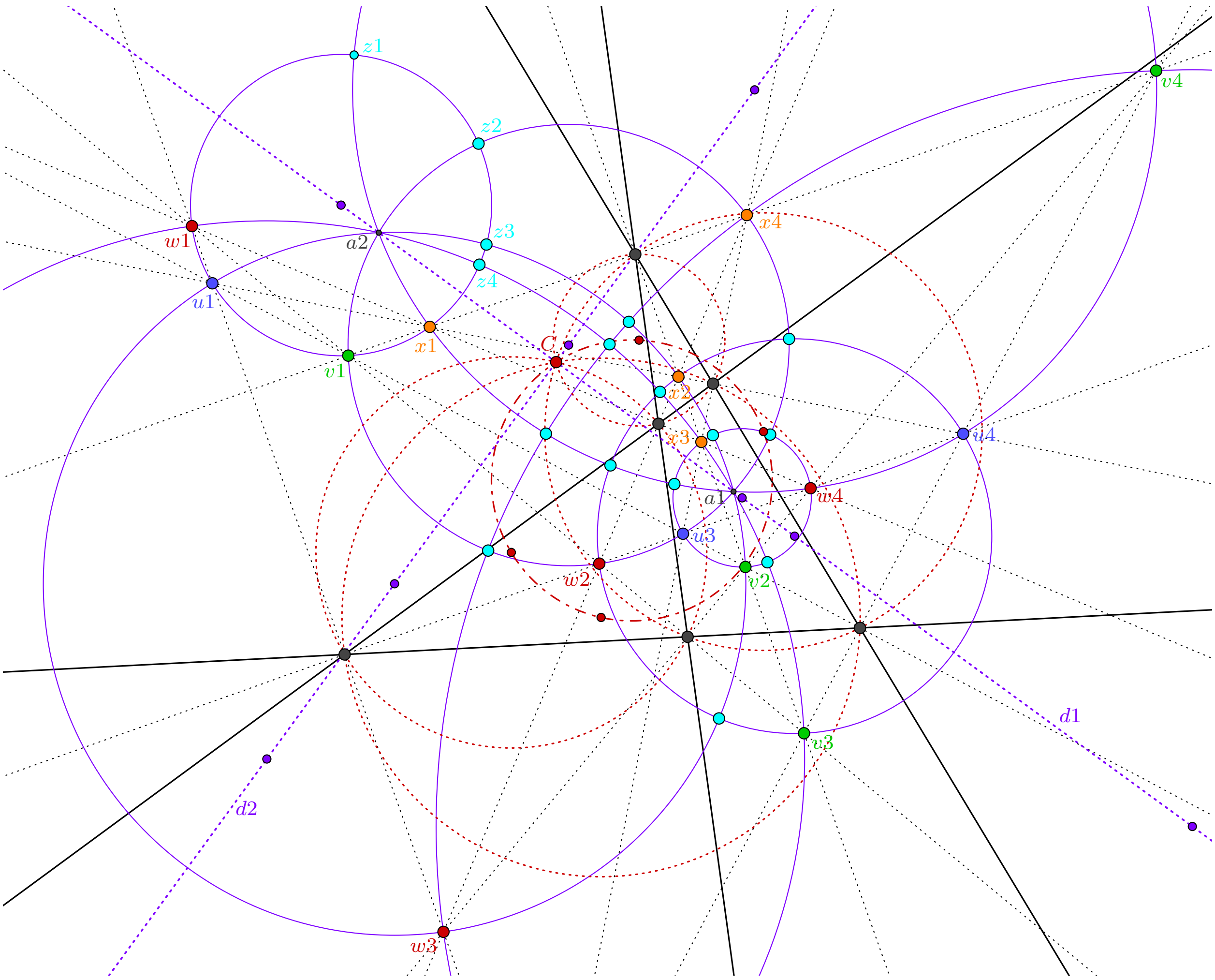

Figure 1

The eight e-circles e_j correspond to the eight E-circles $E_j, j = 1, 2, ..., 8$ in [4], Theorem 1.

In the following we name circles by their centre. The circle with centre a is the a-circle or the circle a, the centre of the circle a is the point a.
By a vertex we mean hereafter always one of the six points of intersection of the four lines in the quadrilateral. We call the incenter and the excentres of the four triangles in a quadrilateral the u-,v-,w-,x-points of the triangles and denote them by $u_j, v_j, w_j, x_j, j = 1, 2, 3, 4$. We call these sixteen points the y-points if we do not want to distinguish. They are intersections of three angle bisectors of a triangle contained in the quadrilateral. The set of all these angel bisectors consists of six pairs of lines, intersecting orthogonally in one of the six vertices. Each of the eight e-circles introduced in proposition 1b) contains eight points through which *one* other e-circle passes (Figure 1). Four of these points are u-,v-,w-,x-points, the other four are not. We call the latter the z-points $z_j, j = 1, 2, ..., 16$. Each of the overall thirty-two y- and z-points lies on two e-circles.

2 Circles in the complete quadrilateral

We illustrate the statements of the first part of Theorem 1 in Figure 2.

The z-points are particularly interesting. All of them are equal in our considerations. We formulate the statements for one of them and give an example of each in Figure 2. In brackets we indicate - also in the following arguments - which elements in the Figure we are talking about.

With suitable numbering, the following applies to the z-point z_1: The z-point z_1 is defined by two e-circles (e_1, e_2), each of which passes through three further z-points $(z_2, z_3, ..., z_7)$ - we call them the z-points belonging to z_1 - and one u-,v-,w- and x-point. Since each pair of intersection points of two e-circles - different from a_1, a_2 - consists of a y-point and a z-point, one of these last points (x_1) lies on both e-circles - we call it the counterpoint of z_1. The set of the remaining six y-points consists of three pairs of u-,v-,w-, or x-points $(u_1, u_4/v_1, v_4/w_1, w_4)$. We call them the y-points belonging to z_1.

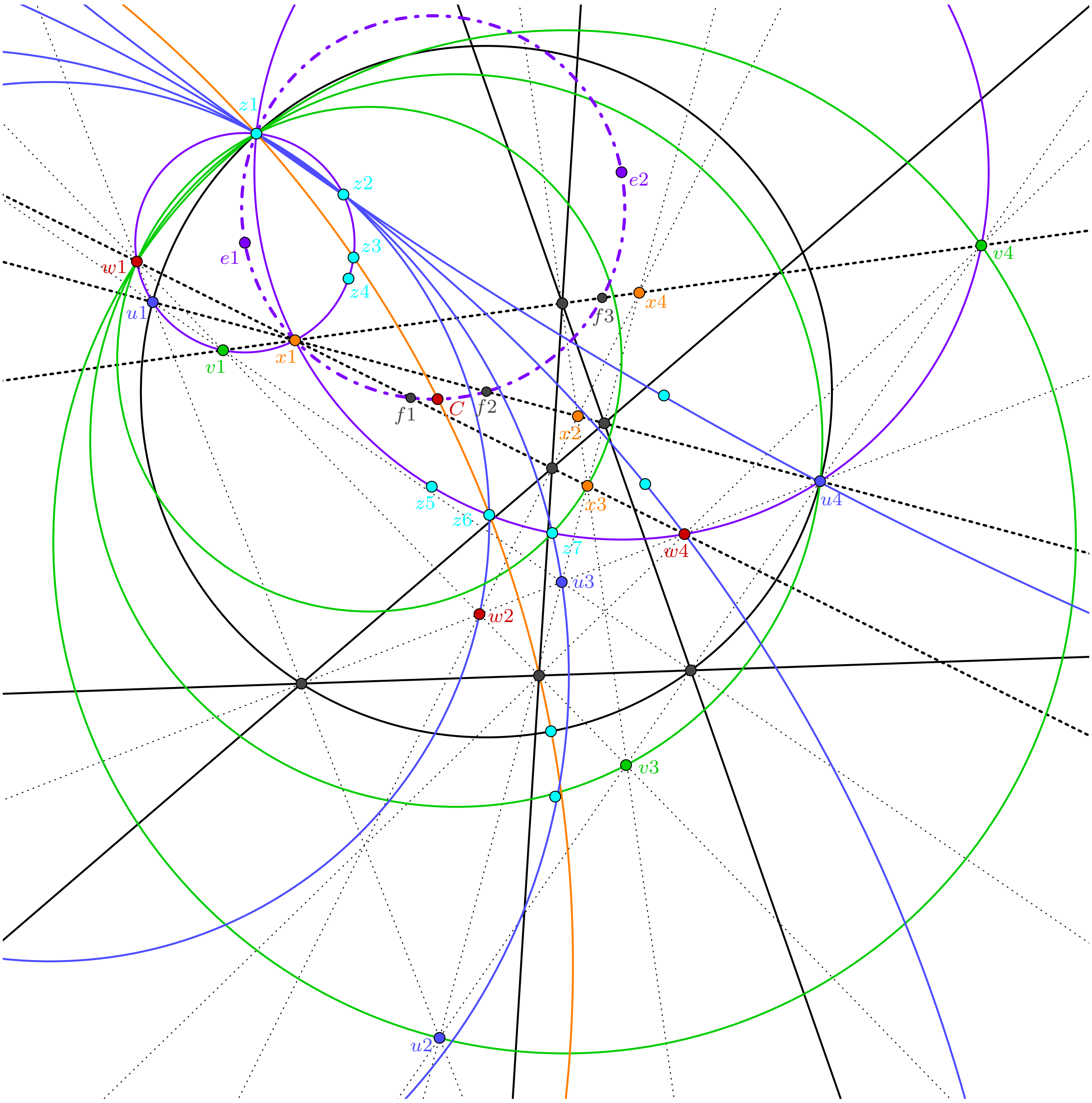

Figure 2

Theorem 1 (b),d) without proof, Figure 2)). *The following thirty-seven circles pass through the z-point z_1 besides the two e-circles (e_1,e_2) defining it:*

a) Three f-circles, each containing in addition another z-point, two vertices and a pair of u-,v-,w-, or x-points. (One through each y-point (u_1/u_4) belonging to z_1).

b) Twelve g-circles each containing three additional points, one out of three different sets of u-,v-,w-,x-,z-points. (Through each y-point belonging to z_1 (w_1) two passing through another such point (u_4,v_4) and one passing through a z-point belonging to z_1 (z_7)).

c) An m-circle containing in addition the two centres of the e-circles defining the point z_1 (e_1/e_2), the Clifford point C, the counterpoint (x_1) of z_1 and the centres of the three f-circles passing through z_1 ($f_1/f_2/f_3$).

d) Eighteen i-circles each containing in addition two more z-points and one y-point. (Through each z-point belonging to z_1 (z_2) two, passing through another such point (z_6,z_7) and two, passing through another y-point belonging to z_1 (u_4,w_4)).

e) Three h-circles each containing in addition three more z-points and one vertex. (Through each z-point belonging to z_1 (z_3) one passing through another such point(z_6)).

For the centers of these circles holds:

i) (Figure 3) Through the centres e_1, e_2 of the e-circles defining the z-point z_1 pass three lines each carrying the centres of three g-circles and of one f-circle passing through z_1. These lines intersect perpendicularly at the f-circle centres and there are three circles each containing two of these g-circle centres and two of these f-circle centers.

ii) Overall there are twenty-four f-circles. Their centres split into four groups of six points which lie on the circumcircle of one of the four triangles contained in the quadrilateral.

iii) There are eight lines, four each parallel to the perpendicularly intersecting lines d_1, d_2 (Proposition 1b)), the sixteen intersections of these eight lines being the centres of the overall sixteen m-circles.

iv) There are twelve circles - the p-circles - each containing in addition four m-circle centres. The centres $p_j, j = 1, 2, ..., 12$ of these twelve p-circles lie on a circle q.

v) There are four points t_1, t_2, t_3, t_4 in which six p-circles meet. These four points also lie on the circle q. Four groups of six p-circles each are thus enclosed in a cardioid with nodes at the points t_1, t_2, t_3, t_4.

3 Proofs first part

In the following we repeatedly use the inscribed angle theorems Eucl. III. 20.

Proof of Theorem 1a). (Figure 3) We consider the y-points u_1, u_4 belonging to the z-point z_1. They lie on the e-circles e_1, e_2 and e_4, e_5 respectively and are the intersection points of two pairs of angel bisectors intersecting orthogonally at the vertices b_1, b_3. We show that the intersection point z_{10} of the two e-circles e_4, e_5 and the vertices b_1, b_3 lie on the circle k_0 passing through the points z_1, u_1, u_4. We have

$$\sphericalangle u_1 z_1 x_1 = 180° - \sphericalangle u_1 v_1 x_1 = \sphericalangle x_1 v_1 b_3$$

$$\sphericalangle x_1 z_1 u_4 = \sphericalangle x_1 v_4 u_4 = \sphericalangle x_1 v_4 b_3 \qquad \sphericalangle u_1 z_1 u_4 = \sphericalangle x_1 v_1 b_3 + \sphericalangle x_1 v_4 b_3$$

The points x_1, v_1, v_4 lie on an angle bisector which is common to the triangles of the v- and x-points and the triangle $v_1 b_3 v_4$ is right-angled. It follows that $\sphericalangle u_1 z_1 u_4 = 90°$. In the same way we get

$$\sphericalangle u_1 z_1 u_4 = \sphericalangle v_1 z_1 v_4 = \sphericalangle w_1 z_1 w_4 = 90° \tag{1}$$

Thus the z-point z_1 lies on the Thales circle k_0 over the points u_1, u_4 and because of $\sphericalangle u_1 b_1 u_4 = \sphericalangle u_1 b_3 u_4 = 90°$ the vertices b_1, b_3 lie also on k_0. But the points u_1, u_4 also belong to the intersection point z_{10} of the other e-circles e_4, e_5 passing through u_1 and u_4 respectively. Thus we have again $\sphericalangle u_1 z_1 u_4 = 90°$ and the point z_{10} also lies on the circle k_0. This is the f-circle f_2, one of the three f-circles passing through z_1. The other two can be obtained by replacing the points u_1, u_4 by v_1, v_4 and w_1, w_4 respectively in the argumentation. □

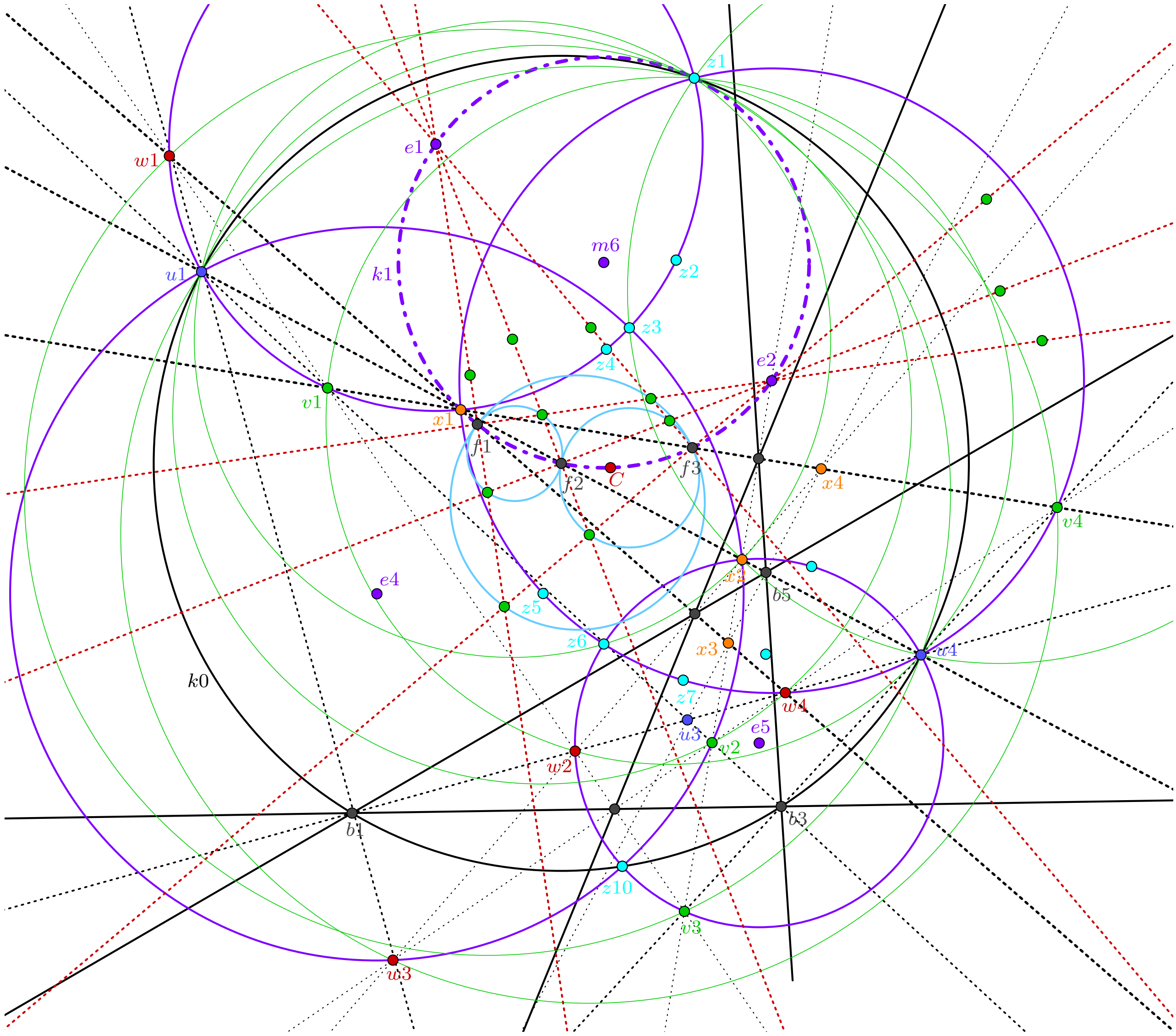

Figure 3

Proof of Theorem 1c). (Figure 3) We have

$$\sphericalangle e_1 z_1 x_1 = 90° - \sphericalangle x_1 u_1 z_1 \qquad \sphericalangle x_1 z_1 e_2 = 90° - \sphericalangle x_1 u_4 z_1$$

So $\sphericalangle e_1 z_1 e_2 = 180° - \{\sphericalangle x_1 u_1 z_1 + \sphericalangle x_1 u_4 z_1\}$. The points u_1, x_1, u_4 also lie on an angle bisector and since, as seen, the triangle $u_1 z_1 u_4$ is right-angled, we get again $\sphericalangle e_1 z_1 e_2 = 90°$. Thus the point z_1 - and with it its counterpoint x_1 - lies on the Thales circle k_1 over the points e_1, e_2. Furthermore the line segments $u_1 z_1$ and $u_4 z_1$ are the common chords of the circles e_1, f_2 and e_2, f_2. According to (1) they are perpendicular to each other and hence so are the corresponding centre lines: $\sphericalangle e_1 f_2 e_2 = 90°$. Using (1) again we find in the same way

$$\sphericalangle e_1 f_j e_2 = 90°, j = 1, 2, 3 \tag{2}$$

Thus the centers f_1, f_2, f_3 of the f-circles belonging to z_1 also lie on the circle k_1. Finally so does the Clifford point C since according to Proposition 1b) we have again $\sphericalangle e_1 C e_2 = 90°$. The circle k_1 is the m-circle m_6 belonging to z_1, respectively to its counterpoint x_1. □

Proof of Theorem 1e). In Figure 4 we consider the two f-circles f_7 and f_8 passing both through the two vertices b_2, b_6 and carrying each a v-point lying on an angular bisector through the vertex b_6 as well as the z-points z_3 and z_6 respectively belonging to z_1. Involving the e-circles e_1, e_2 defining the z-point z_1 we show that the vertex b_2 lies on the circle defined by the points z_1, z_3, z_6. We have

$$\sphericalangle z_6 z_1 z_3 = \sphericalangle z_3 z_1 x_1 - \sphericalangle z_6 z_1 x_1 = \sphericalangle z_3 v_1 x_1 - \sphericalangle z_6 v_4 x_1$$

$$= \sphericalangle z_3 v_1 b_6 - \sphericalangle z_6 v_4 b_6 = \sphericalangle z_3 b_2 b_6 - \sphericalangle z_6 b_2 b_6 = \sphericalangle z_3 b_2 z_6 = \sphericalangle z_6 b_2 z_3$$

which proofs the assertion. In the same way it can be shown that a further z-point z_{10} lies on the circle k_2. The circle k_2 is one of the three h-circles passing through z_1.

□

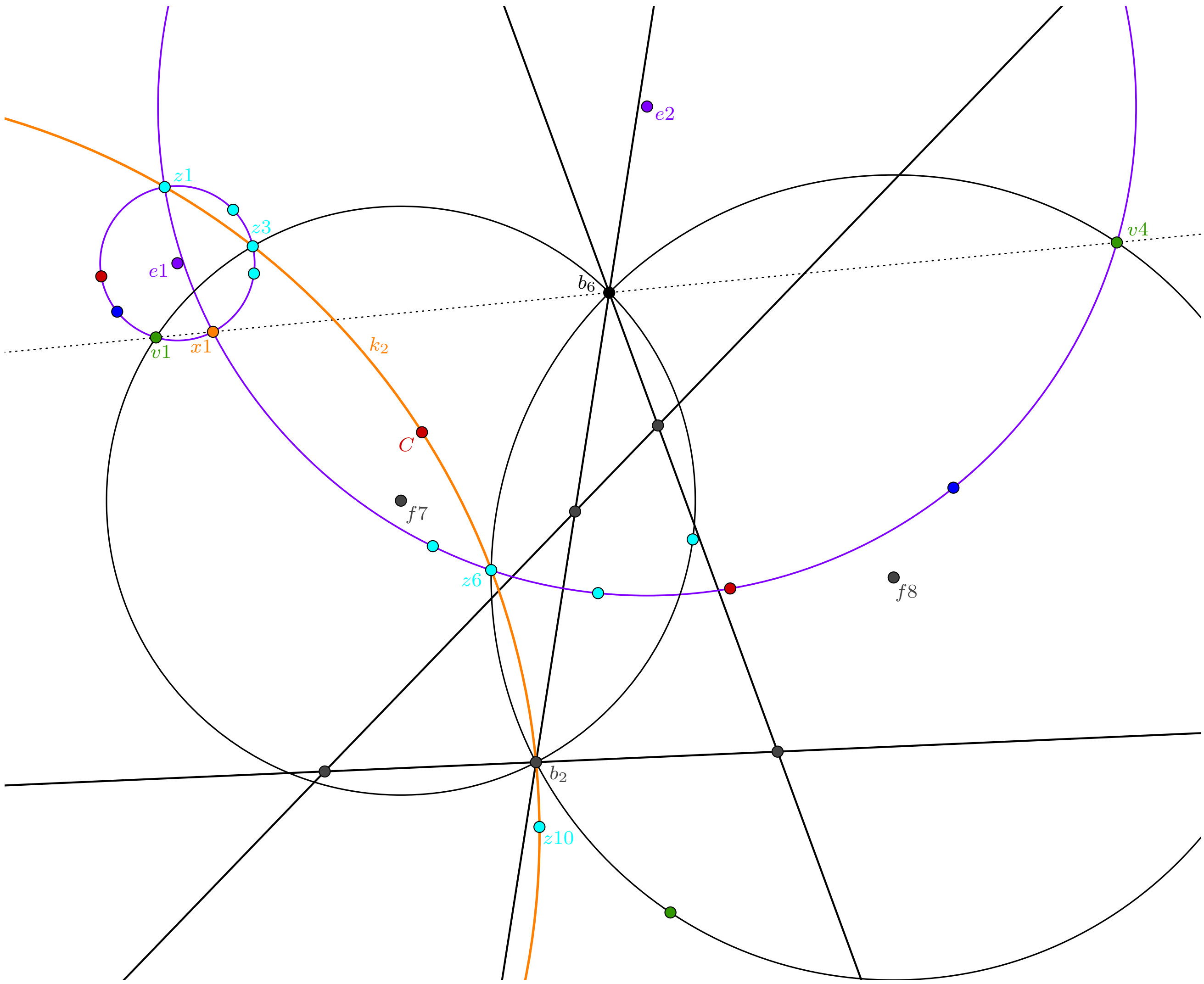

Figure 4

4 Two Propositions

We need two Propositions both interesting in themselves.The first one is certainly well known and its proof is easy to find, so we do not give it here.

Definition. Two excentres or one incentre and one excentre of a triangle lie on angle bisectors of this triangle, which intersect orthogonally at two triangle vertices. We call the corresponding Thales circles over these pairs of circle centres the *k-circles* of the triangle.

Proposition 2 (without proof)**.** *The centres of the six k-circles of a triangle lie on the circumcircle of the triangle and delimit three of its diameters.*

Remark on the f-circles. Every f-circle is in particular a k-circle of a triangle in the quadrilateral. So the overall twenty-four f-circles split into four groups of six circles which are the six k-circles of one of these four triangles.
In the proof of Theorem 1a) we have further seen that the two y-points lying on every f-circle delimit a diameter of the circle. So the centers of the f-circles, are the midpoints of the line segments delimited by these two y-points.

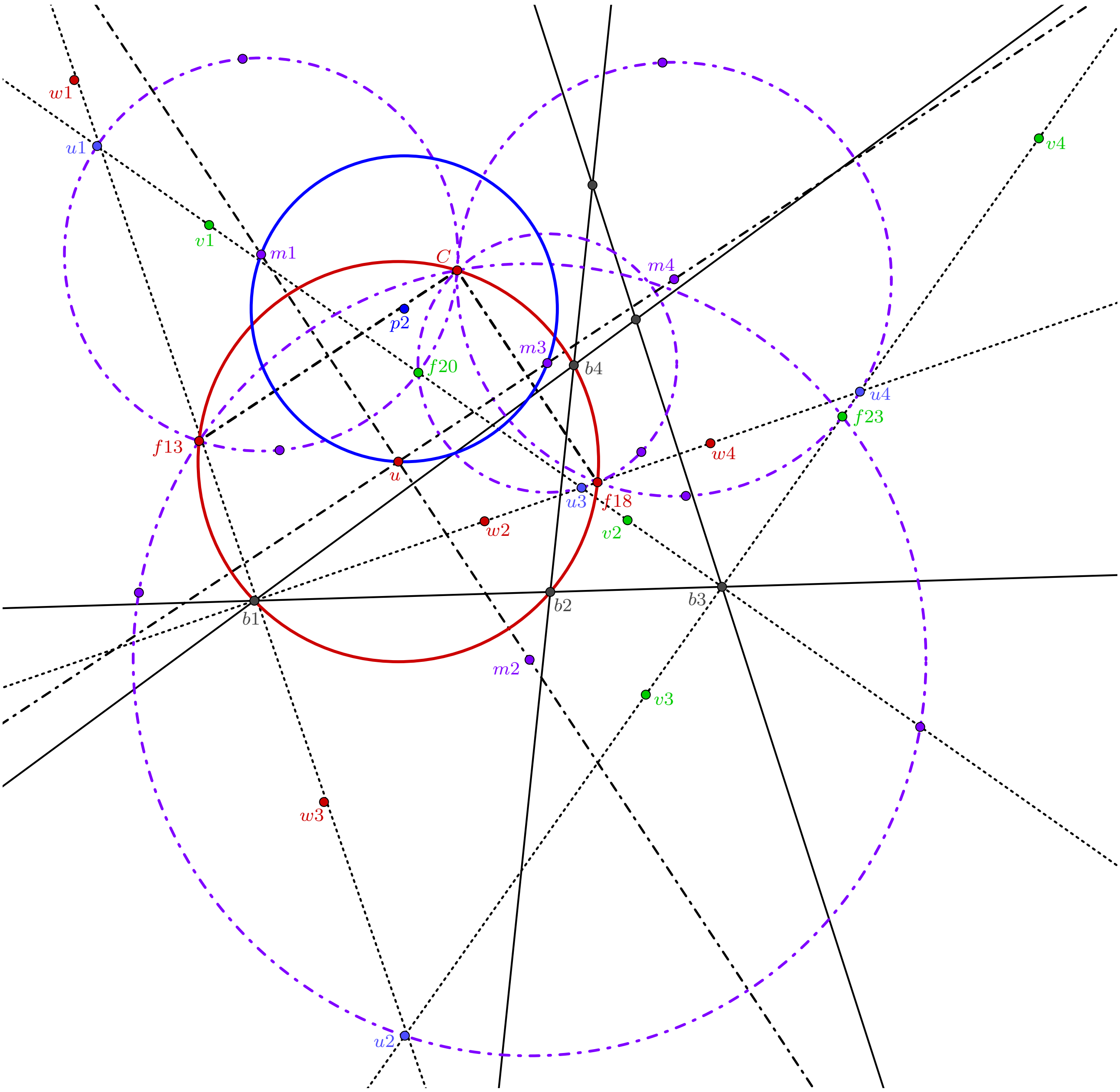

Figure 5

Each angel bisector through a vertex belongs to two different triangles contained in the quadrilateral. It therefore carries two pairs of y-points of different kinds.
Through every one of the overall sixteen y-points, passes exactly one m-circle because the y-point is the counterpoint of exactly one z-point. We call this m-circle the m-circle belonging

to the y-point.

Remark on the m-circles. (Figure 2). Each y-point (x_1) of a certain kind is the intersection of three angle bisectors containing a pair of y-points of a different kind ($u_1, u_4/v_1, v_4/w_1, w_4$). The f-circle centres (f_1, f_2, f_3) on the m-circle (m_6) belonging to the first y-point (x_1) are the centres of the f-circles through the two y-points of the other kind on these bisectors.

Proposition 3. *The centre m_3 of the m-circle m_3 belonging to the incentre u_3 of the triangle $b_1b_3b_5$ is the orthocentre in the triangle formed by the centres m_1, m_2, m_4 of the m-circles belonging to the other three u-points.*

Proof of Proposition 3. We consider in Figure 5 the angle bisectors in b_1 carrying the points u_1, u_2 and w_1, w_3 respectively u_3, u_4 and w_2, w_4. According to the previous remarks on the f- and m-circles, the midpoints f_{13} and f_{18} of the line segments w_1w_3 and w_2w_4 are f-circle centres and they are the points of intersection of the m-circles m_1, m_2 respectively m_3, m_4 belonging to the points u_1, u_2 respectively u_3, u_4. According to Proposition 2, the points f_{13}, f_{18} lie on the circumcircle u (passing through the Clifford point C) of the triangle $b_1b_2b_4$. Since $\sphericalangle f_{13}b_1f_{18} = 90°$, the line segment $f_{13}f_{18}$ is a diameter of this circumcircle and so we have $\sphericalangle f_{13}Cf_{18} = 90°$. Hence the line segments $f_{13}C$ and $f_{18}C$, the common chords of the m-circles m_1, m_2 and m_3, m_4 respectively are perpendicular to each other and thus so are the corresponding centre lines, the lines m_1m_2 and m_3m_4. The same holds for the lines m_1m_3 and m_2m_4, which can be seen by repeating the argumentation using the angle bisectors through the vertex b_3. The point m_3 is the orthocentre of the triangle $m_1m_2m_4$. □

5 Proofs second part

Proof of Theorem 1i). In Figure 3 we consider the f-circle f_2 defined by the u-points u_1, u_4 belonging to z_1 and the g-circles passing through these u-points. The statements about the circle centres on lines follow directly from the fact that the corresponding circles have common chords, namely the line segments u_1z_1 and u_4z_1 respectively for the mentioned circles. Since the f-circle f_2 passes through u_1 *and* u_4, the corresponding centre lines intersect at its centre f_2. They form a right angle there, due to (2). Again, one can argue in the same way for the other two pairs of y-points belonging to z_1, that is for the f-points f_1, f_3. So the three circles whose existence is to be proved are the Thales circles over corresponding g-circle centres for the right angles in the f-circle centres. □

Proof of Theorem 1ii). The assertion follows immediately from the previous remark about the f-circles and Proposition 2. □

Proof of Theorem 1iii). Every m-circle passes through the Clifford point C and through two e-circle centres e_j, e_k, one on the line d_1 the other on d_2. Therefore the m-circle centres are the intersection points of the perpendicular bisectors of the line segments e_jC, e_kC. There are four of these parallel to d_1 and four parallel to d_2. The sixteen intersections of these eight lines are the centres of the sixteen m-circles. □

Proof of Theorem 1iv),v). (Figure 6) The statement of Proposition 3 of course is also valid for the v-, w- and x-points. For this reason, the grid in Figure 6 has several very special properties. For example the points m_3, m_8, m_{11}, m_{16} are the orthocentres of the triangles $m_1m_2m_4, m_5m_9m_{14}, m_6m_7m_{13}, m_{10}m_{12}m_{15}$.

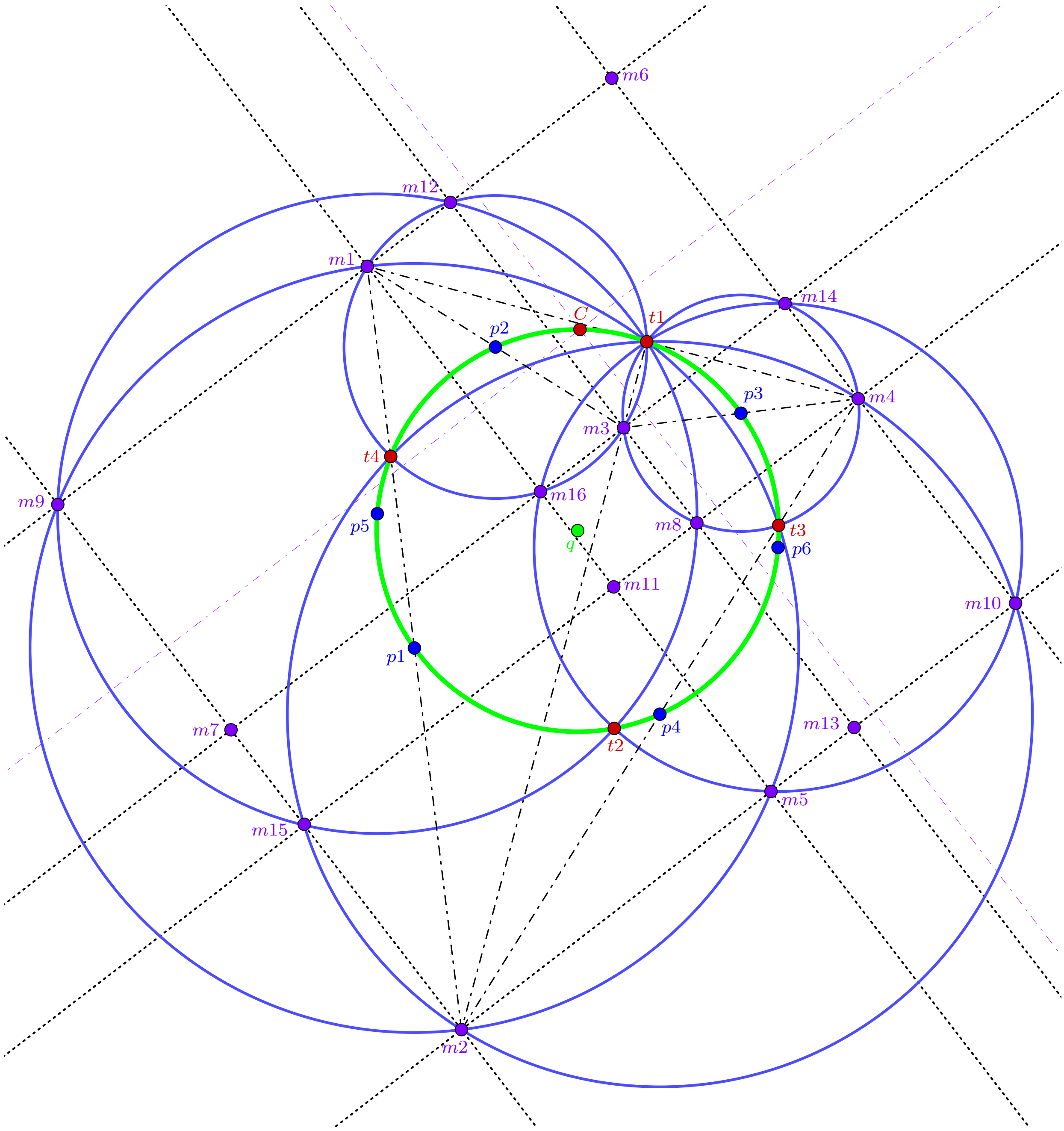

Figure 6

We will show, that the rectangle circumcircles $p_j, j = 1, 2, ..., 6$, in Figure 6, have the required properties. Their centres $p_j, j = 1, 2, ..., 6$ lie on the rectangle centres. Central dilations with factor 2 and centres m_2, m_3 respectively m_1, m_4 show that the lines p_1p_4 and p_2p_3 respectively p_1p_2 and p_3p_4 are parallel to the lines m_1m_4 respectively m_2m_3. These last two lines are perpendicular to each other according to Proposition 3 so that the p-points p_1, p_2, p_3, p_4 form a rectangle. They lie on its circumcircle q - which must be shown to have the required properties - and delimit two of its diameters. If we repeat this argumentation for the points p_2, p_4, p_5, p_6, taking into account that m_{16} is the orthocentre in the m-point triangle $m_{10}m_{12}m_{15}$, we see that the points p_5 and p_6 also lie on the circle q and delimit a third of its diameters.
Since m_3 is the orthocentre in the m-point triangle $m_1m_2m_4$, the circles p_1, p_2, p_3, p_4 intersect at the foot of altitude t_1 of this triangle on the side m_1m_4. We will see that t_1 is indeed one of the four t-points by showing that the circles p_5 and p_6 also pass through it: The circles p_1, p_3 also contain the points m_5, m_9 and m_8, m_{14} respectively belonging to the m-point triangle $m_5m_9m_{14}$ with orthocentre m_8. Thus t_1 is also a foot of altitude of this triangle and so the p-circles p_5, p_6, passing through the points m_8, m_9 and m_5, m_{14} respectively, also pass through t_1. The point t_1

is the intersection of six p-circles.
Next we show that the point t_1 lies on the circle q: Another foot of altitude of the triangle $m_1m_2m_4$ is the intersection t_4 - we will see that it is indeed another t-point - of the circles p_2, p_4. As such it lies on the line m_1m_2. Moreover with

$$\sphericalangle p_2t_1t_4 = 90^\circ - \sphericalangle t_1m_1t_4 \qquad \sphericalangle t_4p_1p_4 = 90^\circ - \sphericalangle t_1m_2t_4$$

we get $\sphericalangle p_2t_1p_4 = 180^\circ - \{\sphericalangle t_1m_1t_4 + \sphericalangle t_1m_2t_4\}$ and since $\sphericalangle m_1t_1m_2 = 90^\circ$ we find $\sphericalangle p_2t_1p_4 = 90^\circ$. Since the line segment p_2p_4 is a diameter of the circle q, the point t_1 also lies on the circle q. Theorem 1iv),v) can now be completely proved by repeating these arguments including the fourth m-point triangle $m_6m_7m_{13}$. □

6 The Steiner circle of the complete quadrilateral

Theorem 2. *The four p-circle intersections t_1, t_2, t_3, t_4 in Theorem 1 are the circumcentres of the four triangles contained in the complete quadrilateral. This means that the circle q is the Steiner circle of the quadrilateral.*
The Steiner circle of a complete quadrilateral contains seventeen special points, namely twelve p-circle centres - delimiting six of its diameters - the circumcircle centres of the four triangles in the quadrilateral and its Clifford point C.

Proof of Theorem 2. (Figures 5 and 6) Of course it is enough to show the assertion for one t-point. We recall certain remarks on Figures 5 and 6 made in the proofs of Preposition 3 and Theorem 1iv),v). The m-circles m_1, m_2 and m_3, m_4 pass through the f-points f_{13} and f_{18} respectively. These two f-points lie on the circumcircle u of the triangle $b_1b_2b_4$ and we have $\sphericalangle f_{13}Cf_{18} = 90^\circ$. Furthermore the m-circle centres m_1, m_3 delimit a diameter of the p-circle p_2. So the line segments $f_{13}C$ and $f_{18}C$, the common chords of the circles m_1, u and m_3, u respectively are perpendicular to each other and hence so are the corresponding centre lines which means that $\sphericalangle m_1um_3 = 90^\circ$. The line segment m_1m_3 being a diameter of the circle p_2, it follows that the circle p_2 passes through the centre u of the triangle circumcircle u. Using the same reasoning, this can be shown for the other five p-circles meeting at t_1. The m-circle intersection t_1 is the circumcentre u. □

Finally we give a conjecture fitting into the context.

Conjecture. *At least ninety-seven special circles presumably pass through the Clifford point C of a complete quadrilateral, namely in addition to*

a) the circumcircles of the four triangles contained in the quadrilateral, its Steiner circle and the overall sixteen m-circles,

presumably also

b) the overall twelve h-circles,

c) forty-eight circles - the j-circles - each containing in addition two of the overall twelve h-circle centres and one of the overall twenty-four f-circle centres,

d) sixteen circles, each containing in addition three j-circle centres and one of the overall sixteen m-circle centres.

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