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A NONTRIVIAL POINT ON THE ISOGONAL PIVOTAL CUBIC WITH PIVOT ON THE CIRCUMCIRCLE

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Abstract. It's well-known that the nine-point center $X(5)$ and its isogonal conjugate point, Kosnita point $X(54)$, lie on the isogonal pivotal cubic $K316$ with pivot Euler reflection point $X(110)$. In this article we generalize this result to any isopivotal cubic with pivot on the circumcircle to find a nontrivial point on it.

1. Introduction

In the '$K316$' page of [2], Bernard Gibert introduced the following result.

Theorem 1.1. *The points $X(1)$, $X(5)$, $X(54)$, $X(110)$, $X(523)$, $X(1113)$, $X(1114)$, $X(2574)$, $X(2575)$ lie on $K316$, namely the isogonal pivotal cubic with pivot $X(110)$.*

However, most of these points are trivial. The incenter $X(1)$ lies on any isogonal pivotal cubic, and $X(110)$ naturally lies on its own isogonal pivotal cubic. For the two intersections of the Euler line and the circumcircle, $X(1113)$ and $X(1114)$, we will reprove the theorem below, which was announced in the 'Table 17' of [2] without proof, as a generalization in Section 3.

Theorem 1.2. *Let P be a point lying on the circumcircle. The two intersections of the parallel at the circumcenter $X(3)$ to the Steiner line of P and the circumcircle lie on the isogonal pivotal cubic with pivot P.*

Because the isogonal pivotal cubic is invariant under isogonal conjugate transformation, we could get $X(523)$, $X(2574)$, $X(2575)$ lie on $K316$ as respective isogonal conjugate points of $X(110)$, $X(1113)$, $X(1114)$.

Now the only part left is $X(5)$ and $X(54)$. It's natural to try to find something similar on general isogonal pivotal cubics. The answer could be stated as follows.

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Theorem 1.3. *Let P be a point lying on the circumcircle. The intersection X of the parallel at $X(5)$ to $X(3)P$ and the parallel at $X(3)$ to the Steiner line of P lies on the isogonal pivotal cubic with pivot P.*

We will prove this theorem in Section 4 by constructing a fourth point on PX different from P, X and the isogonal conjugate point of X.

2. Definitions, notations and auxiliary results

From now on, we denote the reference triangle by $\triangle ABC$ and follow the Kimberling number for the triangle center in [3], such as $X(1)$ for incenter, $X(3)$ for circumcenter, etc. For a point(resp. set of points) P, we denote its image under isogonal conjugate transformation by $\mathbf{g}P$. We will construct this article on the real projective plane and denote the line at infinity by $\mathcal{L}_\infty$. For four points W, X, Y, Z lying on a conic(resp. line) $\mathcal{C}$, their cross-ratio with respect to $\mathcal{C}$ will be denoted by $[W, X; Y, Z]_\mathcal{C}$ when we regard Y, Z as separating W and X. Similarly, for a pencil of four concurrent lines OA, OB, OC, OD, their cross-ratio will be denoted by $O[A, B; C, D]$.

Definition 2.1. *Let P be an arbitrary point. The locus of point X such that X,$\mathbf{g}X$,P are collinear is called the isogonal pivotal cubic with pivot P, i.e. $p\mathcal{K}(X(6), P)$.*

Definition 2.2. *Let P be a point lying on $\odot(ABC)$. The point $X(4)$ and the reflections of P with respect to sidelines of $\triangle ABC$ are collinear, and this line is called the Steiner line of P and denoted by $\mathbf{S}(P)$. Conversely, for any line l passing through $X(4)$, there exists a unique $P \in \odot(ABC)$, which is called the anti-Steiner point of l and denoted by $\mathbf{S}^{-1}(l)$, such that $l = \boldsymbol{S}(P)$, or precisely, $\mathbf{S}^{-1}(l)$ is the intersection of the reflections of l with respect to the sidelines of $\triangle ABC$.*

Definition 2.3. *For an arbitrary point P, let $\mathbf{c}P$ denote the point such that $\overrightarrow{X(2)P} = -2\overrightarrow{X(2)\mathbf{c}P}$. The $\mathbf{c}P$ is called the complement of P and P is called the anticomplement of $\mathbf{c}P$. For a set of points P, we denote its image under complement transformation by $\mathbf{c}P$.*

Remark 2.1. *By $X(3) = \mathbf{c}X(4)$, we could get that $\mathbf{cS}(P)$ is the parallel at $X(3)$ to $\mathbf{S}(P)$, where $P \in \odot(ABC)$.*

Definition 2.4. *A conic(resp. hyperbola) is called a circum-conic(resp. circum-hyperbola) if the vertices of $\triangle ABC$ lie on it.*

Now we introduce some auxiliary results which will be used in the following sections.

Theorem 2.1. *For any line l which does not pass through any vertex of $\triangle ABC$, $\mathbf{g}l$ is a circum-conic. Conversely, for any circum-conic $\mathcal{C}$, $\mathbf{g}\mathcal{C}$ is a line.*

Proof. *We first assume l is a line which does not pass through any vertex of $\triangle ABC$. Consider three points $P, Q, R \in l$. Then*

$$B[\mathbf{g}P, \mathbf{g}Q; \mathbf{g}R, C] = B[P, Q; R, A] = A[P, Q; R, B] = A[\mathbf{g}P, \mathbf{g}Q; \mathbf{g}R, C],$$

so $A, B, C, \mathbf{g}P, \mathbf{g}Q, \mathbf{g}R$ lie on a common conic, i.e. $\mathbf{g}l$ is a circum-conic. And the converse direction is similar. $\square$

Remark 2.2. *In particular,* $\mathbf{g}\mathcal{L}_\infty = \odot(ABC)$.

Corollary 2.1. *Let* X, Y, Z, W *be four points collinear on a line which does not pass through any vertex of* $\triangle ABC$. *Then*

$$[X, Y; Z, W]_l = [\mathbf{g}X, \mathbf{g}Y; \mathbf{g}Z, \mathbf{g}W]_{\mathbf{g}l}.$$

Proof.

$$[X, Y; Z, W]_l = A[X, Y; Z, W] = A[\mathbf{g}X, \mathbf{g}Y; \mathbf{g}Z, \mathbf{g}W] = [\mathbf{g}X, \mathbf{g}Y; \mathbf{g}Z, \mathbf{g}W]_{\mathbf{g}\mathcal{H}}.$$

Theorem 2.2. *(Angle's version of Menelaus's theorem) Let* U, V, W *be three points lies on the sidelines* YZ, ZX, XY *of a* $\triangle XYZ$ *respectively. Then for an arbitrary point* O

$$\frac{\sin\measuredangle YOU}{\sin\measuredangle UOZ} \cdot \frac{\sin\measuredangle ZOV}{\sin\measuredangle VOX} \cdot \frac{\sin\measuredangle XOW}{\sin\measuredangle WOY} = -1.$$

It is a corollary of the standard Menelaus's theorem.

Theorem 2.3. *(Isogonality of the complete quadrilateral) Let* P,Q *be two arbitrary points. Then we have*

$$\mathbf{g}(PQ \cap \mathbf{g}P\mathbf{g}Q) = P\mathbf{g}Q \cap \mathbf{g}PQ.$$

Proof. Set $X := PQ \cap \mathbf{g}P\mathbf{g}Q$ and $Y := P\mathbf{g}Q \cap \mathbf{g}PQ$. By the angle's version of Menelaus's theorem, we could get

$$\frac{\sin\measuredangle \mathbf{g}QAX}{\sin\measuredangle XA\mathbf{g}P} \cdot \frac{\sin\measuredangle \mathbf{g}PAQ}{\sin\measuredangle QAY} \cdot \frac{\sin\measuredangle YAP}{\sin\measuredangle PA\mathbf{g}Q} = -1.$$

Then by $\measuredangle PA\mathbf{g}Q = -\measuredangle \mathbf{g}PAQ$, we have

$$\frac{\sin\measuredangle QAY}{\sin\measuredangle YAP} = \frac{\sin\measuredangle \mathbf{g}QAX}{\sin\measuredangle XA\mathbf{g}P} = \frac{\sin\measuredangle QA\mathbf{g}X}{\sin\measuredangle \mathbf{g}XAP},$$

which means $\mathbf{g}X \in AY$. By symmetry, we get $Y = \mathbf{g}X$. □

The following properties are about rectangular hyperbola.

Theorem 2.4. *A circum-conic* $\mathcal{C}$ *is a rectangular hyperbola if and only if* $X(4) \in \mathcal{C}$.

Proof. Let the two intersections of $\mathbf{g}\mathcal{C}$ and $\odot(ABC)$ be U and V, then

$$X(4) \in \mathcal{C} \iff X(3) \in \mathbf{g}\mathcal{C} \iff AU \perp AV \iff A\mathbf{g}U \perp A\mathbf{g}V,$$

so by $\mathbf{g}U, \mathbf{g}V \in \mathcal{L}_\infty$ the last assertion is equivalent to the two asymptotes of $\mathcal{C}$ is perpendicular. □

Theorem 2.5. *Let* $\mathcal{H}$ *be a rectangular circum-hyperbola and* P *be the fourth intersection of* $\mathcal{H}$ *and* $\odot(ABC)$. *Then* $P, X(4)$ *is a pair of antipodal points on* $\mathcal{H}$.

Proof. Let X, Y be the two intersections of $\mathbf{g}\mathcal{H}$ and $\odot(ABC)$, $\mathbf{g}P = \mathcal{L}_\infty \cap \mathbf{g}\mathcal{H}$, so

$$[X(4), P; \mathbf{g}X, \mathbf{g}Y]_{\mathcal{H}} = [X(3), \mathbf{g}P; X, Y]_{\mathbf{g}\mathcal{H}} = -1,$$

which implies that $X(4)P$ passes through the center of $\mathcal{H}$. □

Consequently, the following corollary holds.

Corollary 2.2. *Let* $\mathcal{H}$ *be a rectangular circum-hyperbola with center* S *and denote its fourth intersection with* $\odot(ABC)$ *by* K. *Then* $\mathbf{c}^{-1}(S)$ *is the antipodal point of* K *with respect to* $\odot(ABC)$.

Most of the proofs of the results above can be also found in [1].

3. Proof of Theorem 1.2

In this section, we will restate Theorem 1.2 by the notations in Section 2 and prove it by straight angle-chasing.

Theorem 3.1. *Let P be a point lying on $\odot(ABC)$ and X be an intersection of $\mathbf{cS}(P)$ and $\odot(ABC)$. Then $X \in p\mathcal{K}(X(6), P)$.*

Proof. See Figure 1. Let U be the reflection of $X(4)$ with respect to AC, V be the second intersection of the parallel at X to BC and $\odot(ABC)$. Set $W := AV \cap \mathbf{S}(P)$ and $Y := AC \cap \mathbf{S}(P)$. Then to prove $\mathbf{g}X \in XP$, it suffices to show AV is parallel to XP, which is equivalent to $\measuredangle AWX(4) = \measuredangle PXX(3) = \frac{\pi}{2} - \measuredangle XAP$. Now we compute $\measuredangle AWX(4) + \measuredangle XAP$. By $\measuredangle XAB = \measuredangle CAV$ we get $\measuredangle AWX(4) + \measuredangle XAP = \measuredangle BAP + \measuredangle AYX(4) = \measuredangle BCP + \measuredangle PYC$. Then $\measuredangle YPC + \measuredangle BCY = \measuredangle X(4)BC + \measuredangle BCA = \frac{\pi}{2}$ completes the proof. □

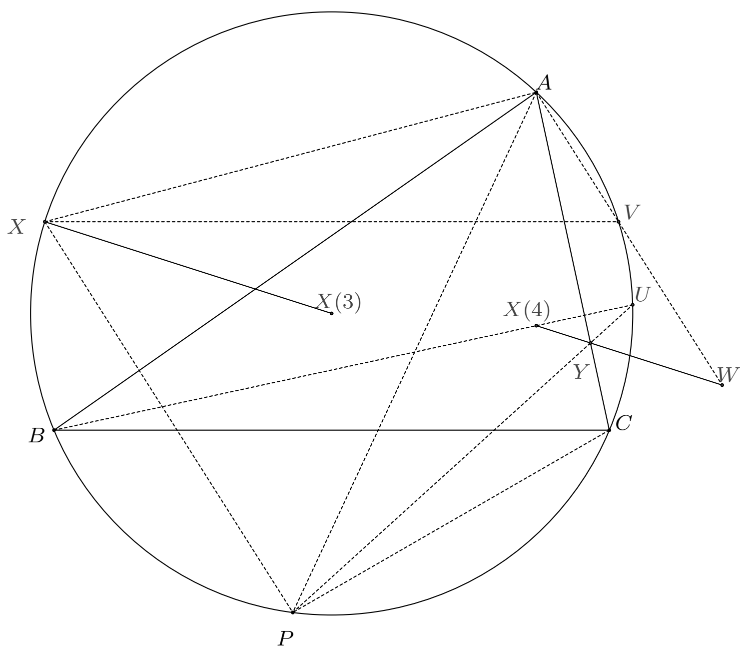

Figure 1. Theorem 3.1

4. Main result

In this section we will establish our main result, Theorem 1.3, from a lemma about the anti-Steiner point and rectangular circum-hyperbola.

Lemma 4.1. *Let $\mathcal{H}$ be a rectangular circum-hyperbola and K be the fourth intersection of it and $\odot(ABC)$. Then for a point $P \in \mathcal{H}$, $\mathbf{S}^{-1}(X(4)P) \in KP$.*

Proof. Set $S_P := \mathbf{S}^{-1}(X(4)P)$, $D := X(4)P \cap BC$, $E := X(4)A \cap BC$ and U be the reflection of $X(4)$ with respect to BC. Then

$$K[A,B;C,P] = [A,B;C,P]_{\mathcal{H}} = X(4)[A,B;C,P] = [E,B;C,D]_{BC}$$

and

$$[E,B;C,D]_{BC} = U[A,B;C,S_P] = [A,B;C,S_P]_{\odot(ABC)} = K[A,B;C,S_P],$$

which implies that K, P, S_P are collinear. □

The other lemma we will use describes the intersection of the tangent of a rectangular circum-hyperbola $\mathcal{H}$ at its fourth intersection with the circumcircle and $\mathbf{g}\mathcal{H}$.

Lemma 4.2. *Let $\mathcal{H}$ be a rectangular circum-hyperbola and K be its fourth intersection with $\odot(ABC)$. Denote the second intersection of $X(3)K$ and $\mathcal{H}$ by Y and its antipodal point with respect to $\mathcal{H}$ by Y'. Then $\mathbf{c}(X(4)Y) = \mathbf{g}\mathcal{H}$ and $T := \mathbf{g}Y'$ lies on the tangent of $\mathcal{H}$ at K.*

Proof. See Figure 2. Let $Z := X(4)K \cap \mathbf{g}\mathcal{H} = X(4)K \cap X(3)\mathbf{g}K$. By the isogonality of the complete quadrilateral, $\mathbf{g}Z \in X(3)K \cap \mathcal{H} \cap X(4)\mathbf{g}K$, so $Y = \mathbf{g}Z$. This implies that $\mathbf{g}K \in X(4)Y$ and proves $\mathbf{c}(X(4)Y) = \mathbf{g}\mathcal{H}$ consequently by $X(3) = \mathbf{g}X(4)$. For the second part, by Theorem 2.5 we conclude $\mathbf{g}K \in KY'$. Then for an arbitrary point $K_1 \in \mathcal{H}$, $\mathbf{g}(KK_1 \cap \mathbf{g}K\mathbf{g}K_1) = K\mathbf{g}K_1 \cap \mathbf{g}KK_1$. Now let $K_1 \to K$ along $\mathcal{H}$, thus $KK_1 \cap \mathbf{g}K\mathbf{g}K_1$ approaches to the intersection of the tangent of $\mathcal{H}$ at K and $\mathbf{g}\mathcal{H}$ and $K\mathbf{g}K_1 \cap \mathbf{g}KK_1$ approaches to be the second intersection of $K\mathbf{g}K$ and $\mathcal{H}$. Hence KT is tangent to $\mathcal{H}$ at K. □

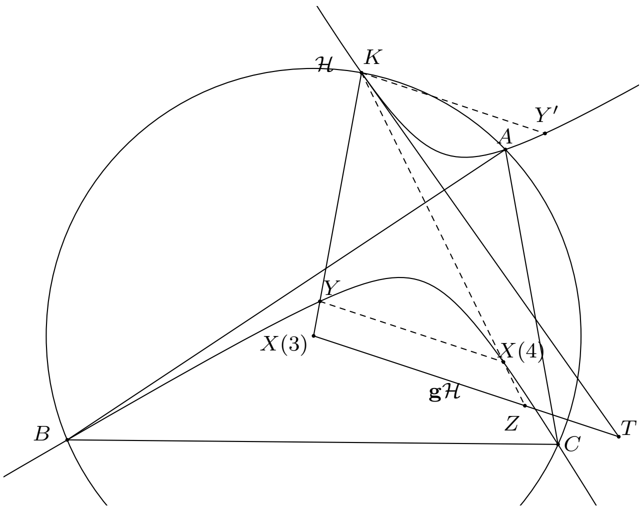

FIGURE 2. Lemma 4.2

Remark 4.1. *Lemma 4.2 through Lemma 4.1 exhibits $K, T, \mathbf{S}^{-1}(X(4)K)$ are collinear.*

We can now formulate our main result.

Theorem 4.1. *Let P be a point lying on the circumcircle. Then $X := \mathbf{c}(X(3)P \cap \mathbf{S}(P)) \in p\mathcal{K}(X(6), P)$.*

Proof. See Figure 3. It suffices to prove $X, \mathbf{g}X, P$ are collinear. Let $\mathcal{H}$ be the rectangular circum-hyperbola $\mathbf{gcS}(P)$ with center S and $\mathcal{H} \cap \odot(ABC) = \{A, B, C, K\}$. Denote the second intersection of $X(3)K$ and $\mathcal{H}$ by Y and its antipodal point with respect to $\mathcal{H}$ by Y'. Set $l := \mathbf{cS}(P)$, $T := \mathbf{g}Y'$, $R := X(3)T \cap X(4)Y'$. Then by the isogonality of the complete quadrilateral we get $\mathbf{g}R = X(3)Y' \cap X(4)T \in \mathcal{H}$.

From Lemma 4.2, we get $X(4)Y // l$ which implies that $X(4)Y = \mathbf{S}(P)$, so through Lemma 4.1 $K, Y, P, X(3)$ are collinear. By definition, $X = \mathbf{c}Y$. In the other hand, $S = \mathbf{g}P$ follows from Corollary 2.2, so $\overrightarrow{YP} = -2\overrightarrow{XS}$. We also have S is the midpoint of YY' and $XS, X(4)Y', X(3)P$ are parallel, thus X is the common midpoint of PY' and $X(3)R$. So

$$[X(4), \mathbf{g}R; \mathbf{g}X, K]_{\mathcal{H}} = [X(3), R; X, \mathbf{g}K]_l = -1.$$

Combine this with $T \in X(4)\mathbf{g}R$ and TK is tangent to $\mathcal{H}$ which has been proved in Lemma 4.2, we must have T is the pole of $K\mathbf{g}X$ with respect to $\mathcal{H}$, i.e. $T\mathbf{g}X$ is tangent to $\mathcal{H}$.

To complete the proof, we have to use the limit argument that we have used in Lemma 4.2. For an arbitrary point $X_1 \in l$, $\mathbf{g}(XX_1 \cap \mathbf{g}X\mathbf{g}X_1) = X\mathbf{g}X_1 \cap \mathbf{g}XX_1$. When $X_1 \to X$ on l, we conclude that $Y' = \mathbf{g}T$ is the second intersection of $X\mathbf{g}X$ and $\mathcal{H}$, hence $Y', X, \mathbf{g}X, P$ are collinear. $\square$

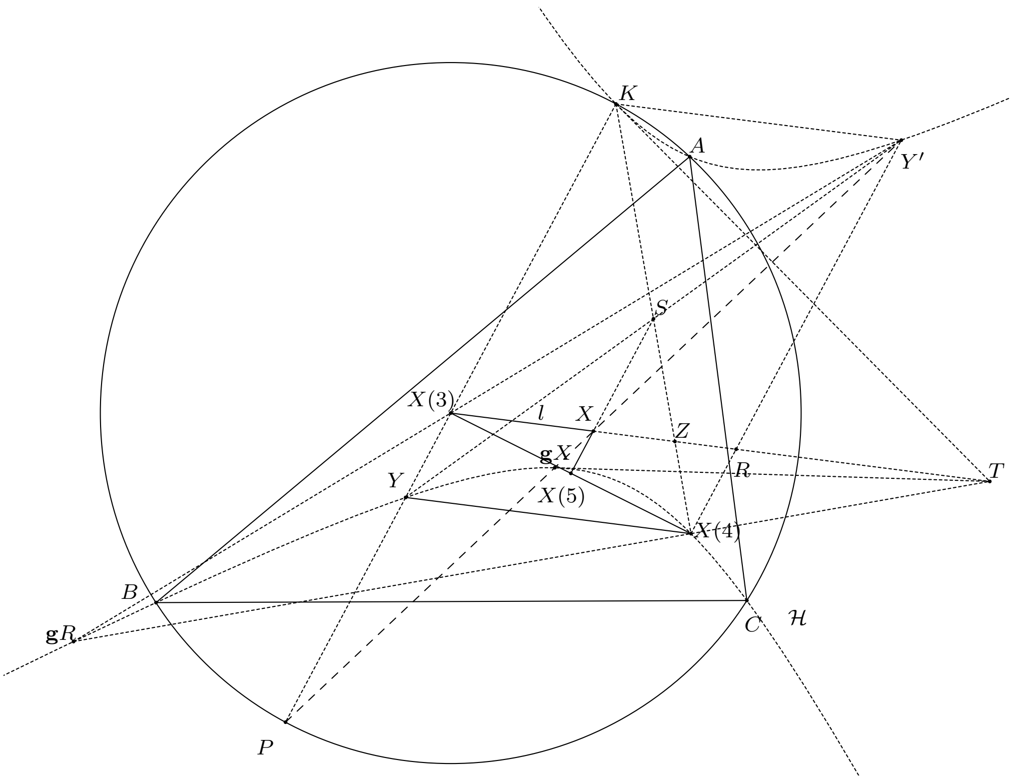

FIGURE 3. Theorem 4.1

5. Conclusion and further problem

In conclusion, our theorem gives the description of all three intersections of $\mathbf{cS}(P)$ and $p\mathcal{K}(X(6), P)$, where P is a point lying on circumcircle. As we could see in Section 4, its proof depends on the limit argument strongly. Furthermore, if we introduce the Cundy-Parry Φ transformation in the 'CL037' page of [2], the point $X(3)P \cap \mathbf{S}(P)$ will turn out to be the image of the antipodal point of P with respect to the circumcircle under Φ, this may be a way to generalize our theorem to the isopivotal cubic with pivot not lying on the circumcircle.

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