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# CUBIC CURVES AND CAYLEY RESIDUALS

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**Abstract.** The notion of *residuals* was introduced by Cayley in the context of cubic curves. Subsequently the idea was extended far beyond their original context and has led to many developments in modern algebraic geometry. Here it is only their original purpose which will be considered and, using residuals as a foundation, some of the classical results are re-visited. Perhaps cubic curves are nowadays not as well known as they deserve to be. This is certainly not due to the lack of fascinating properties that they possess.

## 1. Introduction.

The theory of cubic curves, that is plane curves which can be represented by ternary cubics (homogeneous polynomials of degree three in three variables) has an illustrious history. Newton and Maclaurin are the two most famous contributors but even they found them to be challenging; Newton missing some cases in his attempt to classify such curves. The high point in the interest in their purely geometric properties was in the 19th century, for by the next century it was their abstract algebraic properties which dominated. This was in part due to the work of Cayley. For example, in [1] the name of Cayley occurs 164 times. In this note it is only the original concept of Cayley which is considered. The abstract developments and higher order curves (and surfaces) will not even be mentioned.

## 2. Residuals and Cubic Curves.

Let $U$ be a cubic curve, $P$ and $A_1, \ldots, A_n$ any points on $U$ with $n = 1, 2, 4, 5, 7$ or $8$. The possibility of a point having multiple tangents or being a point of inflexion was ignored by [2] (although such points are extensively considered elsewhere in this book). The same rather cavalier attitude is taken here; more formally *all points considered here have a unique tangent which meets $U$ again in a different point.* Furthermore, the point at which the tangent at $P$ meets $U$ again is called the *tangential point* of $P$ and is denoted by $P^*$. The residual of an unordered collection of points such as $P, A_1, A_2, A_3$ on $U$ is a single point on $U$ denoted by $[P, A_1, A_2, A_3]$ and has

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the following properties:

- $[P] = P$.
- $[A_1, A_2]$ is the point on $U$ where the chord $A_1A_2$ meets $U$ again.
- Any conic through $A_1, A_2, A_3, A_4$ will meet $U$ in two other points, say $B_1, B_2$. Then $[A_1, A_2, A_3, A_4] = [B_1, B_2]$ and is independent of the choice of the conic.
- $[A_1, A_2, A_3, A_4, A_5]$ is the point at which the (unique) conic through these five points meets $U$ again.
- Any cubic curve through $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ will meet $U$ in two other points, say $B_1, B_2$. Then $[A_1, A_2, A_3, A_4, A_5, A_6, A_7] = [B_1, B_2]$ and is independent of the choice of the cubic.
- $[A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8]$ is the common point of all cubic curves through these eight points (and necessarily lies on $U$).
- $[A_1, A_2, A_3, A_4] = [\,[A_1, A_2], [A_3, A_4]\,]$.
- $[A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8] = [\,[A_1, A_2, A_3, A_4], [A_5, A_6, A_7, A_8]\,]$.
- $[P, [A_1, \ldots, A_n]\,] = [P, A_1, \ldots, A_n]$ for $n = 1, 4, 7$ and any point $P$
- $[P, [P, A_1, \ldots, A_n]\,] = [A_1, \ldots, A_n]$ for $n = 1, 4, 7$ and any point $P$.
- As limiting cases, $[P, P] = P^*$ and $[P, P^*] = P$.
- $[A_1, \ldots, A_n]^* = [A_1^*, \ldots, A_n^*]$

The proofs of some of the above results require much ingenuity but ultimately rely upon only two properties of cubic curves, viz.

(1) Any line meets a cubic curve in three points.
(2) If two cubic curves have eight points in common then they also have a ninth common point. Furthermore, any cubic curve through these eight points will also pass through this ninth point and may be expressed as a linear combination of the original two curves.

The *modus operandi* adopted here is to presume the truth of the above results and to deduce theorems; some will be familiar but others perhaps less so.

**Theorem 2.1.** *If $A, B, C$ are collinear points on a cubic curve, then their tangential points are also collinear.*

**Proof.** Since $C$ lies on the line $A, B$ and on $U$ it follows that

$$C = [A, B] \Rightarrow C^* = [A, B]^* = [A^*, B^*]$$

and so $A^*, B^*, C^*$ are collinear.

Two further results may be established by similar arguments:

(1) Suppose that infinitely many cubic curves may be drawn through a given set of nine points on $U$. Then infinitely many cubic curves may be drawn through their nine tangential points.
(2) If six points on a cubic curve also lie on a conic then their six tangential points also lie on a conic.

**Theorem 2.2.** *The points $A, B, C$ lie on a cubic curve and the six points $A, A^*, B, B^*, C, C^*$ also lie on a conic. Then $A, B, C$ are collinear.*

**Proof.** Since $C^*$ is the sixth point at which the conic through $A, A^*, B, B^*, C$ meets $U$ we have

$$\begin{aligned} C^* &= [A, A^*, B, B^*, C] = [\,[A, A^*, B, B^*],\, C] = [\,[\,[A, A^*], [B, B^*]\,],\, C] \\ &= [\,[A, B],\, C] \\ \Rightarrow [A, B] &= [C, C^*] = C \end{aligned}$$

and so $A, B, C$ are collinear.

**Theorem 2.3.** *The points $A, B, C$ lie on the cubic curve $U$ and the lines $BC$, $CA$, $AB$ meet $U$ again at $D, E, F$ respectively. Infinitely many cubic curves may be drawn so as to touch $U$ at $A, B, C$ and pass through $D, E, F$.*

**Proof.** Now

$$\begin{aligned} [A, A, B, B, C, C, E, F] &= [\,[\,[A, E],\, [A, F]\,],\, [\,[B, C],\, [B, C]\,]\,] \\ &= [\,[C, B],\, [D, D]\,] = [D, D^*] \\ &= D \end{aligned}$$

which is all that is required.

**Theorem 2.4.** *The point $P$ lies on the cubic curve $U$ and three of the points of contact of the tangents from $P$ to $U$ (disregarding the tangent at $P$) are $A, B, C$. Let the lines $BC$, $CA$, $AB$ meet $U$ again at $D, E, F$ respectively. Then the lines $AD, BE, CF$ are concurrent at a point on $U$.*

**Proof.** Since $P = A^* = B^* = C^*$ we have

$$D = [B, C] \Rightarrow D^* = [B^*, C^*] = [P, P] = P^*$$

and so the tangents at $D, E, F$ are concurrent at $P^*$. Also

$$[A, D]^* = [A^*, D^*] = [P, P^*] = P$$

and so the tangents at $[A, D]$, $[B, E]$, $[C, F]$ all pass through $P$. But only four points have this property and three of them are $A, B, C$. Thus the points $[A, D]$, $[B, E]$, $[C, F]$ coincide (at a point on $U$).

The converse may be proved in a similar way:
*The points $A, B, C$ lie on a cubic curve and the lines $BC$, $CA$, $AB$ meet $U$ again at $D, E, F$ respectively. If $AD, BE, CF$ are concurrent at a point on $U$ then the tangents to $U$ at $A, B, C$ are concurrent at a point on $U$.*

This also gives Maclaurin's theorem which is perhaps the most wonderful result of all:
*The point $P$ lies on the cubic curve $U$ and the points of contact of the tangents from $P$ to $U$ (disregarding the tangent at $P$) are $A, B, C, D$. The three diagonal points of the quadrangle $ABCD$ lie on $U$ and the tangents to $U$ at these three points and at $P$ have a common point on $U$.*

**Theorem 2.5.** *The point $P$ lies on the cubic curve $U$ and the points of contact of the tangents from $P$ to $U$ are $A, B, C, D$. The diagonal points of the quadrangle $ABCD$ are $E, F, G$. Then infinitely many cubic curves may be drawn through the nine points $A, B, C, D, E, F, G, P, P^*$.*

**Proof.** We may choose

$$E = [A, B] = [C, D],\ \ F = [B, C] = [A, D],\ \ G = [A, C] = [B, D]$$

and then

$$\begin{aligned}[A,B,C,D,E,F,G,P] &= [\,[A,P],\ [B,E],\ [C,G],\ [D,F]\,] \\ &= [A,A,A,A] \\ &= (A^*)^* = P^*\end{aligned}$$

as required.

**Theorem 2.6.** *The points $A$, $B$, $C$ lie on the cubic curve $U$ and the lines $BC$, $CA$, $AB$ meet $U$ again at $D$, $E$, $F$ respectively. The tangents to $U$ at $A, B, C$ are concurrent if and only if the lines $AD, BE, CF$ are concurrent.*

**Proof.** Suppose that the tangents at $A, B, C$ are concurrent at $P$. From Theorem 2.3, infinitely many cubic curves may be drawn so as to touch $U$ at $A, B, C$ and pass through $D, E, F$. Now choose the one, say $\tilde{U}$, which also passes through $P$. By Theorem 2.4 the lines $AD, BE, CF$ are concurrent at a point on $\tilde{U}$.

The converse result may be established by choosing the new cubic to pass through the common point of the lines $AD, BE, CF$.

**Theorem 2.7.** *Let $A, B, C$ be any points on a cubic curve $U$. Let the lines $BC$, $CA$, $AB$ meet $U$ again at $D$, $E$, $F$ respectively. Then the lines $A^*D$ and $EF$ meet at a point on $U$.*

**Proof.** We have

$$[A,A,B,C] = [A^*,[B,C]\,] \quad \text{and} \quad [A,A,B,C] = [\,[A,B],[A,C]\,]$$

and so $[A^*,D] = [F,E]$ as required.

**Theorem 2.8.** *The points $A$, $B$, $C$ lie on the cubic curve $U$ and the lines $BC$, $CA$, $AB$ meet $U$ again at $D$, $E$, $F$ respectively. Then $A^*$, $B^*$, $C^*$ (which are presumed to be distinct) are collinear if and only if $D, E, F$ are collinear.*

**Proof.** If $A^*$, $B^*$, $C^*$ are collinear then $A^* = [B^*, C^*]$ and so

$$D^* = [B,C]^* = [B^*,C^*] = A^*.$$

The previous result now gives

$$[F,E] = [A^*,D] = [D^*,D] = D$$

and so the points $D$, $E$, $F$ are collinear.

On the other hand, if $D, E, F$ are collinear then

$$\begin{aligned}F &= [D,E] = [\,[B,C],[A,C]\,] \\ &= [\,[A,B],[C,C]\,] = [F,C^*] \\ \Rightarrow C^* &= F^* = [A,B]^* = [A^*,B^*]\end{aligned}$$

and so $A^*, B^*, C^*$ are collinear.

Notice that if $B^*$ and $C^*$ coincide then the first statement in the proof is false.

**Theorem 2.9.** *Let the points of contact of the tangents from the point $P_0$ (which is not on $U$) to the cubic curve $U$ be $P_1, \dots, P_6$. Let the chord $P_iP_j$ $(i \neq j)$ meet $U$ again at $Q_{ij}$. Then the six points*

$$P_1^{**},\ Q_{1i}\ (2 \leq i \leq 6)$$

*(where $P_1^{**} = (P_1^*)^*$) lie on a conic.*

**Proof.**

$$
\begin{aligned}
[Q_{12},\ldots,Q_{16}] &= [\,[Q_{12},\ldots,Q_{15}],Q_{16}]\,] \\
&= [\,[\,[P_1,P_2],\ldots,[P_1,P_5]\,],Q_{16}] \\
&= [\,[\,[P_1,P_1,P_1,P_1],[P_2,P_3,P_4,P_5]\,],Q_{16}\,] \\
&= [\,[P_1^*,P_1^*],[P_1,P_6]\,],Q_{16}\,] \\
&= [\,[P_1^{**},Q_{16}],Q_{16}] = P_1^{**}
\end{aligned}
$$

since $[P_2,P_3,P_4,P_5]=[P_1,P_6]=Q_{16}$.

## References

[1] Dolgachev, Igor, *Classical Algebraic Geometry: a modern view*, C.U.P., 2012.
[2] Salmon, George, *A treatise on the higher plane curves*, third edition, Hodges, Foster & Figgis, Dublin, 1879.

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