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SOME OSCILLATIONS OF HELICOIDAL TREES ON A CIRCULAR CYLINDER IN THE THREE DIMENSIONAL EUCLIDEAN SPACE

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Abstract. We obtain a generalization of the mechanical solution given by S. Gueron and R. Tessler w.r. to the weighted Fermat-Torricelli problem on a circular cylinder S, which derives a new structure of solution, which may be called "an oscillatory Fermat-Torricelli tree of helices." The weighted Fermat-Torricelli problem on S states that: Given three points and a positive real number (weight), which correspond to each point, find the point (weighted Fermat-Torricelli point), such that the sum of the weighted distances to these three points (length of helices) is minimized. By applying the mechanical device of Pick and Polya the oscillatory tree solution is a new solution w.r to the weighted Fermat-Torricelli problem for a given isosceles geodesic triangle on S with corresponding two equal weights at the vertices of the base segment. It is worth mentioning that after time t, the oscillatory knot of the mechanical system passes through the weighted Fermat-Torricelli point with non zero velocity. Furthermore, we give a numerical example to verify the structure of an oscillatory Fermat-Torricelli tree for a given isosceles triangle with equal weights.

1. Introduction

We start by stating the weighted Fermat-Torricelli problem in $\mathbb{R}^2$.

Problem 1. *Given three points $A_i = (x_i, y_i)$, and a positive real number (weight) w_i, which correspond to each A_i, for $i = 1, 2, 3$, find a point O which minimizes the objective function*

$$f(x,y) = \sum_{i=1}^{3} w_i \sqrt{(x-x_j)^2 + (y-y_j)^2} \tag{1.1}$$

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The solution of the weighted Fermat-Torricelli problem (Problem 1) is called the weighted Fermat-Torricelli tree, which consists of the union of three edges (branches) A_1O, A_2O, A_3O, which meet at the weighted Fermat-Torricelli point O.

By replacing $w_1 = w_2 = w_3$ in (1.1), we obtain the (unweighted) Fermat-Torricelli tree. The (unweighted)Fermat-Torricelli problem was first stated by Pierre de Fermat (1643) and first solved by E. Torricelli.

The existence and uniqueness of the weighted Fermat-Torricelli tree and a complete characterization of the "floating case" and "absorbed case" has been established by Y. S Kupitz and H. Martini (see [7], theorem 1.1, reformulation 1.2 page 58, theorem 8.5 page 76, 77). A particular case of this result for three non-collinear points in $\mathbb{R}^2$, is given by the following theorem:

Theorem 1. [1],[7] *Let there be given three non-collinear points* $A_1, A_2, A_3 \in \mathbb{R}^2$ *with corresponding positive weights* w_1, w_2, w_3.
(a) The weighted Fermat-Torricelli point O *exists and is unique.*
(b) If for each point $A_i \in \{A_1, A_2, A_3\}$

$$\| \sum_{j=1, i\neq j}^{3} w_j \vec{u}(A_i, A_j) \| > w_i, \tag{1.2}$$

for $i, j = 1, 2, 3$ *holds, then*
(b_1) the weighted Fermat-Torricelli point O *(weighted floating equilibrium point) does not belong to* $\{A_1, A_2, A_3\}$ *and*
(b_2)

$$\sum_{i=1}^{3} w_i \vec{u}(O, A_i) = \vec{0}, \tag{1.3}$$

where $\vec{u}(A_k, A_l)$ *is the unit vector from* A_k *to* A_l, *for* $k, l \in \{0, 1, 2, 3\}$ *(Weighted Floating Case).*
(c) If there is a point $A_i \in \{A_1, A_2, A_3\}$ *satisfying*

$$\| \sum_{j=1, i\neq j}^{3} w_j \vec{u}(A_i, A_j) \| \leq w_i, \tag{1.4}$$

then the weighted Fermat-Torricelli point O *(weighted absorbed point) coincides with the point* A_i *(Weighted Absorbed Case).*

By replacing $w_1 = w_2 = w_3$ in Theorem 1, we get:

Corollary 1. *If* $w_1 = w_2 = w_3$ *and all three angles of the triangle* $\triangle A_1A_2A_3$ *are less than* 120°, *then* O *is the isogonal point (interior point) of* $\triangle A_1A_2A_3$ *whose sight angle to every side of* $\triangle A_1A_2A_3$ *is* 120°.

Corollary 2. *If* $w_1 = w_2 = w_3$ *and one of the angles of the triangle* $\triangle A_1A_2A_3$ *is equal or greater than* 120°, *then* O *is the vertex of the obtuse angle of* $\triangle A_1A_2A_3$.

Regarding an excellent historical exposition for the solution of the weighted Fermat-Torricelli problem, we mention the works of [1], [7], [3] and [4] and for further generalizations two elegant classical works are given in [5] and [6].

In 2002, S Gueron and R. Tessler invented a mechanical solution in the sense of Polya and Varignon, by applying the following construction:

Suppose that $\{A_1, A_2, A_3\}$ lie on a horizontal table , and that holes are drilled at the vertices, where smooth pulleys are attached. The three massless strings referring to A_1, A_2 and A_3 that emanate from the knot O are passed through the pulleys, and three masses w_1, w_2, w_3, are suspended from the ends of these strings.

Assume that the system is released and reaches its mechanical equilibrium, and that the knot stops at the interior point O. Then, by applying the minimum energy principle at equilibrium, they obtain ta weighted Fermat-Torricelli tree solution. Thus, the mechanical equilibrium of the system gives the condition for vectorial balance:

$$w_1\vec{u}(O, A_1) + w_2\vec{u}(O, A_2) + w_3\vec{u}(O, A_3) = \vec{0}. \tag{1.5}$$

We note that the mechanical system reaches its mechanical equilibrium, by taking into account friction.

In 2014, we find the exact location of the weighted Fermat-Torricelli point for a geodesic triangle on flat surfaces of revolution (Circular cylinder, circular cone, Euclidean plane) ([11]).

In this paper, we generalize the mechanical solution of S. Gueron and R. Tessler, for the case of an isosceles geodesic triangle $\triangle A_1A_2A_3$ on a circular cylinder S where $w_1 = 1$ and $w_2 = w_3$ by introducing the oscillatory Fermat-Torricell tree solution of the corresponding mechanical system, by assuming it is frictionless.

We note that after time t the oscillatory knot of the mechanical system passes through the weighted Fermat-Torricelli point with non zero velocity, by releasing the mechanical system from the vertex A_1 with zero velocity.

Furthermore, we give a numerical example to verify the structure of an oscillatory Fermat-Torricelli tree for a given isosceles triangle with equal weights in $\mathbb{R}^2$.

2. A generalization of the mechanical solution of S. Gueron and R. Tessler

We take into account the parametric form of a (right) circular cylinder S of radius one and axis (axis of revolution) the z-axis:

$\vec{r}(u, v) = (\cos v, \sin v, u)$.

The geodesics of the circular cylinder are the straight lines on the circular cylinder parallel to the z-axis, the circles obtained by intersecting the circular cylinder with planes parallel to the xy-plane and circular helices of the parametric form $\vec{r}(t) = (\cos t, \sin t, bt + c)$.

Let $\triangle A_1A_2A_3$ be an isosceles geodesic triangle, which is composed of three circular helixes, such that $A_1A_2 = A_1A_3$.

Without loss of generality, we set:

$A_1 = (1, 0, 0)$, $A_2 = (\cos\omega_2, \sin\omega_2, z_2)$ and $A_3 = (\cos\omega_3, \sin\omega_3, z_3)$

$$\omega_2 > \omega_3$$

and

$$\omega_2, \omega_3 \in (0, \pi).$$

We denote by $\vec{r}_{ij} = (\cos t, \sin t, b_{ij} t)$ the circular helix on S from A_i to A_j for $i, j = 1, 2, 3, i \neq j$.

Thus, we get: $b_{12} = \frac{z_2}{\omega_2}$ and $b_{13} = \frac{z_3}{\omega_3}$.

The length of the circular helix from $\vec{r}_{23}$ from A_2 to A_3 is ([10, Example 1.4.1,pp. 15]):

$$(a_{23})_g = \sqrt{1 + b_{23}^2}(\omega_2 - \omega_3). \tag{2.1}$$

The length of the circular helix from $\vec{r}_{12}$ from A_1 to A_2 is :

$$(a_{12})_g = \sqrt{1 + b_{12}^2}(\omega_2). \tag{2.2}$$

The length of the circular helix from $\vec{r}_{13}$ from A_1 to A_3 is:

$$(a_{13})_g = \sqrt{1 + b_{13}^2}(\omega_3). \tag{2.3}$$

Taking into account (2.2)and (2.3), $(a_{12})_g = (a_{13})_g$ yields

$$\omega_3 = \omega_2 \frac{\sqrt{1 + b_{12}^2}}{\sqrt{1 + b_{13}^2}}. \tag{2.4}$$

$$b_{23} = \frac{b_{13}\omega_3 - b_{12}\omega_2}{\omega_2 - \omega_3} \tag{2.5}$$

By replacing (2.4) in (2.5), we obtain:

$$b_{23} = \frac{b_{13}\sqrt{1 + b_{12}^2} - b_{12}\sqrt{1 + b_{13}^2}}{\sqrt{1 + b_{13}^2} - \sqrt{1 + b_{12}^2}}. \tag{2.6}$$

We shall use the same mechanical system of S. Gueron and R. Tessler in the spirit of Pick and Polya, in order to solve the following mechanical problem:

Problem 2. *A (right) half circular cylinder is drilled with three holes corresponding to three given points A_1, A_2, A_3 which form an isosceles geodesic triangle $A_1A_2A_3$ where $A_1A_2 = A_1A_3 = a$ and three strings are tied together in a knot with mass m_0 at one knot and the loose ends are passed through the three holes attached to the physical weights $w_1 = 1$ from A_1, w_2 from A_2 and w_3 from A_3, where $w_2 = w_3$. If we release m_0 from A_1 with zero velocity find the motion of the knot O.*

Definition 1. *We call the motion of the knot with mass m_0 w.r. to the mechanical system of Problem 2 an oscillatory Fermat-Torricelli tree of three helices.*

We shall verify the oscillation of the knot with mass m_0 numerically in example 1.

We denote by O the corresponding weighted Fermat-Torricelli point of the isosceles geodesic triangle $\triangle A_1A_2A_3$ where $w_2 = w_3$ and $w_1 = 1$. The point O belongs to the height (helix) A_1A_4 w.r. to the base A_2A_3. By applying theorem 1 for $w_2 = w_3$ and $w_1 = 1$ we get:

Lemma 1. *If* $\angle A_4A_1A_3 < \frac{\arccos(\frac{1}{2w_2^2}-1)}{2}$,
then the weighted Fermat-Torricelli point O *of* $\triangle A_1A_2A_3$, *belongs to the height* A_1A_4 *w.r. to the base* A_2A_3 *and*

$$\angle A_4OA_3 = \frac{\arccos(\frac{1}{2w_2^2}-1)}{2}. \tag{2.7}$$

We assume that $\angle A_4A_1A_3 < \frac{\arccos(\frac{1}{2w_2^2}-1)}{2}$, such that Lemma 1 holds.

Suppose that we release mass m_0 from the vertex A_1 with zero velocity $\dot{x}(0) = 0$. After time t, m_0 reaches at the point S which lies on A_1O, because $F_2 = F_3 = w_2$ Thus, the knot will move along the geodesic arc defined by A_1O (helix) via the force $\vec{F}_{23} - \vec{F}_1$, such that $\vec{F}_{23} = \vec{F}_2 + \vec{F}_3$ and $\Delta F = F_{23} - F_1 = 2\cos\angle OSA_3 - 1$

We set $\angle OSA_3 := \phi(t)$, $x[t] := A_1S$ and $\angle OA_1A_3 := \phi(0)$.

Theorem 2. *The solution of the mechanical system of Problem 2 is an oscillatory Fermat-Torricelli tree of three helices which is described by the motion of the knot with mass* m_0 *along the helix defined by the* A_1O *having non zero velocity after time* t_0 *at the weighted Fermat-Torricelli point* O *of* $\triangle A_1A_2A_3$ *which is given by:*

$$\dot{x}(t_0) = \sqrt{\frac{2}{m_0}(2w_2\sqrt{1+b_{13}^2}\omega_3 - x(t_0) - \frac{w_2\sqrt{1+b_{23}^2}(\omega_2-\omega_3)}{\sin(\angle A_4OA_3)})}. \tag{2.8}$$

Proof. Unrolling the cylinder S in terms of the vertex A_1, we obtain an isometric mapping from S to $\mathbb{R}^2$ (see fig. 1) Therefore, we get:

$$(a_{ij})_g = (a_{ij})_0, for \quad i,j = 1,2,3, \tag{2.9}$$

$$A_1 = (0,0),$$
$$A_2 = (\omega_2, z_2)$$
$$A_3 = (\omega_3, z_3)$$
$$A_4 = (\frac{\omega_2+\omega_3}{2}, \frac{z_2+z_3}{2},)$$
$$O = (\omega_0, z_0),$$

and

$$\omega_2 > \omega_3.$$

By applying the minimum energy principle, the knot will move along the helix A_1S, because the energy of the path is minimized on the geodesic arc A_1A_4 ([9, PartIII, 12,pp.70-73]). Minimization of energy functionals yield

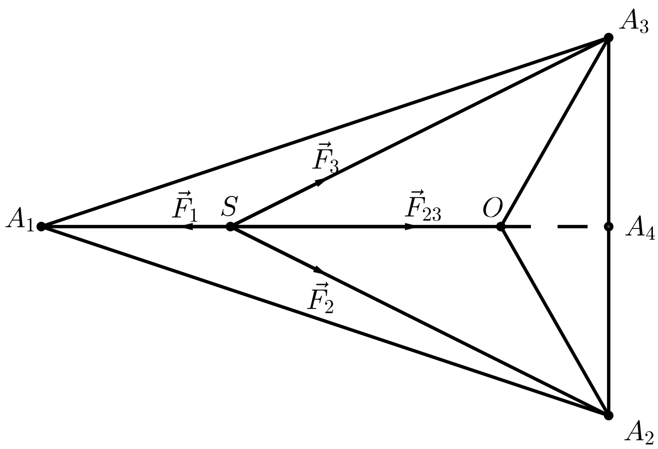

FIGURE 1

a minimization of length of functionals and a balancing condition at the corresponding weighted Fermat-Torricelli point ([8, Theorem 4]). We shall show that the knot passes through O with non-zero velocity.

By applying the sine law in $\triangle SOA_2$, we obtain:

$$\frac{OS}{\sin(\frac{\angle A_4OA_3}{2} - \phi(t))} = \frac{OA_3}{\sin\phi(t)} \tag{2.10}$$

where

$$OS = a(\cos\phi(0) - \frac{\sin\phi(0)}{\tan(\frac{\angle A_4OA_3}{2})}) - x \tag{2.11}$$

By replacing (2.11) in (2.10), we get:

$$x(t) = a\cos\phi(0) - a\sin\phi(0)\cot\phi(t). \tag{2.12}$$

At time t the force along the path A_1O is given by:

$$m_0\ddot{x} = 2w_2\cos\phi(t) - 1. \tag{2.13}$$

By differentiating (2.12), we have:

$$dx = \frac{a\sin\phi(0)}{\sin^2\phi(t)}d\phi \tag{2.14}$$

Differentiating twice $x(t)$ w.r. to t yields:

$$\ddot{x} = \dot{x}\frac{d\dot{x}}{dx}.$$

Thus, by integrating both parts of (2.13) from A_1 to S, w.r. to x and taking into account (2.12), we obtain:

$$m_0 \frac{(\dot{x}(t))^2 - \dot{x}(0))^2}{2} = \int_{\phi(0)}^{\phi} 2aw_2 \sin\phi(0) \frac{\cos\phi}{\sin^2\phi(t)} d\phi - \int_0^x dx \tag{2.15}$$

or

$$m_0 \frac{(\dot{x}(t))^2}{2} = 2aw_2 - 2aw_2 \frac{\sin\phi(0)}{\sin\phi} - \int_0^x dx \tag{2.16}$$

By setting $t = t_0$, taking into account that (2.9) and by replacing (2.3), (2.1) in (2.16), we obtain (2.8).

□

Remark 1. *Substituting (2.6) in (2.8), the velocity $\dot{x}(t_0)$ depends on b_{13}, ω_2, ω_3 and w_2.*

Corollary 3. *The solution of the mechanical system of Problem 2 in $\mathbb{R}^2$ is an oscillatory Fermat-Torricelli tree, which is described by the motion of the knot with mass m_0 along the line defined by the A_1O having non zero velocity after time t_0 at the weighted Fermat-Torricelli point O of $\triangle A_1A_2A_3$ which is given by:*

$$\dot{x}(t_0) = \sqrt{\frac{2}{m_0}(2aw_2 - x(t_0) - \frac{2aw_2 \sin\phi(0)}{\sin(\angle A_4OA_3)})}. \tag{2.17}$$

Corollary 4. *For $w_2 = w_3 = 1$, the velocity of the knot with mass m_0 which passes through the unweighted Fermat-Torricelli point is given by:*

$$\dot{x}(t_0) = \sqrt{\frac{2}{m_0}(2 - x(t_0) - \frac{4a\sin\phi(0)}{\sqrt{3}})}. \tag{2.18}$$

Proof. By replacing $w_2 = w_3 = 1$, and $\phi = 60^\circ$, we derive (2.18). □

Proposition 1. *The movement of the mechanical system is determined by the following differential equation:*

$$m_0(\frac{a\sin\phi(0)}{\sin^2\phi(t)}\ddot{\phi} - 2a\frac{\sin\phi(0)}{\sin^3\phi}\cos\phi\dot{\phi}) = 2w_2\cos\phi - 1 \tag{2.19}$$

with initial conditions $\phi(0) = \phi_0$ and $\dot{\phi} = 0$.

Proof. Differentiating twice (2.12) and by replacing in (2.13) we derive (2.19). □

Proposition 2. *The work of the force of the mechanical system along the path A_1O starting from A_1 with $\dot{x}(0) = 0$ is given by:*

$$W = 2w_2(a - A_2O) - A_1O \neq 0. \tag{2.20}$$

Proof. We start with the work of the force $F_{23} - F_1$ along A_1O :

$$W = \int_0^{A_1O} (F_{23} - F_1)dx. \tag{2.21}$$

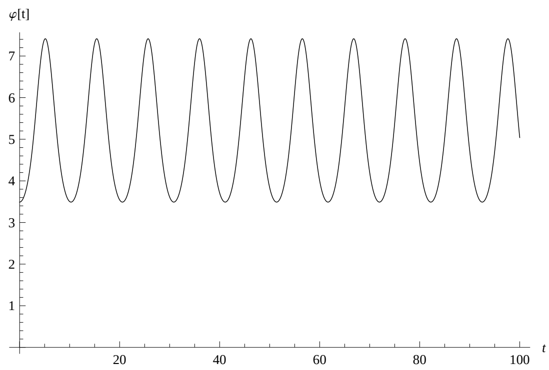

FIGURE 2

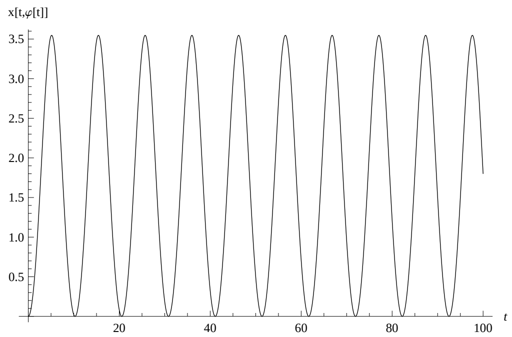

FIGURE 3

By replacing (2.14) in (2.21), we get:

$$W = \int_{\phi(0)}^{\angle A_4OA_3} 2w_2 \cos\phi \frac{a \sin\phi(0)}{\sin^2\phi} d\phi - A_1O, \tag{2.22}$$

or

$$W = 2w_2(a - a\frac{\sin\phi(0)}{\sin\angle A_4OA_3}) - 1$$

which yields (2.20).

□

For $w_2 = w_3 = 1$, we get:

Corollary 5. *The work of the force of the mechanical system along the path A_1O starting from A_1 with $\dot{x}(0) = 0$ is given by:*

$$W = 2(a - A_2O) - A_1O \neq 0, \tag{2.23}$$

where O is the unweighted Fermat-Torricelli point.

Example 1. *Given an isosceles triangle $\triangle A_! A_2 A_3$ where $a = 5$, $\phi(0) = 40°$, $w_1 = w_2 = w_3 = 1$, we derive that $\angle A_4OA_3 = 60°$.*

Suppose that we release mass m_0 from the vertex A_1 with zero velocity $\dot{x}(0) = 0$. After time t, m_0 reaches at the point S which lies on A_1O. By replacing $a, \phi(0)$ in (2.19), we obtain a numerical solution using Mathematica of $\phi(t) and x(t)$ (see fig. 1, 2).

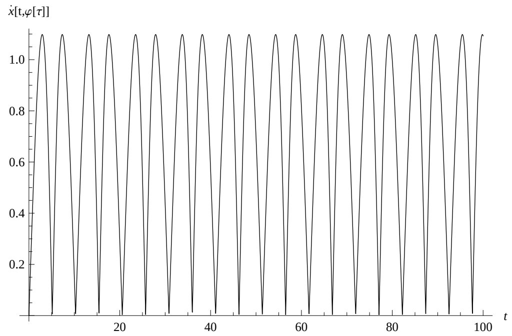

FIGURE 4

By replacing $a, \phi(0), m_0 = 1$ in (2.18), we derive a numerical solution using Mathematica of $\dot{x}(t)$ (see fig. 4).

$$x[t] \approx 1.77363 + 1.77363 \sin(0.61133(-2.56947 + t))$$

$$\dot{x}(t) \approx \|1.08427 \cos(0.61133(-2.56947 + t))\|$$

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