



LOCI OF THE BROCARD POINTS OVER SELECTED TRIANGLE FAMILIES

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Abstract. We study the loci of the Brocard points over two selected families of triangles: (i) 2 vertices fixed on a circumference and a third one which sweeps it, (ii) Poncelet 3-periodics in the homothetic ellipse pair. Loci obtained include circles, ellipses, and teardrop-like curves. We derive expressions for both curves and their areas. We also study the locus of the vertices of Brocard triangles over the homothetic and Brocard-poristic Poncelet families.

1. INTRODUCTION

Keywords and phrases: (2021)Mathematics Subject Classification: The Brocard points Ω_1 and Ω_2 [10] are a well-studied pair of points interior to a triangle, introduced by August L. Crelle in 1816, given a construction by Karl F.A. Jacobi in 1825, and rediscovered by Henri Brocard in 1875 [17, Brocard Points]; see Figure 1.

A subject less studied is their loci over selected triangle families and whether these possess interesting properties. We consider the following two families, and describe rather interesting properties of their Brocard loci:

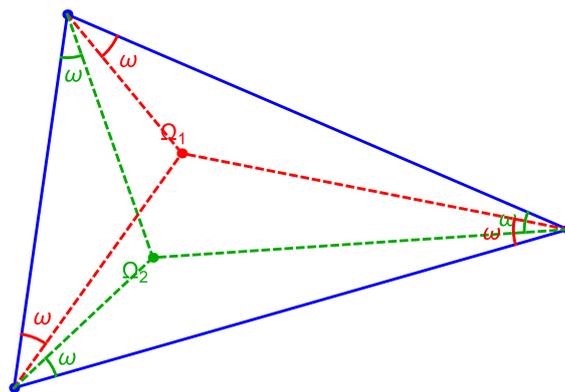


FIGURE 1. The Brocard Points Ω_1 (resp. Ω_2) are where sides of a triangle concur when rotated about each vertex by the Brocard angle Ω . When sides are traversed and rotated clockwise (resp. counterclockwise), one obtains Ω_1 (resp. Ω_2).

Keywords and phrases: Poncelet, Brocard, Homothetic, Porism, Locus.

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- Circle-Mounted: two vertices fixed at points on a circumference (or ellipse boundary) and a third one which slides over it.
- Homothetic Pair: Poncelet 3-Periodics interscribed in a homothetic, concentric pair of ellipses. This family conserves the Brocard angle [15].

Main Results. We (i) derive explicit expressions for the Brocard loci over the above families, (ii) study their areas with respect to the curves in which they are inscribed, (iii) study the locus of vertices of the Brocard Triangles¹ (there are 7 of them, see [9]) under two families: homothetic and Brocardporistic (whose Brocard points are stationary [1]). In the former case the only the vertices of the First Brocard is an ellipse, and in the latter 4 out of 7 describe circles, both results stemming directly from known homotheties of said triangles.

Related Work. Ferréol describes a construction for a right strophoid as the locus of the orthocenter of a family of triangles with two fixed vertices and a third one revolving on a circumference [5]. Odehnal has studied loci of triangle centers for the poristic triangle family [12]; similarly, Pamfilos proves properties of the family of triangles with fixed 9-point and circumcircle [13]. We have studied the loci of triangle centers for 3-periodics in the elliptic billiard (confocal Poncelet pair), identifying a few centers whose loci are elliptic [7], and built and interactive app to visualize loci of centers of several triangle families [3]. Loci of triangle centers over other Poncelet families are studied in [6]. In [14] we study an infinite, converging sequence of Brocard porisms brought about by successive computations of the second Brocard triangle of a given family.

Below when referring to triangle centers we adopt Kimberling's X_k notation [11].

2. CIRCLE-MOUNTED TRIANGLES

Let a family of triangles be defined with two vertices V_1, V_2 stationary with respect to a circle of radius a (say centered at the origin) and a third one $P(t)$ which executes on revolution over the the circumference, $P(t) = a[\cos t, \sin t]$.

Referring to Figure 2 (left):

Proposition 2.1. *The locus of Ω_1 (resp. Ω_2) with $V_1 = (0, 0)$ and $V_2 = (0, a)$ is a circle of radius $\frac{a}{3}$ (resp. a teardrop curve) of area $\frac{\pi a^2}{9}$ (resp. $\frac{2\pi a^2}{9}$).*

Proof.

In this case we have that:

$$\Omega_1(t) = a \left[\frac{\cos t}{5 - 4 \sin t}, \frac{2 - a \sin t}{5 - 4 \sin t} \right]$$

$$\Omega_2(t) = a \left[\frac{2 \cos t - \sin 2t}{5 - 4 \sin t}, \frac{2 \sin t + \cos 2t}{5 - 4 \sin t} \right]$$

¹The 7th Brocard was invented by one author and Peter Moses during this research: its vertices are the intersections of cevians through X_3 with the Brocard Circle. The inspiration was the 2nd Brocard whose vertices lie at intersections of cevians through X_6 with said circle [17, Second Brocard Triangle].

Remark 2.1. *The above loci intersect at $a[\pm\sqrt{3}/6, 1/2]$; along with $V_2 = (0, a)$ they define an equilateral. This stems from the fact that when $P(t) = a[\pm\sqrt{3}/2, 1/2]$, $V_1V_2P(t)$ is equilateral and the two Brocard points coincide at the Barycenter X_2 .*

Referring to Figure 2 (right):

Proposition 2.2. *The locus of Ω_1 and Ω_2 with $V_1 = (0, a)$ and $V_2 = (a, 0)$ are a pair of inversely-identical², skewed teardrop shapes with the following equations:*

$$\Omega_1(t) = a \left[\frac{\sin^2 t + \cos t - \sin t}{(\sin t - 2) \cos t - 2 \sin t + 3}, \frac{1 - \cos t}{(\sin t - 2) \cos t - 2 \sin t + 3} \right]$$

$$\Omega_2(t) = a \left[\frac{1 - \sin t}{(\cos t - 2) \sin t - 2 \cos t + 3}, \frac{(\cos^2 t - \cos t + \sin t)}{(\sin t - 2) \cos t - 2 \sin t + 3} \right]$$

Remark 2.2. *Let $D(x, y) = (y, x)$, i.e., the reflection about the diagonal. It follows that $\Omega_2(t) = (D \circ \Omega_1)(t - \frac{\pi}{2})$.*

In Figure 3 we show the shape of the locus varies in a complicated way when $V_2 = (0, a)$ and $V_1 = (x, 0)$, with $0 \leq x \leq a$.

Referring to Figure 4 (left), Robert Ferréol has kindly contributed [4]:

Proposition 2.3. *With $V_1 = (-a, 0)$ and $V_2 = (a, 0)$, the loci of the Brocards are a pair of inversely-identical teardrop shapes whose areas are $\pi a^2/\sqrt{5}$. The one with a cusp on V_1 is given by the following quartic:*

$$(1) \quad x^4 - 2x^3 + 2y^2x^2 + 2x - 2y^2x - 1 + y^4 + 4y^2 = 0$$

Proof.

$$\Omega_1(t) = \left[\frac{-a \cos 2t - 8a \cos t + a}{\cos 2t - 9}, \frac{-2a \sin 2t - 4a \sin t}{\cos 2t - 9} \right]$$

$$\Omega_2(t) = \left[\frac{-8a \cos t + a \cos 2t - a}{\cos(2u) - 9}, \frac{2a \sin 2t - 4a \sin t}{\cos 2t - 9} \right]$$

Let $R(x, y) = (-x, y)$. Then $\Omega_2(t) = (R \circ \Omega_1)(t)$. The implicit form of Ω_2 is given by

$$B_2(x, y) = a^2(a^2 - 2ax - 4y^2) + 2ax(x^2 + y^2) - (x^2 + y^2)^2 = 0.$$

Analogously, $B_1(x, y) = B_2(-x, y) = 0$ is the implicit form of Ω_1 .

The area of the region bounded by Ω_i is given by $\frac{1}{2} \int_{\Omega_i} xdy - ydx$. It follows that $A(\Omega_i) = \frac{\sqrt{5}\pi a^2}{5}$.

Figure 4 (right) depicts the loci of Ω_1 and Ω_2 with $P(t)$ on an ellipse with semi-axes (a, b) and with $V_1 = (-a, 0)$ and $V_2 = (a, 0)$. These are a pair of symmetric teardrop curves whose complicated parametric equations we omit.

²Identical modulo inverse similarity [17, Inversely Similar].

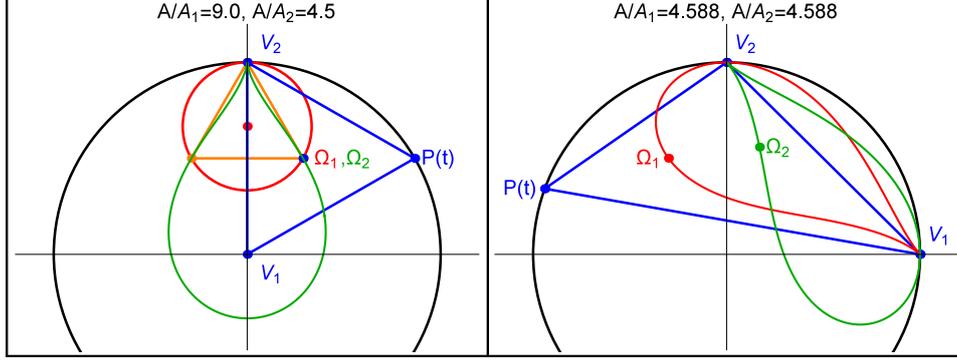


FIGURE 2. **Left:** V_1 and V_2 are affixed to the center and top vertex of the unit circle and a third one $P(t)$ revolves around the circumference. The locus of the Brocard points Ω_1, Ω_2 are a circle (red) and a teardrop (green) whose areas are $1/9$ and $2/9$ that of the generating circle. The sample triangle (blue) shown is equilateral, so the two Brocard points coincide. Notice the curves' two intersections along with the top vertex form an equilateral (orange). **Right:** V_1, V_2 are now placed at the left and top vertices of the unit circle. The Brocard points of the family describe to inversely-similar teardrop shapes. [Video](#), [Live](#)

Proposition 2.4. *The locus of Ω_1 and Ω_2 with $V_1 = (x_1, 0)$, $|x_1| \leq a$, $V_2 = (-a, 0)$ and $V_3 = (a \cos t, a \sin t)$ are a pair of singular teardrop curves with the following areas:*

$$A_1 = \frac{4(x_1 + a)^2 a^5 \pi}{(3a^2 + x_1^2)^2 \sqrt{4a^2 + x_1^2}}$$

$$A_2 = \frac{(2a^2 - ax_1 + x_1^2)(x_1 + a)^3 a^2 \pi}{(3a^2 + x_1^2)^2 \sqrt{4a^2 + x_1^2}}$$

When $x_1 = a$, the ratio of A_1 and A_2 by the area of the circle $a^2 \pi$ both reduce to $1/\sqrt{5} \simeq 0.4472$.

Proof. The above is obtained with direct integration and simplification with a computer algebra system (CAS).

3. HOMOTHETIC PAIR LOCI

Consider an origin-centered ellipse with semi-axes (a, b) and an internal concentric, axis-aligned one with semi-axes $(a', b') = (a/2, b/2)$. This pair is associated with a 3-periodic Poncelet porism since the condition $a'/a + b'/b = 1$ is satisfied [8].

Referring to Figure 5:

Proposition 3.1. *The loci of the Brocard points over 3-periodics in the homothetic pair are ellipses E_1 and E_2 which are reflected images of each other about either the x or y axis. Furthermore these are concentric and homothetic to the ellipses in the pair.*

Proof. The loci are given by

$$E_1(x, y) = \frac{(7a^4 + 6a^2b^2 + 3b^4)x^2}{a^2(a^2 - b^2)^2} + \frac{(3a^4 + 6a^2b^2 + 7b^4)y^2}{b^2(a^2 - b^2)^2} - \frac{4\sqrt{3}(a^2 + b^2)xy}{ab(a^2 - b^2)} - 1$$

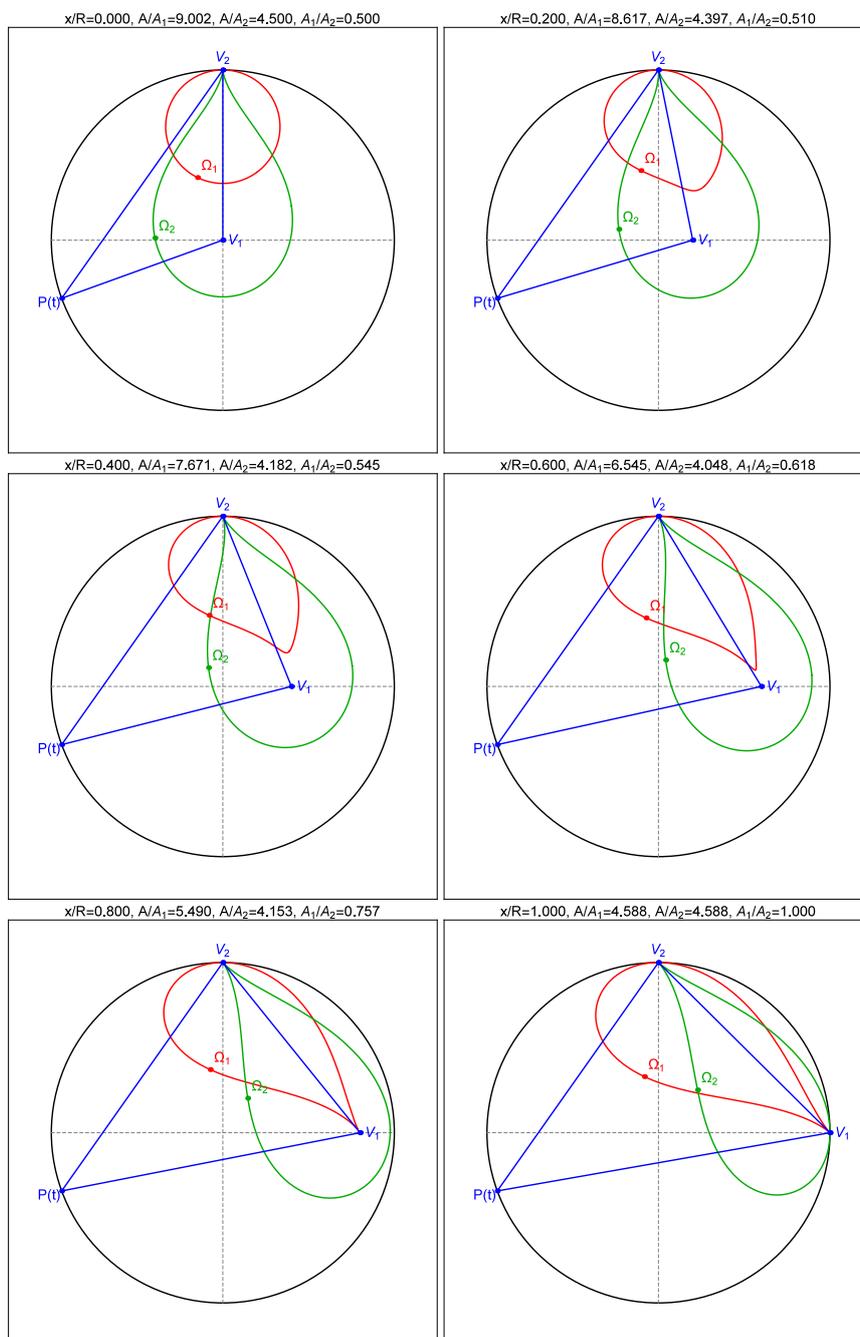


FIGURE 3. Shape of loci of Brocard Point Ω_1 (red) and Ω_2 (green) for V_2 fixed at $(0, 1)$, as V_1 moves from the origin along the x axis toward $(1, 0)$. The loci are obtained over a complete revolution of $P(t)$ on a unit circle (black). **Top left:** $V_1 = (0, 0)$, the locus of Ω_1 (resp. Ω_2) is a perfect circle (resp. a teardrop curve) of $1/9$ (resp. $2/9$) the area of the external. **Bottom right:** when $V_1 = (a, 0)$ the two loci are inversely-similar copies of each other, whose areas are $1/\sqrt{5} \approx 0.447$ that of the circle. [Video](#), [Live](#)

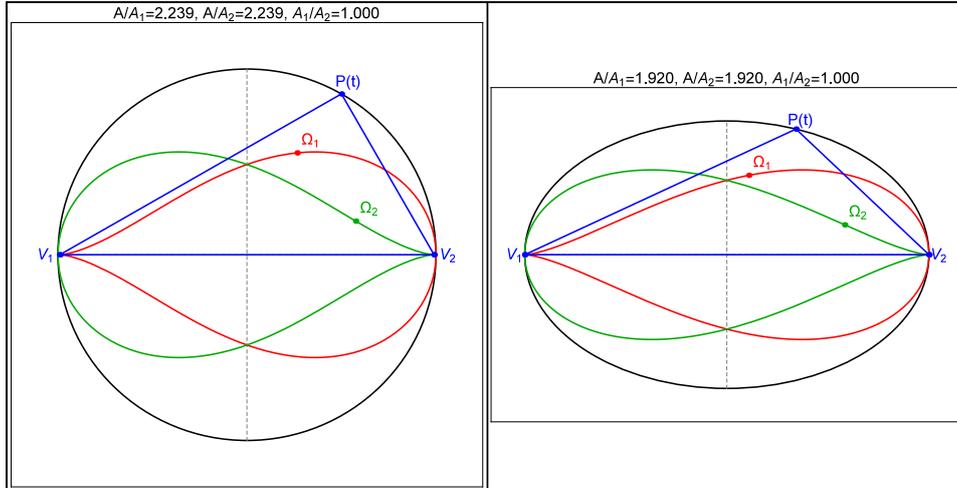


FIGURE 4. **Left:** With antipodal V_1 and V_2 and $P(t)$ revolving on the circumference, the loci of the Brocards are symmetric teardrops whose area are $1/\sqrt{5}$ that of the circle. **Right:** With V_1, V_2 at the major vertices of an ellipse of axes (a, b) , and $P(t)$ revolving on its boundary, the the Brocard loci (red and green) are still symmetric (though stretched) teardrop shapes. In this case $a/b = 1.5$. [Live](#)

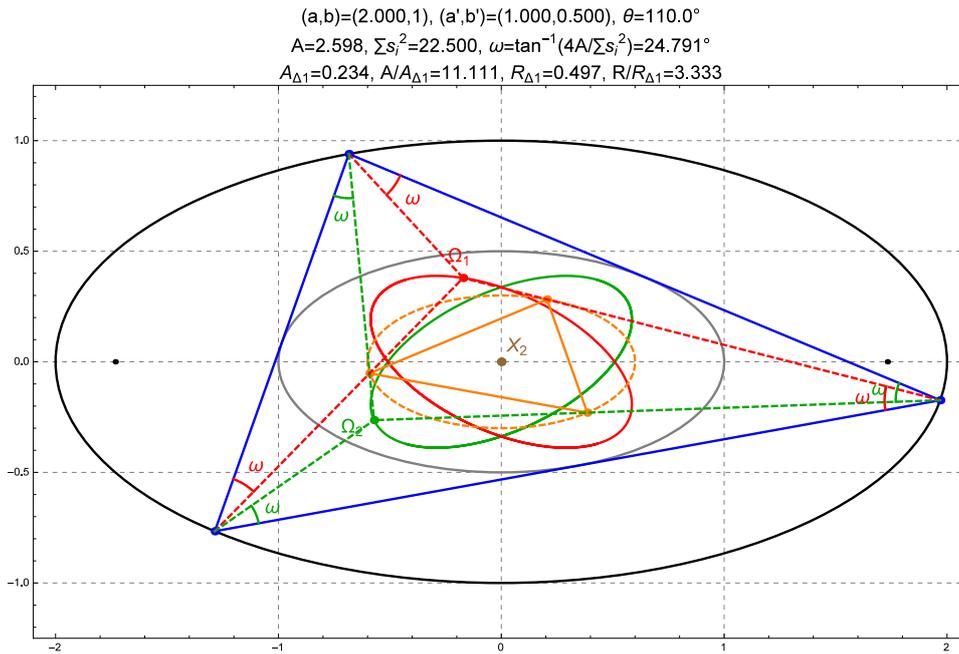


FIGURE 5. The 1d Poncelet 3-periodic family interscribed in the homothetic pair conserves sum of squared sidelengths, area, and Brocard angle ω [15]. The loci of the two Brocard points Ω_1 and Ω_2 are tilted ellipses (red and green) of aspect ratio equal to those in the pair [Video](#). The locus (dashed orange) of the vertices of the first Brocard triangle (orange) is an axis-aligned ellipse also homothetic to the pair. [Video](#), [Live](#)

$$E_2(x, y) = \frac{(7a^4 + 6a^2b^2 + 3b^4)x^2}{a^2(a^2 - b^2)^2} + \frac{(3a^4 + 6a^2b^2 + 7b^4)y^2}{b^2(a^2 - b^2)^2} + \frac{4\sqrt{3}(a^2 + b^2)xy}{ab(a^2 - b^2)} - 1$$

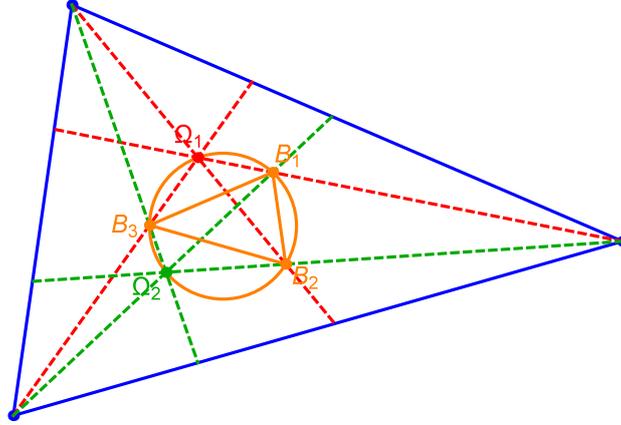


FIGURE 6. Construction for the First Brocard Triangle (orange) taken from [17, First Brocard Triangle]. It is inversely similar to the reference one (blue), and their barycenters X_2 are common. Its vertices B_1, B_2, B_3 are concyclic with the Brocard points Ω_1 and Ω_2 on the Brocard circle (orange).

The angle θ between the axes of ellipses E_1 and E_2 is given by

$$\tan \theta = \frac{4\sqrt{3}(a^2 + b^2)ab}{3a^4 + 2a^2b^2 + 3b^4}.$$

In no other concentric Poncelet pairs studied so far (poristic, incircle, inellipse, dual, confocal) is the locus of either Brocard point an ellipse.

Remark 3.1. *At $a/b = \sqrt{5}$ the elliptic loci of the Brocard points over the homothetic family are internally tangent to the inner ellipse.*

Remark 3.2. *At $a/b \simeq 3.8$ the elliptic loci of the Brocard points over the homothetic family intersect the y axis at $b/2$, i.e., at the top vertex of the caustic.*

In Appendix A we derive a few properties of the Brocard circle with respect to the homothetic family.

4. BROCARD TRIANGLES

Consider a triangle $T = P_1P_2P_3$ with Brocard points Ω_1 and Ω_2 . Referring to Figure 6:

Definition 4.1 (First Brocard Triangle). *The vertices P'_1, P'_2, P'_3 of the First Brocard Triangle T_1 are defined as follows: P'_1 (resp. P'_2, P'_3) is the intersection of $P_2\Omega_1$ (resp. $P_3\Omega_1, P_1\Omega_1$) with $P_3\Omega_2$ (resp. $P_1\Omega_2, P_2\Omega_2$).*

Known properties of the T_1 include that (i) it is inversely similar to T , (ii) its barycenter X_2 coincides with that of the reference triangle, and (iii) its vertices are concyclic with Ω_1, Ω_2, X_3 , and X_6 on the Brocard circle [17, Brocard Circle], whose center is X_{182} .

4.1. **Homothetic Pair.** Referring to Figure 5:

Proposition 4.1. *Over 3-periodics in the homothetic pair, the locus of the vertices of T_1 is an axis-aligned, concentric ellipse, homothetic to the ones in the pair and interior to the caustic. Its axes are given by:*

$$a' = \frac{a(a^2 - b^2)}{2(a^2 + b^2)}, \quad b' = \frac{b(a^2 - b^2)}{2(a^2 + b^2)}$$

Proof. The locus must be an ellipse since T_1 is inversely similar to the 3-periodics whose vertices are inscribed in an ellipse and their barycenters coincide. A vertex of the Brocard triangle is parametrized by

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$$

It can be shown $a' < a/2$ and $b' < b/2$ therefore the locus is interior to the caustic, i.e., the stationary Steiner inellipse.

Since homothetic 3-periodics conserve area [15], so must T_1 (inversely similar). Its area can be computed explicitly:

Remark 4.1. *Over 3-periodics in the homothetic pair, the area of T_1 is invariant and given by*

$$\frac{3\sqrt{3}ab(a^2 - b^2)^2}{16(a^2 + b^2)^2}$$

Corollary 4.1. *The similarity ratio of homothetic 3-periodics to the T_1 is invariant and given by*

$$\frac{2(a^2 + b^2)}{a^2 - b^2}.$$

4.2. Vertices of the T_1 over the homothetic family. Let the $P_1P_2P_3$ be the vertices of a 3-periodic in the homothetic pair, and $P'_1P'_2P'_3$ those of T_1 . These are given by:

$$\begin{aligned} P'_1 &= k_1 R_x P_2 \\ P'_2 &= k_1 R_x P_3 \\ P'_3 &= k_1 R_x P_1 \end{aligned}$$

where $k_1 = \frac{a^2 - b^2}{2(a^2 + b^2)}$ and $R_x(x, y) = (-x, y)$ is a reflection.

4.3. Brocard Porism. This is a 3-periodic family inscribed in a fixed circumcircle and a fixed inellipse, known as the Brocard Inellipse [17, Brocard Inellipse]. A remarkable property of this family³ is that the Brocard points are stationary at the foci of said inellipse [1].

Given the axes (a, b) of the latter, we have shown elsewhere that the circumcenter X_3 , circumradius R , and (conserved) Brocard angle ω of the family are given by [15]:

³Polygon porisms with fixed symmedian and Brocard points were discovered and studied in the late XIX century, and were known as “harmonic” polygons [2, 16]. The Brocard porism is the $N = 3$ case of such families.

The first remark implies a “russian doll” nesting of Brocard porisms composed of the original family and successive second Brocard triangles, each giving rise to its own private porism, treated in [14]; see this [video](#).

5. CONCLUSION

A few questions are posed to the reader:

- Are there other Poncelet ellipse pairs, concentric or not, whose Brocard points of their 3-periodics trace out ellipses? (so far he have found none)
- In the homothetic pair are there other derived triangles (besides the First Brocard), non-homothetic to the 3-periodics, whose vertices trace out conics?

A list of animations of some of the results above appears on Table 1.

Exp	Title	Video youtu.be/<.>	App bit.ly/<.>
01	Loci of Brocard Points on Circle-Mounted Triangles: Center-Top	Ms8jC9y0KU4	3jX6FII
02	Loci of Brocard Points on Circle-Mounted Triangles: Left-Right	n/a	3m53uQT
03	Loci of the Brocard Points and Vertices of 1st Brocard Triangle over the Homothetic Family	13i3JGY-fK4	3iaISog
04	Circular locus of Brocard Triangles, T_j , $j = 1, 2, 5, 7$	_bK-BCQv24A	32GFvQu
05	Russian-Doll Nesting of Second Brocard Triangles	Z3Y1EbCFbnA	n/a

TABLE 1. Animations of phenomena, the last two columns contain links to YouTube videos and/or a browser-based simulation.

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APPENDIX A. BROCARD CIRCLE OF THE HOMOTHETIC PAIR

Recall the Brocard circle is the circumcircle of the T_1 [17, Brocard Circle].

Corollary A.1. *Over 3-periodics in the homothetic pair, the ratio of areas of the circumcircle of 3-periodics to that of the Brocard circle is invariant and given by*

$$\frac{4(a^2 + b^2)^2}{(a^2 - b^2)^2}.$$

Proof. The result follows from Corollary 4.1. Straightforward calculations yield that the circumradius of 3-periodics is given by:

$$R^2 = -\frac{(a^2 - b^2)^3 \cos(6t)}{32a^2b^2} + \frac{(a^2 + b^2)(a^4 + 14a^2b^2 + b^4)}{32a^2b^2}.$$

Proposition A.1. *Over the homothetic family, the Brocard circle is given by:*

$$E(x, y) = \frac{4(a^2 + b^2)^2 x^2}{a^2(a^2 - b^2)^2} + \frac{4(a^2 + b^2)^2 y^2}{b^2(a^2 - b^2)^2} - 1$$

REFERENCES

- [1] Bradley, C., Smith, G., *On a construction of Hagge*, Forum Geometricorum, **7** (2007), 231–247.
- [2] Casey, J., *A sequel to the first six books of the Elements of Euclid*, Hodges, Figgis & Co., 5th ed., Dublin, 1888.
- [3] Darlan, I., Reznik, D., *An app for visual exploration, discovery, and sharing of Poncelet 3-periodic phenomena*, (2021), [arXiv:2106.04521](https://arxiv.org/abs/2106.04521).
- [4] Ferréol, R., *Brocard's teardrop equation*, Private Communication, (2021).
- [5] Ferréol, R., *Right strophoid*, Mathcurve Portal, (2020), <https://bit.ly/3xcYi3n>.
- [6] Garcia, R., Reznik, D., *Discovering Poncelet Invariants in the Plane*, IMPA Editora, Rio de Janeiro, 2021. ISBN:9786589124436.
- [7] Garcia, R., Reznik, D., Koiller, J., *Loci of 3-periodics in an elliptic billiard: why so many ellipses?*, (2020). [arXiv:2001.08041](https://arxiv.org/abs/2001.08041).
- [8] Georgiev, V., Nedyalkova, V., *Poncelet's porism and periodic triangles in ellipses*, Dynamat, (2012), <https://bit.ly/2G33bFQ>.
- [9] Gibert, B., *Brocard triangles and related cubics*, Catalogue of Triangle Cubics (CTC), (2020), <https://bit.ly/3kW1ZFK>.
- [10] Johnson, R. A., *Advanced Euclidean Geometry (2nd. ed)*, John W. Young (editor), Dover, New York, 1960.
- [11] Kimberling, C., *Encyclopedia of triangle centers (ETC)*, (2021), <https://bit.ly/3m00ver>.
- [12] Odehnal, B., *Poristic loci of triangle centers*, J. Geom. Graph., **15:1** (2011), 45–67.
- [13] Pamfilos, P., *Triangles sharing their Euler circle and circumcircle*, Intl. J. of Geom., **9:1** (2020), 5–24.
- [14] Reznik, D., Garcia, R., *An infinite, converging, sequence of Brocard porisms*, (2020), [arXiv:2010.01391](https://arxiv.org/abs/2010.01391).
- [15] Reznik, D., Garcia, R., *Related by similarity II: Poncelet 3-periodics in the homothetic pair and the Brocard porism*, Intl. J. of Geom., **10:4** (2021), 18–31.
- [16] Simmons, T. C., *The recent geometry of the triangle: Cosymmedian and co-Brocardal triangles*. In: Companion to the Weekly Problem Papers ((J. Milne, ed.), chapt. VII–IX. McMillan, London, 1888. pp. 147–184.
- [17] Weisstein, E., *MathWorld—A Wolfram Web Resource*, (2021), mathworld.wolfram.com.

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