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DYNAMIC GEOMETRY OF KASNER TRIANGLES WITH A FIXED WEIGHT

D. ANDRICA, O. BAGDASAR, and D.-Ş. MARINESCU

Abstract. In this article we consider the geometrical iterative process defined by the Kasner triangles with a fixed real weight. The main results about the convergence of this process are given in Section 2. The proofs are elementary and use the general theory of the second order linear recurrences. In the remark after the proof a higher level approach is presented.

1. Introduction

Given a fixed plane configuration $\mathcal{F}_0$ and a sequence of plane transformations $(T_n)_{n\geq 0}$, we consider the iterative process described by

$$\mathcal{F}_0 \xrightarrow{T_0} \mathcal{F}_1 \xrightarrow{T_1} \mathcal{F}_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} \mathcal{F}_n \xrightarrow{T_n} \mathcal{F}_{n+1} \xrightarrow{T_{n+1}} \cdots$$

This means that $\mathcal{F}_0$ is transformed by T_0 in $\mathcal{F}_1$ which is transformed by T_1 in $\mathcal{F}_2$, etc. Clearly, after n steps the initial configuration $\mathcal{F}_0$ is tranformed in $\mathcal{F}_n$ by the composition $T_{n-1}\circ T_{n-2}\circ\cdots\circ T_0$. We call a such iterative process the *dynamic geometry* generated by $\mathcal{F}_0$ and the sequence $(T_n)_{n\geq 0}$. The initial configuration $\mathcal{F}_0$ could be any general pattern defined using polygons (triangles, quadrilaterals, etc.), circles, and associated geometric elements.

There are some key problems arising in the study of a dynamic geometry:

(1) Describe the n-step configuration $\mathcal{F}_n$ and its geometric elements;
(2) Study the convergence of the sequence $(\mathcal{F}_n)_{n\geq 0}$;
(3) Study the convergence in shape of the sequence $(\mathcal{F}_n)_{n\geq 0}$;
(4) If the above sequence is convergent, find the convergence order of some of its elements;
(5) Obtain properties of the initial configuration $\mathcal{F}_0$ from the study of the geometry of $\mathcal{F}_n$ for some $n \geq 1$.

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In case of convergence, it seems that finding the limit of a certain iterative geometric process is a much more difficult problem than the one concerning the limiting shape. One reason is that in some configurations the angle computations are easier than computations involving distances, ratios, etc. Some examples of iterative processes inspired by simple geometrical configurations are reviewed in the expository article [4]: the Kasner triangles in various situations, the dynamic geometry generated by the incircle and the circumcircle of a triangle, the pedal triangle, the orthic triangle, and the incentral triangle. Such recursive systems describing some dynamic geometries are considered by S. Abbot [1], G. Z. Chang, P. J. Davis [5], R. J. Clarke [6], J. Ding, L. R. Hitt, X-M. Zhang [8], S. Donisi, H. Martini, G. Vincenzi, G. Vitale [9], R. Hitt, X-M. Zhang [10], D. Ismailescu, J. Jacobs, [11], J. G. Kingston, J. L. Synge [12], P. Pech [13], O. Roeschel [14], I. J. Schoenberg [15], B. M. Stewart [16], M. de Villiers [17], and B. Ziv [18].

The main purpose of the present paper is to characterize all real numbers α such that the sequence of Kasner triangles with the fixed weight defined by α is convergent (Section 2). When this process is convergent we determine the order of convergence.

In the study of an iterative geometric process, a useful method is to associate complex coordinates to the points and to transfer the geometrical problem in one in the complex plane. This is the principal technique used in Sections 2.

2. KASNER TRIANGLES WITH A FIXED WEIGHT

Perhaps the simplest example of this type of construction uses the median triangles. In this process we consider the initial configuration $\mathcal{F}_0$ to be the triangle $A_0B_0C_0$ and one forms $\mathcal{F}_1$ which is the triangle $A_1B_1C_1$ whose vertices are the mid-points of the edges of $\mathcal{F}_0$. Then the third configuration $\mathcal{F}_2$ is the triangle $A_2B_2C_2$ formed by the midpoints of the edges of $\mathcal{F}_1$. Continuing this iterations one obtains a sequence of triangles connected by the corresponding transformations which in this case are affine transformations.

The geometry of $\mathcal{F}_n$ is very simple since the triangle $A_nB_nC_n$ is similar to $A_0B_0C_0$ with the similarity ratio $1/2^n$.

Considering the complex coordinates of the vertices of the triangle $\mathcal{F}_n$, we obtain that the process is described by the points $A_n(a_n), B_n(b_n), C_n(c_n)$, where for $n = 0, 1, \ldots$, we have

$$\begin{cases} a_{n+1} = \frac{1}{2}(b_n + c_n) \\ b_{n+1} = \frac{1}{2}(c_n + a_n) \\ c_{n+1} = \frac{1}{2}(a_n + b_n) \end{cases} \tag{1}$$

and a_0, b_0, c_0 are fixed complex numbers. An elementary approach of system (1) is to add the equations and obtain

$$a_{n+1} + b_{n+1} + c_{n+1} = a_n + b_n + c_n = \ldots = a_0 + b_0 + c_0 = 3g_0,$$

where g_0 is the complex coordinates of the centroid G of the initial triangle $A_0B_0C_0$ (see the book of T. Andreescu and D. Andrica [2]). By replacing

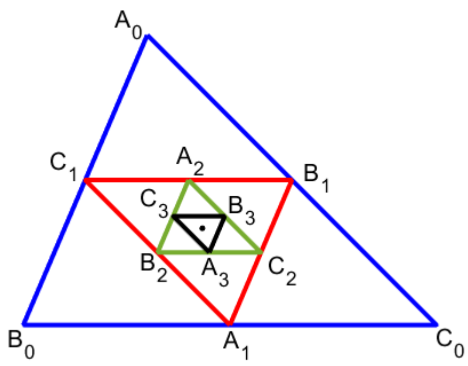

FIGURE 1. Sequence of median triangles.

in the first equation of (1), it follows $b_n + c_n = 3g_0 - a_n$, hence

$$a_{n+1} = \frac{1}{2}(3g_0 - a_n), n = 0, 1 \dots,$$

That is

$$a_{n+1} - g_0 = -\frac{1}{2}(a_n - g_0), n = 0, 1 \dots .$$

From the last relation we obtain

$$a_n - g_0 = \frac{(-1)^n}{2^n}(a_0 - g_0), n = 0, 1 \dots, \tag{2}$$

and similar formulas for the sequences $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$. These formulas show the convergence

$$a_n \to g_0, \ b_n \to g_0, \ c_n \to g_0,$$

that is the limit of sequence $(\triangle A_n B_n C_n)_{n\geq 0}$ is the degenerated triangle at G. The order of convergence of $(a_n)_{n\geq 0}$ is obtained also from (2)

$$\lim_{n\to+\infty} 2^n |a_n - g_0| = |a_0 - g_0|.$$

Similarly, for $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ we have

$$\lim_{n\to+\infty} 2^n |b_n - g_0| = |b_0 - g_0|, \lim_{n\to+\infty} 2^n |c_n - g_0| = |c_0 - g_0|.$$

A generalization of the previous construction is the following. Consider a real number $\alpha \in \mathbb{R}^*, \alpha \neq \frac{1}{2}$, and the initial triangle $A_0B_0C_0$. Now construct the triangle $A_1B_1C_1$ such that A_1, B_1, C_1 divides the segments $[B_0C_0], [C_0A_0]$, respectively $[A_0B_0]$ in the ratio $1-\alpha : \alpha$. Continuing this process one obtains a sequence of triangles $A_nB_nC_n$ connected by the corresponding transformations which in this case are also affine transformations. The terms of this sequence are called Kasner triangles (after E. Kasner (1878-1955)), also known as nested triangles in other references.

A natural problem is to determine all real numbers α for which the sequence $(A_nB_nC_n)_{n\geq 0}$ is convergent. A similar problem is mentioned by D.

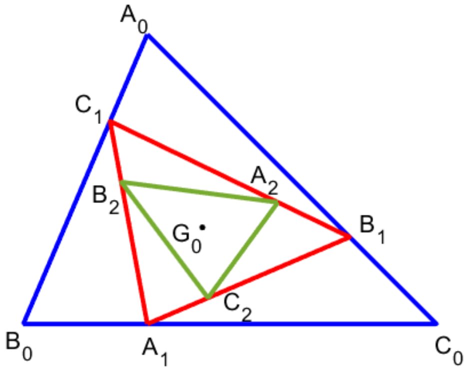

FIGURE 2. Sequence of Kasner triangles for $\alpha = 0.3$.

Ismailescu and J. Jacobs [11], who explore the values of the division ratio α for which the above triangle sequence is divergent (in shape).

The following result completely solved this problem.

Theorem. *1) The sequence* $(\triangle A_nB_nC_n)_{n\geq 0}$ *is convergent if and only if* $\alpha \in (0,1)$. *When the sequence is convergent, its limit is the degenerated triangle at* G_0, *the centroid of* $\triangle A_0B_0C_0$.

2) When $\alpha \in (0,1)$, *the order of convergence to* g_0 *of the sequences* $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$, *and* $(c_n)_{n\geq 0}$ *is* $\left(\frac{1}{3\alpha^2-3\alpha+1}\right)^{n/2}$.

Proof. 1) The case $\alpha = \frac{1}{2}$ has been discussed above in details that why we may assume that $\alpha \neq \frac{1}{2}$. The complex coordinates a_n, b_n, c_n of the vertices of the triangles $A_nB_nC_n$ satisfy the recurrent system

$$\begin{cases} a_{n+1} = \alpha b_n + (1-\alpha)c_n \\ b_{n+1} = \alpha c_n + (1-\alpha)a_n \\ c_{n+1} = \alpha a_n + (1-\alpha)b_n \end{cases} \tag{3}$$

where a_0, b_0, c_0 are fixed complex numbers. Let us observe again that for every $n \geq 0$, we have

$$a_n + b_n + c_n = a_0 + b_0 + c_0 = 3g_0,$$

hence $c_n = 3g_0 - a_n - b_n$, where g_0 is the complex coordinate of the centroid G_0 of the initial triangle $A_0B_0C_0$. From the system (3) we obtain

$$\begin{cases} a_{n+1} = \alpha b_n + (1-\alpha)(3g_0 - a_n - b_n) \\ b_{n+1} = \alpha(3g_0 - a_n - b_n) + (1-\alpha)a_n \end{cases} \tag{4}$$

The system (4) is equivalent to

$$\begin{cases} a'_{n+1} = (\alpha-1)a'_n + (2\alpha-1)b'_n \\ b'_{n+1} = (1-2\alpha)a'_n - \alpha b'_n \end{cases} \tag{5}$$

where $a'_n = a_n - g_0$ and $b'_n = b_n - g_0, n = 0, 1, \ldots$. From the first equation of (5) it follows

$$b'_n = \frac{1}{2\alpha - 1}(a'_{n+1} - (\alpha - 1)a'_n),$$

and we get the following second order linear recursive relation for the sequence $(a'_n)_{n\geq 0}$

$$a'_{n+2} = -a'_{n+1} - (3\alpha^2 - 3\alpha + 1)a'_n, n = 0, 1, \ldots \tag{6}$$

The characteristic equation of (6) is

$$t^2 + t + (3\alpha^2 - 3\alpha + 1) = 0$$

with the roots

$$t_1 = \frac{1}{2}(-1 + i(2\alpha - 1)\sqrt{3}), \quad t_2 = \frac{1}{2}(-1 - i(2\alpha - 1)\sqrt{3}).$$

A convenient expression for the roots is the following

$$t_1 = \alpha\omega + (1 - \alpha)\omega^2, \quad t_2 = \alpha\omega^2 + (1 - \alpha)\omega,$$

where $\omega = \exp\left(\frac{2\pi}{3}i\right)$ is the third root of unity satisfying the relations $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$.

From the general theory of such recurrences (see [3]) we have

$$a'_n = ut_1^n + vt_2^n,$$

where the coefficients u, v are determined from the linear system

$$\begin{cases} u + v = a'_0 \\ t_1 u + t_2 v = a'_1. \end{cases} \tag{7}$$

After simple computations, we obtain

$$\begin{aligned} u = \frac{a'_1 - t_2 a'_0}{t_1 - t_2} &= \frac{\alpha b_0 + (1 - \alpha)c_0 - g_0 - t_2(a_0 - g_0)}{i(2\alpha - 1)\sqrt{3}} \\ &= \frac{(a_1 - g_0) - (a_0 - g_0)\left(\alpha\omega^2 + (1 - \alpha)\omega\right)}{(2\alpha - 1)(2\omega + 1)} \end{aligned}$$

and

$$\begin{aligned} v = \frac{t_1 a'_0 - a'_1}{t_1 - t_2} &= \frac{t_1(a_0 - g_0) - \alpha b_0 - (1 - \alpha)c_0 + g_0}{i(2\alpha - 1)\sqrt{3}} \\ &= \frac{(a_0 - g_0)\left(\alpha\omega + (1 - \alpha)\omega^2\right) - (a_1 - g_0)}{(2\alpha - 1)(2\omega + 1)}. \end{aligned}$$

It follows that

$$a_n = g_0 + ut_1^n + vt_2^n, \tag{8}$$

hence the sequence $(a_n)_{n\geq 0}$ is convergent if and only if $|t_{1,2}| < 1$. On the other hand we have $|t_{1,2}|^2 = \frac{1}{4}(1 + 3(2\alpha - 1)^2) = 3\alpha^2 - 3\alpha + 1$ and the inequality $|t_{1,2}|^2 < 1$ is equivalent to $\alpha(\alpha - 1) < 0$. The conclusion follows for the sequence $(a_n)_{n\geq 0}$.

Clearly, in the case of convergence from (8) we get $\lim_{n\to+\infty} a_n = g_0$.

We use the same argument for the sequences $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$.

2) Because $|t_1| = |t_2|$, using formula (8) we have

$$\frac{1}{|t_1|^n}|a_n - g_0| = |u + v\left(\frac{t_2}{t_1}\right)^n| \le |u| + |v|, \tag{9}$$

that is

$$\left(\frac{1}{3\alpha^2 - 3\alpha + 1}\right)^{n/2} |a_n - g_0| \le |u| + |v|,$$

hence $|a_n - g_0| = O((3\alpha^2 - 3\alpha + 1)^{n/2})$.

Similarly, we obtain $|b_n - g_0| = O((3\alpha^2 - 3\alpha + 1)^{n/2})$ and $|c_n - g_0| = O((3\alpha^2 - 3\alpha + 1)^{n/2})$, and the conclusion follows. □

Corollary. *The highest convergence speed of the process is obtained for $\alpha = \frac{1}{2}$. Moreover, the convergence speed is strictly increasing for $\alpha \in (0, \frac{1}{2})$ and strictly decreasing for $\alpha \in (\frac{1}{2}, 1)$.*

Proof. The conclusion easily follows from the property that the quadratic function $\alpha \mapsto 3\alpha^2 - 3\alpha + 1$, is strictly decreasing on the interval $(0, \frac{1}{2})$ and strictly increasing on $(\frac{1}{2}, 1)$, and it has a minimum point at $\alpha = \frac{1}{2}$. □

The first terms of two Kasner sequences are shown in Figure 3.

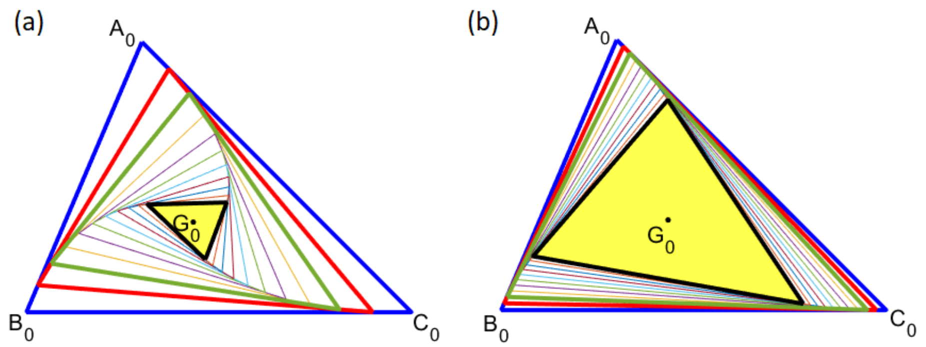

FIGURE 3. Sequence of Kasner triangles $(\triangle A_n B_n C_n)$ with $n = 0, \ldots, 10$, computed for (a) $\alpha = 0.1$; (b) $\alpha = 0.025$.

We mention that the sufficient condition $\alpha \in (0, 1)$ for the convergence of the sequence $(\triangle A_n B_n C_n)_{n \ge 0}$ appears in Theorem 1.2.1 of the book of P. J. Davis [7], where the proof uses a geometrical Lyapunov function associated to this iterative process.

Remarks. 1) A higher level approach of the above proof of property 1) is the following. It is easy to see that system (3) is equivalent to the following matrix relation

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 1-\alpha \\ 1-\alpha & 0 & \alpha \\ \alpha & 1-\alpha & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \tag{10}$$

therefore

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = U^n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}, \tag{11}$$

where U is the double stochastic circulant matrix (see [7]) given by

$$U = \begin{pmatrix} 0 & \alpha & 1-\alpha \\ 1-\alpha & 0 & \alpha \\ \alpha & 1-\alpha & 0 \end{pmatrix}. \tag{12}$$

The matrix U has the characteristic polynomial $p_U(t) = (t-1)(t^2+t+3\alpha^2-3\alpha+1)$, hence its eigenvalues are $t_0 = 1$ and $t_{1,2} = \frac{1}{2}(-1 \pm i(2\alpha-1)\sqrt{3})$. It follows that we have

$$U = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{pmatrix} P^{-1}, \tag{13}$$

for some nonsingular matrix P, hence for every positive integer $n \geq 0$ we have

$$U^n = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_1^n & 0 \\ 0 & 0 & t_2^n \end{pmatrix} P^{-1}. \tag{14}$$

The relation (11) shows that the sequences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}, (c_n)_{n\geq 0}$ are convergent if and only if the matrix sequence $(U^n)_{n\geq 0}$ is convergent. But the convergence of $(U^n)_{n\geq 0}$ is reduced by relation (14) to $t_1^n \to 0$ and $t_2^n \to 0$, which is equivalent to $|t_{1,2}| < 1$. On the other hand, we have $|t_2|^2 = |t_3|^2 = t_2 t_3 = 3\alpha^2 - 3\alpha + 1$, and we obtain the convergence of $(U^n)_{n\geq 0}$ if and only if $\alpha \in (0,1)$.

To find the limit of this sequence of triangles, let us consider $\alpha \in (0,1)$ and $a_n \to a, b_n \to b, c_n \to c$. Taking limits in the system (3) we obtain

$$\begin{cases} a = \alpha b + (1-\alpha)c \\ b = \alpha c + (1-\alpha)a \\ c = \alpha a + (1-\alpha)b, \end{cases} \tag{15}$$

hence $a - c = \alpha(b-c), b - a = \alpha(c-a), c - b = \alpha(a-b)$. If $a \neq b, b \neq c, c \neq a$, then by multiplying the above relation it follows $\alpha^3 = -1$, a contradiction. If we have one equality, for instance $a = b$, then we get $b = c$, and finally $a = b = c$. Now, passing to the limit in $a_n + b_n + c_n = 3g_0$, one obtains $a = b = c = g_0$. The conclusion is that the limit of sequence $(\triangle A_n B_n C_n)_{n\geq 0}$ is the degenerated triangle at point G.

2) Writing $1 + i(2\alpha-1)\sqrt{3} = \sqrt{1-3\alpha+3\alpha^2}\exp(\theta i)$, we have

$$\theta = \cos^{-1}\frac{1}{\sqrt{1-3\alpha+3\alpha^2}}.$$

If $\theta = (p/q)\pi$, then from (9) it follows

$$\left(\frac{1}{3\alpha^2-3\alpha+1}\right)^{n/2}|a_n - g_0| = |u + v\exp(2n\theta i)| = \left|u + v\exp\left(\frac{2np}{q}i\right)\right|,$$

hence the left hand side sequence has exactly q convergent subsequences.

3) Concerning this process, the following interesting result is stated in the paper of D. Ismailescu and J. Jacobs [11]. Let

$$\eta = \cos^{-1}\left(-1 + \frac{1}{2(1-3\alpha+3\alpha^2)}\right)$$

If $\eta = (p/q)\pi$, where p and q are positive integers, then the shape sequence is periodical. More precisely, for every $k \geq 0$, triangles $A_kB_kC_k$ and $A_{k+2q}B_{k+2q}C_{k+2q}$ are similar.

3. Non-convergent iterations

For the sake of completeness, we now discuss some details regarding the non-convergent iterations obtained for $\alpha \notin (0,1)$.

3.1. **Divergent orbits.** For $\alpha < 0$ and $\alpha > 1$ the orbits are divergent. We now show two divergent iterations obtained for $\alpha = 1.4$ depicted in Figure 4 and $\alpha = -0.3$, illustrated in Figure 5. Notice that in this case the points $A_{n+1}, B_{n+1}, C_{n+1}$ still belong to the support lines of the sides of the triangle $A_nB_nC_n$, but are now outside the segments $[B_nC_n]$, $[C_nA_n]$ and $[A_nB_n]$. Also, the orientations of the spirals generated by the vertices for values $\alpha > 1$ and $\alpha < 0$, opposite to each other.

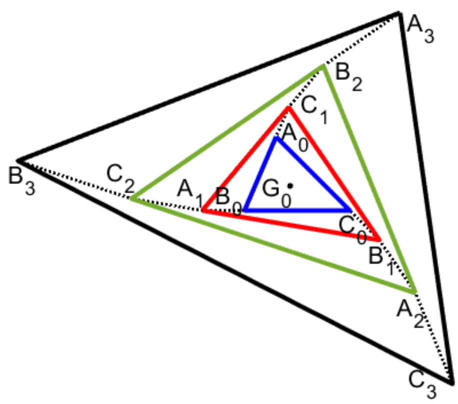

Figure 4. Sequence of Kasner triangles for $\alpha = 1.4$.

3.2. **Periodic orbits.** When $\alpha = 0$ and $\alpha = 1$ the orbits are periodic of period 3.

Indeed, for $\alpha = 0$, by (3) one has $a_{n+3} = c_{n+2} = b_{n+1} = a_n$, $b_{n+3} = a_{n+2} = c_{n+1} = b_n$, and $c_{n+3} = b_{n+2} = a_{n+1} = c_n$, hence the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$ are given explicitly by

$$\begin{cases} a_n: & a_0, c_0, b_0, a_0, c_0, b_0, a_0, \ldots \\ b_n: & b_0, a_0, c_0, b_0, a_0, c_0, b_0, \ldots \\ c_n: & c_0, b_0, a_0, c_0, b_0, a_0, c_0, \ldots . \end{cases} \tag{16}$$

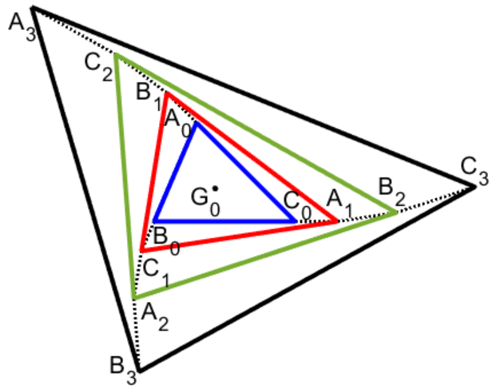

FIGURE 5. Sequence of Kasner triangles for $\alpha = -0.3$.

Similarly, for $\alpha = 1$, by (3) one has $a_{n+3} = b_{n+2} = c_{n+1} = a_n$, $b_{n+3} = c_{n+2} = a_{n+1} = b_n$, and $c_{n+3} = a_{n+2} = b_{n+1} = c_n$, therefore the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$ are given explicitly by

$$\begin{cases} a_n : & a_0, b_0, c_0, a_0, b_0, c_0, a_0, \ldots \\ b_n : & b_0, c_0, a_0, b_0, c_0, a_0, b_0, \ldots \\ c_n : & c_0, a_0, b_0, c_0, a_0, b_0, c_0, \ldots . \end{cases} \tag{17}$$

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BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
CLUJ-NAPOCA 400084, ROMANIA
E-mail address: dandrica@math.ubbcluj.ro

SCHOOL OF COMPUTING AND ENGINEERING
UNIVERSITY OF DERBY
KEDLESTON ROAD, DE22 1GB, UK
E-mail address: o.bagdasar@derby.ac.uk

"IANCU DE HUNEDOARA" NATIONAL COLLEGE
HUNEDOARA, ROMANIA
E-mail address: marinescuds@gmail.com