LATTICE EQUABLE QUADRILATERALS II:

KITES, TRAPEZOIDS & CYCLIC QUADRILATERALS

CHRISTIAN AEBI AND GRANT CAIRNS

Abstract. We show that up to Euclidean motions, there are 4 infinite families of lattice equable kites given by corresponding Pell or Pell-like equations, exactly 5 lattice equable trapezoids (2 isosceles, 2 right, 1 singular) and 4 lattice equable cyclic quadrilaterals. We also show that with one exception, the interior diagonals of lattice equable quadrilaterals are irrational.

1. INTRODUCTION

A *lattice equable quadrilateral* (LEQ for short) is a quadrilateral whose vertices lie on the integer lattice $\mathbb{Z}^2$ and which is equable in the sense that its area equals its perimeter. The general context here is the field of study related to Heronian triangles and their generalisations. Recall that a *Heronian triangle*, or *Heron triangle* is a triangle with integer side lengths and integer area [4, 10]. Similarly, quadrilaterals with integer side lengths, integer diagonals and integer area have been studied for some time; for the rational version, see [7] and [5, pp. 216–221]. *Brahmagupta quadrilaterals*, which are cyclic quadrilaterals with integer side lengths, integer diagonals and integer area, have received particular attention [11, 8]. The objects of our study are somewhat different. While LEQs have integer side lengths (see [1, Remark 2]), they do not necessarily have diagonals of integer lengths (see Theorem 1.4 below). On the other hand, the requirement that the vertices be lattice points is often a significant restriction.

This paper is a continuation of the work [1] which treated lattice equable parallelograms. It is shown in [1] that lattice equable parallelograms are quite

abundant, and naturally form a forest of three infinite trees all of whose vertices have degree 4, bar the roots. This present paper studies LEQs with other specials properties: kites, trapezoids and cyclic quadrilaterals.

Recall that a *kite* is a planar quadrilateral having two disjoint pairs of adjacent sides of equal length; see [13, Chap. 7]. A *dart* is a concave kite. If a lattice equable kite Q has vertices $O(0,0)$, $A(x,y)$, $B(z,w)$, $C(u,v)$ and is symmetrical about the diagonal OB, then we can cut Q along OB and rearrange to form the parallelogram $O(0,0), A(x,y), B(z,w), C'(z-x, w-y)$, which by construction is a lattice equable parallelogram; see Figure 1. By the reverse procedure, a lattice equable parallelogram $OABC'$ produces a lattice equable kite $OABC$ provided the reflection C of A in the diagonal OB is a lattice point. Thus lattice equable kites correspond to a particular subfamily of lattice equable parallelograms. Not all lattice equable parallelograms produce lattice equable kites; for example, as shown in Figure 2, the 3×6 rectangle does not produce a lattice equable kite. Here the point $C = (-\frac{9}{5}, \frac{12}{5})$ is not a lattice point. In fact, there is no lattice equable kite with side lengths 3 and 6, as we will see from the corollary of the following theorem.

Theorem 1.1. *The four families* K1 – K4 *given in the following table provide the list, without redundancy, of all lattice equable kites, up to Euclidean motions. In each case, the points $O(0,0), A, B, C$ are the vertices of a lattice equable kite with axis of symmetry OB, and M is the midpoint of the segment AC. In each family, the nonnegative integers n, i are solutions to a given Pell or Pell-like equation. In family K2 we require $n > 1$.*

Case	Equation	M	A	B	C
K1	$n^2 - 5i^2 = 4$	$\frac{1}{2}(n+5i)(2,1)$	$M + (2,-4)$	$n(2,1)$	$M - (2,-4)$
K2	$n^2 - 5i^2 = 1$	$(2n+5i)(2,1)$	$M + (1,-2)$	$4n(2,1)$	$M - (1,-2)$
K3	$n^2 - 2i^2 = 1$	$(n+2i)(2,2)$	$M + (2,-2)$	$4n(1,1)$	$M - (2,-2)$
K4	$2n^2 - i^2 = 1$	$(4n+3i)(\frac{3}{2},\frac{3}{2})$	$M + (\frac{3}{2},-\frac{3}{2})$	$12n(1,1)$	$M - (\frac{3}{2},-\frac{3}{2})$

Corollary 1.1. *Up to Euclidean motions, there are only three convex lattice equable kites. They are:*

(a) The rhombus of side length 5 of Figure 6.

(b) The 4×4 square.

(c) The lattice equable kite with side lengths 3 and 15 shown in Figure 7.

The proof of this theorem and its corollary are given in the next Section; it replies heavily on the results of [1]. Section 3 gives more details about the families K1 – K4 of Theorem 1.1.

We now turn to equable trapezoids. As discussed in [13], the term 'trapezoid' is not universal. For us, a *trapezoid* is a quadrilateral that has exactly one pair of parallel sides. An *isosceles trapezoid* is a trapezoid for which the
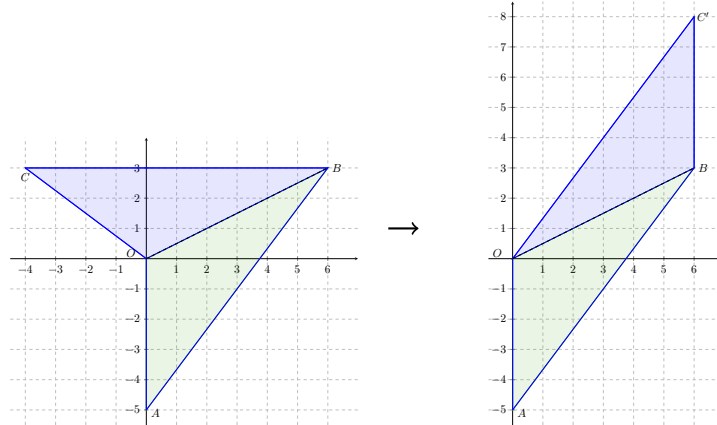
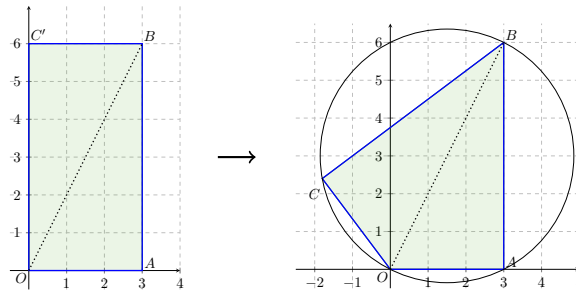FIGURE 1. Lattice equable kites produce lattice equable parallelograms

FIGURE 2. A lattice equable parallelogram that fails to produce a lattice equable kite

nonparallel sides have equal length. Thus an isosceles trapezoid is the truncation of an isosceles triangle by a line parallel to the base. A *right trapezoid* is a trapezoid having two adjacent right angles.

In [6], Dolan briefly outlines how equable trapezoids might be classified by reducing the problem to the study of certain triangles; [6] concludes with the words: "A nice exercise for an interested reader is then to determine precisely which triangles produce equable trapezia in this fashion". Indeed, this nice exercise gives the following result.

Theorem 1.2. *Up to Euclidean motions, there are only five equable trapezoids having integer sides and area. They are the two right trapezoids shown in Figure 3, the two isosceles trapezoids shown in Figure 4, and the trapezoid of side lengths 20,4,15,3, shown in Figure 5.*

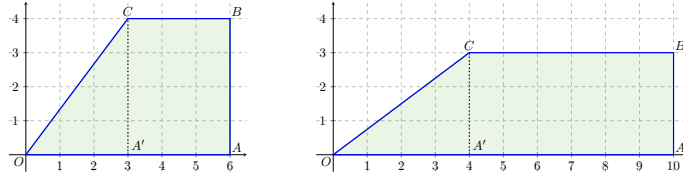The proof of this theorem is given in the Section 4.

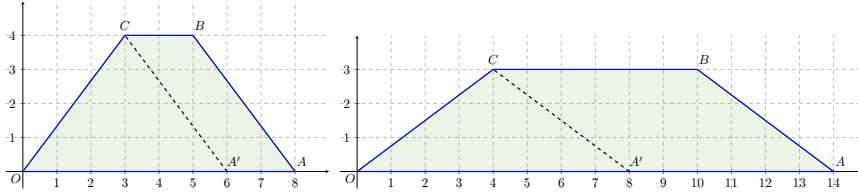FIGURE 3. The 6,4,3,5 and 10,3,6,5 lattice equable right trapezoids

FIGURE 4. The 8,5,2,5 and 14,5,6,5 lattice equable isosceles trapezoids
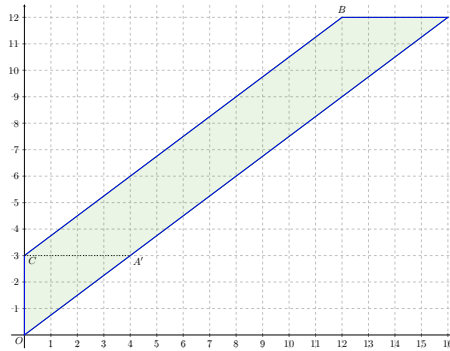

FIGURE 5. The 20,4,15,3 lattice equable trapezoid

Remark 1.1. *The five equable trapezoids of Theorem 1.2 are all LEQs. None of them are obtuse. The trapezoid of side lengths 20,4,15,3 is particularly interesting. First, it has two orthogonal non-adjacent edges. Secondly, it cannot be drawn as a LEQ with horizontal parallel sides.*

A *cyclic quadrilateral* is a quadrilateral whose vertices lie on a circle. The classification of cyclic LEQs that we give below in Theorem 1.3 is not really new. The determination of the possible integer side lengths for equable cyclic quadrilaterals was established by Nelson [9]. It also appears in the web pages of Balmoral Software [2] where the problem is shown to amount to a finite number of cases that is large but can be performed by computer. The proof of Theorem 1.3 that we give in Section 5 is virtually identical to that of [9] until near the end where a simplification allows us to reduce the problem to just 63

cases, that the authors have resolved both by computer and by "bare-hands". The LEQ hypothesis is then only used to exclude certain cyclic orders of the sides.

Theorem 1.3. *Up to Euclidean motions, there are only four cyclic LEQs; they are the 4×4 square, the 3×6 rectangle, and the two isosceles trapezoids shown in Figure 4.*

The paper concludes with a general result. It is shown in [1] that the diagonals of all lattice equable parallelograms are irrational. This is not the case for all LEQs. For example, for the concave LEQ with vertices

$$O(0,0), A(20,15), B(8,10), C(8,15),$$

the external diagonal AC has length 12. Nevertheless, one has the following result, which is proved in Section 6.

Theorem 1.4. *For a lattice equable quadrilateral $OABC$, the interior diagonals are irrational, except if $OABC$ is congruent to the right trapezoid of side lengths 6,4,3,5 shown on the left of Figure 3.*

Notation. In this paper, a quadrilateral $OABC$ is defined by four vertices O, A, B, C, no three of which are colinear, such that the line segments joining the vertices have no interior points of intersection; that is, our quadrilaterals have no self-intersections. We will always write the vertices O, A, B, C in positive (counterclockwise) cyclic order. We use the notation $K(OABC)$ for area and $P(OABC)$ for perimeter. For ease of expression, we often simply write K for $K(OABC)$, and P for $P(OABC)$, and we abbreviate the triangle areas $K(OAB), K(BCO)$ as K_A, K_C respectively, and the perimeter $P(BCO)$ as P_C. By abuse of notation, we write OA, AB, BC, CO for both the sides, and their lengths; the meaning should be clear from the context. We also usually denote OA, AB, BC, CO by the letters a, b, c, d. The lengths of the diagonals OB, AC are denoted p, q, respectively. We use vector notation, such as $\overrightarrow{AB}$. But we use the same symbol, A say, for the vertex A and its position vector $\overrightarrow{OA}$. In this paper, we employ the term *positive* in the strict sense. So $\mathbb{N} = \{n \in \mathbb{Z} \mid n > 0\}$.

2. Proof of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1. First note that in each family, C is visibly the reflection of A in the line through OB, so $OABC$ is a kite. Secondly, the vertices A, B, C are lattice points. Indeed, this is obvious for K2 and K3, while for K1, it follows from the fact that if $n^2 - 5i^2 = 4$, then n and i have the same parity. For K4, if $2n^2 - i^2 = 1$, then i is odd and so M is not a lattice point, but A, B, C are. To see that $OABC$ is equable, we need to show that K_A is the sum of the length $a = OA$ and $b = AB$. Note that $K_A = \frac{1}{2}OB \cdot MA$. Routine calculations give the following values:

Case	K_A	a	b
K1	$5n$	$\frac{5}{2}(n+i)$	$\frac{5}{2}(n-i)$
K2	$10n$	$5(n+2i)$	$5(n-2i)$
K3	$8n$	$4(n+i)$	$4(n-i)$
K4	$18n$	$3(3n+2i)$	$3(3n-2i)$

From the above table, $K_A = a + b$, so the members of the families K1-K4 are all equable.

Conversely, consider a lattice equable kite with vertices

$$O(0,0), A(x,y), B(z,w), C(u,v),$$

that is symmetrical about the diagonal $p = OB$. By translation, and reflection in the x-axis, y-axis and the lines $y = \pm x$, if necessary, we may assume without loss of generality that the diagonal p lies in the first quadrant and has gradient ≤ 1. Furthermore, by reflection in the x and y axes and translating, we may assume that $a \geq b$ and that O, A, B, C are in positive cyclic order.

Notice that $O(0,0), A(x,y), B(z,w), C'(z-x, w-y)$ is a lattice equable parallelogram. Suppose initially that the diagonal $p = OB$ is the shortest of the two diagonals. From [1, Lemma 2], $p^2 = a^2 + b^2 - 2\sqrt{a^2b^2 - 4(a+b)^2}$ and thus from [1, Lemma 6b], the diagonal $AC = q$ satisfies

$$q^2 = \frac{16(a+b)^2}{p^2} = \frac{16(a+b)^2}{a^2 + b^2 - 2\sqrt{a^2b^2 - 4(a+b)^2}}.$$

Furthermore, from [1, Section 3], there are positive integers m, n with $m \leq n$ for which we have the Vieta relations

$$(1) \quad ab = k(m^2 + n^2), \quad a + b = kmn, \quad \sqrt{a^2b^2 - 4(a+b)^2} = k(n^2 - m^2),$$

where k is either 5, 8 or 9. It follows that, as in the proof of [1, Theorem 5], $p^2 = k^2 m^2 n^2 - 4kn^2$. Hence

$$(2) \quad q^2 = \frac{16(a+b)^2}{a^2 + b^2 - 2\sqrt{a^2b^2 - 4(a+b)^2}} = \frac{16k^2 m^2 n^2}{k^2 m^2 n^2 - 4kn^2} = \frac{16km^2}{km^2 - 4}.$$

Since q^2 is the square of the distance between two lattice points, it is a positive integer, though q may well be irrational. Thus $16km^2 = q^2(km^2 - 4)$, so $q^2 \geq 16$ and

$$m^2 = \frac{4q^2}{k(q^2 - 16)}.$$

As $m \geq 1$, we have $q^2 \leq \frac{16k}{k-4}$. Thus, as $k \in \{5, 8, 9\}$, we have three cases:

(a) For $k = 5$, we have $m^2 = \frac{4q^2}{5(q^2 - 16)}$ and $q^2 \leq 80$. Calculations show that the only possibilities are (a1) $m = 1, q^2 = 80$ and (a2) $m = 2, q^2 = 20$.

(b) For $k = 8$, we have $m^2 = \frac{q^2}{2(q^2 - 16)}$ and $q^2 \leq 32$. Calculations give just the one possibility $m = 1, q^2 = 32$.

(c) For $k = 9$, we have $m^2 = \frac{4q^2}{9(q^2-16)}$ and $q^2 \leq \frac{144}{5}$ so $q^2 \leq 29$. Calculations give just the one possibility $m = 2, q^2 = 18$.

We now examine these four possibilities in greater detail:

(a1). For $k = 5, m = 1, q^2 = 80$: one has $a = \frac{1}{2}(5n + \sqrt{5n^2 - 20})$ from (1). So we require that $5n^2 - 20$ be a square, say $5n^2 - 20 = j^2$. Note that j must be a multiple of 5, say $j = 5i$. So $n^2 - 4 = 5i^2$. Thus $a = \frac{1}{2}(5n + 5i)$. In this case $b = \frac{1}{2}(5n - 5i)$, again from (1). Note that $q^2 = 80$ is the sum of two squares in only one possible way: $80 = 8^2 + 4^2$. So, as O, A, B, C are in positive cyclic order, the vector $\overrightarrow{AC}$ is necessarily $(-4, 8)$. From above, $p^2 = k^2m^2n^2 - 4kn^2 = 5n^2$. Since OB is orthogonal to AC, we have $B = n(2, 1)$. Since a and b are fixed, the triangle OAB is thus uniquely determined. The vertices A, B, C are therefore those of family K1.

(a2). For $k = 5, m = 2, q^2 = 20$: one has $a = 5n + 2\sqrt{5n^2 - 5}$. So we require that $5n^2 - 5$ be a square, say $5n^2 - 5 = j^2$. Once again, j must be a multiple of 5, say $j = 5i$. So $n^2 - 1 = 5i^2$. Thus $a = 5n + 10i$ and $b = 5n - 10i$. Note that $q^2 = 20$ is the sum of two squares in only one possible way: $20 = 4^2 + 2^2$. So the vector $\overrightarrow{AC}$ is necessarily $(-2, 4)$. The diagonal p satisfies $p^2 = k^2m^2n^2 - 4kn^2 = 80n^2$. Since OB is orthogonal to AC, we have $B = 4n(2, 1)$. Since a and b are fixed, the triangle OAB is thus uniquely determined. The vertices A, B, C are therefore those of family K2.

(b). For $k = 8, m = 1$: one has $a = 2(2n + \sqrt{2n^2 - 2})$. So we require that $2n^2 - 2$ be a square, say $2n^2 - 2 = j^2$. Note that j must be even, say $j = 2i$. So $n^2 - 1 = 2i^2$. Thus $a = 4(n + i)$ and $b = 4(n - i)$. Note that $q^2 = 32$ is the sum of two squares in only one possible way: $32 = 4^2 + 4^2$. So the vector $\overrightarrow{AC}$ is necessarily $(-4, 4)$. The diagonal p satisfies $p^2 = k^2m^2n^2 - 4kn^2 = 32n^2$. Since OB is orthogonal to AC, we have $B = 4n(1, 1)$. Since a and b are fixed, the triangle OAB is thus uniquely determined. The vertices A, B, C are therefore those of family K3.

(c). For $k = 9, m = 2$: one has $a = 3(3n + 2\sqrt{2n^2 - 1})$. So we require that $2n^2 - 1$ be a square, say $2n^2 - 1 = i^2$. So $a = 3(3n + 2i)$ and $b = 3(3n - 2i)$. Note that $q^2 = 18$ is the sum of two squares in only one possible way: $18 = 3^2 + 3^2$. So the vector $\overrightarrow{AC}$ is necessarily $(-3, 3)$. The diagonal p satisfies $p^2 = k^2m^2n^2 - 4kn^2 = 288n^2$. Since OB is orthogonal to AC, we have $B = 12n(1, 1)$. Since a and b are fixed, the triangle OAB is thus uniquely determined. The vertices A, B, C are therefore those of family K4 with $n \geq 2$. Notice that there is a member of K4 with $n = 1$ which is not included here as $n \geq m = 2$. We will find the missing family member below.

The above calculations assumed that p is the shortest diagonal of the parallelogram $OABC'$. Similarly, if p is the longest diagonal, then $n^2 = \frac{4q^2}{k(q^2-16)}$, and essentially the same calculations as before gives the possibilities:
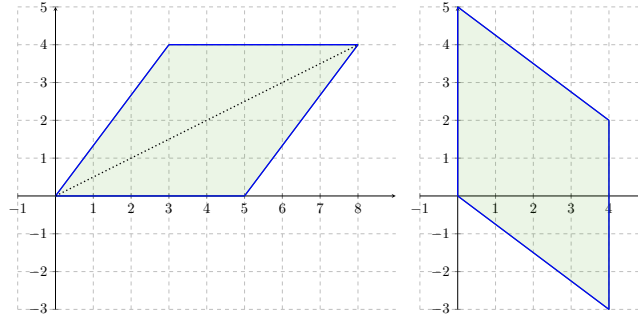
FIGURE 6. Equable Rhombi

(i) $k = 5$; $n = 1, q^2 = 80$ or $n = 2, q^2 = 20$.
(ii) $k = 8$; $n = 1, q^2 = 32$.
(iii) $k = 9$; $n = 2, q^2 = 18$.

As $m \leq n$, this gives six potential cases:

(i1) $k = 5$; $m = n = 1$. But then (1) has no integer solutions for a, b.
(i2) $k = 5$; $m = 1, n = 2$. Then $a = 5, b = 5$.
(i3) $k = 5$; $m = n = 2$. But then (1) has no integer solutions for a, b.
(ii) $k = 8$; $m = n = 1$. Then $a = 4, b = 4$.
(iii1) $k = 9$; $m = 1, n = 2$. Then $a = 15, b = 3$.
(iii2) $k = 9$; $m = n = 2$. But then (1) has no integer solutions for a, b.

So it remains to consider the (i2), (ii), (iii1). In Case (i2), $k = 5, m = 1, n = 2, a = b = 5$ and long diagonal p given by $p^2 = k^2 m^2 n^2 - 4km^2 = 80$. But 80 is the sum of two squares in only one possible way: $80 = 4^2 + 8^2$. So B is necessarily $(8, 4)$. Then A is forced to be $(5, 0)$ and C is $(3, 4)$, as shown on the left of Figure 6. The rhombus of side length 5 also appeared as member $n = 2, i = 0$ of K1; this is shown on the right of Figure 6. Obviously, in Figure 6, the figure on the left can be moved onto the figure on the right by a Euclidean motion.

Case (ii) is the 4×4 square, so its short and long diagonals are equal: this appeared as member $n = 1, i = 0$ of K3.

The kite of Case (iii1) has $k = 9, m = 1, n = 2, a = 15, b = 3$ and long diagonal p given by $p^2 = k^2 m^2 n^2 - 4km^2 = 288$. But 288 is the sum of two squares in only one possible way: $288 = 12^2 + 12^2$. So B is necessarily $(12, 12)$. Then A is forced to be $(12, 9)$ and C is $(9, 12)$, as shown in Figure 7. This kite is the first member of the family K4, with $n = 1, i = 1$. We have thus shown that the table in the statement of the theorem is complete.

It remains to see that there is no redundancy in the table of the theorem. Within each family there is certainly no redundancy, as one can see from the formula for B in the theorem. Note that from [1], for a lattice equable kite with
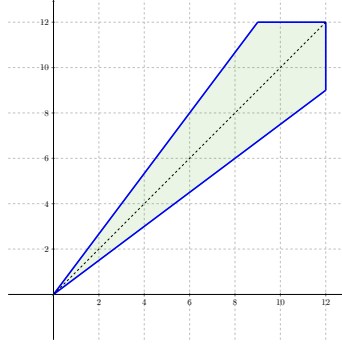
FIGURE 7. Lattice equable kite with side lengths 3 and 15

vertices $O(0,0), A, B, C$ and $a = OA$ and $b = AB$, one has $\gcd(a,b) = 3,4,5$ according to whether $k = 9,8,5$ respectively, where k is as defined above. Recall that the members of K1 – K4 have $k = 5,5,8,9$ respectively. It follows that if a kite is congruent to both a member of Kx and a member of Ky, for $x < y$, then the only possibility is that $x = 1, y = 2$. Suppose we have a kite Q_1 in K1, with $n_1^2 - 5i_1^2 = 4$, that is congruent to a kite Q_2 in K2, with $n_2^2 - 5i_2^2 = 1$. From the table at the beginning of the proof of the theorem, we have from the formulas for the areas that $n_1 = 2n_2$. It follows that, as $n_1^2 - 5i_1^2 = 4$ and $n_2^2 - 5i_2^2 = 1$, we have $i_1 = 2i_2$. But then for Q_1, we have $b = \frac{5}{2}(n_1 + i_1) = 5(n_2 + i_2)$, while for Q_2, we have $b = 5(n_2 + 2i_2)$. But then $i_2 = 0$ and hence $n_2 = 1$, which is excluded in the statement of the theorem. This completes the proof of Theorem 1.1.   □

Proof of Corollary 1.1. The convex kites itemized in the corollary are the first members of the families K1, K3 and K4 respectively. We claim all the other kites exhibited in Theorem 1 are darts. To see this, it suffices to note that in all these cases, the vector $\overrightarrow{MB}$ lies in the third quadrant. For the family K1, when $n > 2$, one has $i > 0$ and so $n^2 = 5i^2 + 4 < 25i^2$, and hence $n - 5i < 0$. Thus, from the formulas for B, M in the theorem, $\overrightarrow{MB} = \frac{1}{2}(n - 5i)(2,1)$ is in the third quadrant.

Similarly, for the other three cases, it is enough to note that:

(1) For the family K2, when $n > 1$, one has $i > 0$ and so $4n^2 = 20i^2 + 4 < 25i^2$ and hence $2n - 5i < 0$.
(2) For the family K3, when $n > 1$, one has $i > 0$ and so $n^2 = 2i^2 + 1 < 4i^2$ and hence $n - 2i < 0$.
(3) For the family K4, when $n > 1$, one has $i > 2$ and so $16n^2 = 8i^2 + 8 < 9i^2$ and hence $4n - 3i < 0$.

□
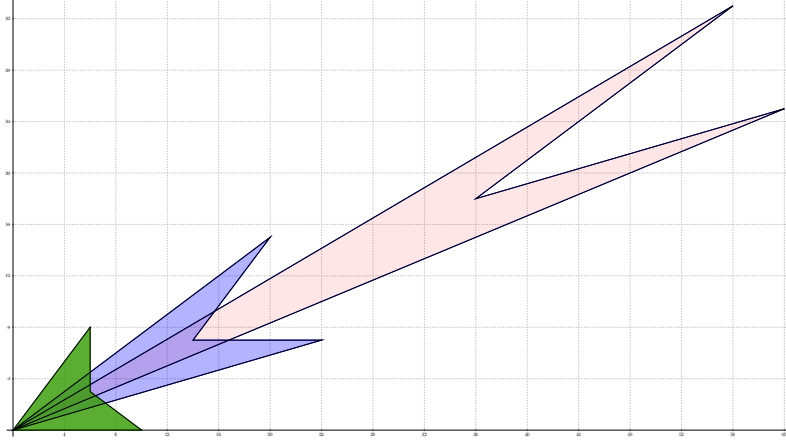
FIGURE 8. K1 lattice equable kites with $n = 3, 7, 18$

3. DISCUSSION OF THE FAMILIES K1 – K4

We now describe the families K1 – K4 in greater detail. Each case is determined by a very common Pell or Pell-like equation.

For K1, the defining condition is the Pell-like equation $n^2 - 5i^2 = 4$. It is well known that its solutions, when denoted (n_j, i_j), satisfy the following recurrence relation:

$$(n_j, i_j) = 3(n_{j-1}, i_{j-1}) - (n_{j-2}, i_{j-2}).$$

The first few values of n are: 2,3,7,18,47,123,... This is OEIS entry A005248 [12]. Here are the first 4 solutions:

n	i	A	B
2	0	(4,-3)	(4,2)
3	1	(10,0)	(6,3)
7	3	(24,7)	(14,7)
18	8	(60,25)	(36,18)

The kites with $n = 3, 7, 18$ are shown in Figure 8.

For K2, the defining condition is the Pell equation $n^2 - 5i^2 = 1$. Its solutions, when denoted (n_j, i_j), satisfy the following recurrence relation:

$$(n_j, i_j) = 18(n_{j-1}, i_{j-1}) - (n_{j-2}, i_{j-2}).$$

The first few values of n are: 1, 9, 161, 2889, 51841,... This is OEIS entry A023039 [12]. Here are the first 3 cases:

n	i	A	B
9	4	(77, 36)	(72,36)
161	72	(1365, 680)	(1288,644)
2889	1292	(24477, 12236)	(23112,11556)

For K3, the defining condition is the Pell equation $n^2 - 2i^2 = 1$. Its solutions, when denoted (n_j, i_j), satisfy the following recurrence relation:

$$(n_j, i_j) = 6(n_{j-1}, i_{j-1}) - (n_{j-2}, i_{j-2}).$$

The first few values of n are: 1, 3, 17, 99, 577,... This is OEIS entry A001541 [12]. Here are the first 3 cases:

n	i	A	B
1	0	(4,0)	(4,4)
3	2	(16, 12)	(12,12)
17	12	(84, 80)	(68,68)

For K4, the defining condition is the Pell-like equation $2n^2 - i^2 = 1$. Its solutions, when denoted (n_j, i_j), satisfy the following recurrence relation:

$$(n_j, i_j) = 6(n_{j-1}, i_{j-1}) - (n_{j-2}, i_{j-2}).$$

The first few values of n are: 1, 5, 29, 169, 985,... This is OEIS entry A001653 [12]. Here are the first 3 cases:

n	i	A	B
1	1	(12, 9)	(12,12)
5	7	(63, 60)	(60,60)
29	41	(360, 357)	(348,348)

4. Proof of Theorem 1.2

We follow a strategy suggested in [6]. Consider a trapezoid with vertices $OABC$ in counterclockwise order, such that OA is the longest side, having an acute angle at 0; see Figure 9. We will not assume that $OABC$ is a LEQ, but we do assume that it is equable with sides of integer length.
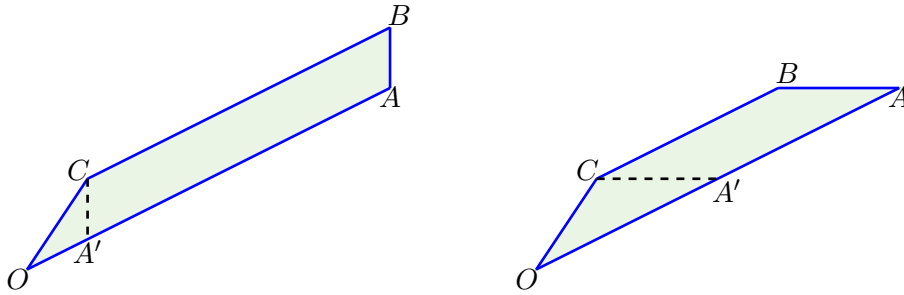

FIGURE 9. Trapezoids; obtuse and acute

Let A' be the point of the side OA such that $A'C$ is parallel to AB. Let a, c denote the lengths of sides OA, BC respectively, and let h be the distance

between the parallel side OA, BC. Note that $K = h(a+c)/2$, so h is rational. The triangle $OA'C$ has integer sides. Its area is $h(a-c)/2$, so it is rational. Hence its area is an integer (so the triangle is Heronian) by the following well known lemma, whose proof we include for completeness (see [3]).

Lemma 4.1. *If a triangle has integer sides and rational area, then its area is an integer.*

Proof. Consider a triangle with integer sides x, y, z and rational area T. By Heron's formula,

$$4T = \sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}.$$

Since $(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$ is an integer, and its square root, $4T$, is rational, $4T$ is necessarily an integer. Notice that the terms $(x+y+z), (-x+y+z), (x-y+z), (x+y-z)$ have the same parity. So to see that T is an integer, it remains to show that the perimeter $S := x+y+z$ is even. We have

$$(4T)^2 = S(S-2x)(S-2y)(S-2z) \equiv S^4 - 2S^3(x+y+z) \equiv -S^4 \pmod{4}.$$

But this is possible only if $4T$ and S are even. $\square$

Since $OABC$ is equable its area is greater than the sum of the lengths of the parallel sides; that is, $h(a+c)/2 > a+c$. So $h > 2$. Now the area K is $K(OA'C) + hc$ and the perimeter P is $P(OA'C) + 2c$. Hence, as $OABC$ is equable,

$$P(OA'C) = K(OA'C) + (h-2)c > K(OA'C).$$

In the language of [2], $OA'C$ is a *perimeter-dominant* Heronian triangle. Heronian triangles with this property have been classified [6, 2]. The complete list is given in Tables 1 and 2. We will see that the triangles of Table 1 arise as the triangle $OA'C$ of an equable trapezoid $OABC$, while the triangles of Table 2 do not. For an equable trapezoid $OABC$, as above, let f denote the length OA'. Then from above $P(OA'C) = K(OA'C) + (h-2)c = K(OA'C) + (\frac{2K(OA'C)}{f} - 2)c$. Rearranging, we have

$$(3) \qquad c = \frac{f\,(Perimeter - Area)}{2(Area - f)}.$$

The constraint that eliminates triangles is the requirement that c is an integer. Table 1 gives three special perimeter-dominant Heronian triangles. The table gives the side lengths, perimeter and area, and in the 4th column, we have the values of c, given by (3), corresponding respectively to the three values of f given in the first column.

For the 3, 4, 5 triangle, given in the first row of Table 1, each choice of side for OA' results in an equable trapezoid.

Side Lengths f	Perimeter	Area	c
(3,4,5)	12	6	(3,6,15)
(5,5,6)	16	12	$(\frac{10}{7}, \frac{10}{7}, 2)$
(5,5,8)	18	12	$(\frac{15}{7}, \frac{15}{7}, 6)$

TABLE 1. Three special perimeter-dominant Heronian triangles

- If $f = 3$, (3) gives $c = 3$, so $OABC$ has parallel sides of length $c = 3$ and $a = c + f = 6$. The other sides have length 4 and 5. This trapezoid is the lattice equable right trapezoid shown on the left of Figure 3.
- If $f = 4$, we obtain $c = 6$, so $OABC$ has parallel sides of length $c = 6$ and $a = c + f = 10$. The other sides have length 3 and 5. This trapezoid is the lattice equable right trapezoid shown on the right of Figure 3.
- If $f = 5$, we obtain $c = 15$, so $OABC$ has parallel sides of length $c = 15$ and $a = c + f = 20$. The other sides have length 3 and 4. This trapezoid is the lattice equable trapezoid shown in Figure 5.

For the 5, 5, 6 triangle, only the choice of $f = 6$ produces an integer value of c. Here (3) gives $c = 2$, so $OABC$ has parallel sides of length $c = 2$ and $a = c + f = 8$. The other sides each have length 5. This trapezoid is the lattice equable isosceles trapezoid shown on the left of Figure 4.

Similarly, for the **5, 5, 8 triangle**, only the choice of $f = 8$ produces an integer value of c. Here (3) gives $c = 6$, so $OABC$ has parallel sides of length $c = 6$ and $a = c + f = 14$. The other sides each have length 5. This trapezoid is the lattice equable isosceles trapezoid shown on the right of Figure 4.

Side Lengths f	Restrictions	Perimeter	Area
$(3, 3x^2 + 1, 3x^2 + 2)$	$x \geq 7,\ x^2 + 1 = 2y^2$	$6x^2 + 6$	$6xy$
$(3, 3x^2 - 2, 3x^2 - 1)$	$x \geq 3,\ x^2 - 1 = 2y^2$	$6x^2$	$6xy$
$(4, 2x^2 + 1, 2x^2 + 3)$	$x \geq 5,\ x^2 + 2 = 3y^2$	$4x^2 + 8$	$6xy$
$(4, 4x^2 - 3, 4x^2 - 1)$	$x \geq 2,\ x^2 - 1 = 3y^2$	$8x^2$	$12xy$

TABLE 2. Infinite families of perimeter-dominant Heronian triangles

It remains for us to show that none of the perimeter-dominant Heronian triangles of Table 2 give rise to an equable trapezoid. In Table 2, the variable x is a positive integer subject to a certain Pell, or Pell-like, equation, as given in the Restrictions column of the table. We examine in turn the four cases given by the four rows of Table 2, and in each case we consider the three given possible values of the side lengths f. In each case, we assume that the value c given by (3) is an integer, and we derive a contradiction. We will on occasion employ the following elementary result.

Lemma 4.2. *Consider positive real constants $\alpha, \beta, \gamma, \delta$ and the real function g defined here below on the open interval $(\delta/\gamma, +\infty)$. Then*

$$g(r) = \frac{\alpha r + \beta}{\gamma r - \delta} < 1, \qquad for \quad r > \frac{\beta + \delta}{\gamma - \alpha}.$$

1st Row: Here $x \geq 7$ and $x^2 + 1 = 2y^2$. In particular, x is odd. The perimeter is $6x^2 + 6$, and the area is $6xy$. So by (3),

$$c = \frac{f(6x^2 + 6 - 6xy)}{2(6xy - f)} = \frac{f(3x^2 + 3 - 3xy)}{6xy - f}.$$

- If $f = 3$, we have $c = \frac{3x^2 + 3 - 3xy}{2xy - 1}$. Using $y > \frac{x}{\sqrt{2}}$, Lemma 4.2 gives

$$c = \frac{3x^2 + 3 - 3xy}{2xy - 1} < \frac{(3 - \frac{3}{\sqrt{2}})x^2 + 3}{\sqrt{2}x^2 - 1} < 1,$$

for $x^2 > \frac{3+1}{\sqrt{2} - (3 - \frac{3}{\sqrt{2}})} = \frac{4\sqrt{2}}{5 - 3\sqrt{2}} \cong 7.47$. Thus $c < 1$ for $x \geq 3$, contradicting the assumption that c is a positive integer.

- If $f = 3x^2 + 1$, the value f is even since x is odd. Furthermore, f is not divisible by 3 or any divisor of x. Let z be an odd prime common divisor of f and $6xy - f$. So z divides $6xy$ and hence z divides y. But $x^2 + 1 = 2y^2$, so z divides $x^2 + 1$ and $f = 3x^2 + 1$, which is impossible as z does not divide x. Hence $\gcd(f, 6xy - f) = 2$. Thus, as c is an integer, $3xy - f/2$ divides $3x^2 + 3 - 3xy$, say $2(3x^2 + 3 - 3xy) = k(6xy - f)$, for some positive integer k. Using $y > \frac{x}{\sqrt{2}}$, Lemma 4.2 gives

$$k = \frac{2(3x^2 + 3 - 3xy)}{6xy - 3x^2 - 1} < \frac{2((3 - \frac{3}{\sqrt{2}})x^2 + 3)}{(\frac{6}{\sqrt{2}} - 3)x^2 - 1} < 2,$$

for $x^2 > \frac{3+1}{(\frac{6}{\sqrt{2}} - 3) - (3 - \frac{3}{\sqrt{2}})} = \frac{4\sqrt{2}}{9 - 6\sqrt{2}} \cong 11.0$. Thus $k < 2$ for $x \geq 4$. So the only possibility is $k = 1$, but then

$$2(3x^2 + 3 - 3xy) = 6xy - 3x^2 - 1 \Leftrightarrow 9x^2 - 12xy + 7 = 0,$$

which is impossible mod 3.

- If $f = 3x^2 + 2$, the value f is odd since x is odd. Furthermore, f is not divisible by 3 or any divisor of x. Let z be a prime common divisor of f and $6xy - f$. So z divides y. But $x^2 + 1 = 2y^2$, so z divides $2x^2 + 2$ and $f = 3x^2 + 2$, which is again impossible as z does not divide x. Hence $\gcd(f, 6xy - f) = 1$. Thus, as c is an integer, $6xy - f$ divides $3x^2 + 3 - 3xy$, say $3x^2 + 3 - 3xy = k(6xy - f)$, for some positive integer k. Using $y > \frac{x}{\sqrt{2}}$,

Lemma 4.2 gives

$$k = \frac{3x^2 + 3 - 3xy}{6xy - 3x^2 - 2} < \frac{(3 - \frac{3}{\sqrt{2}})x^2 + 3}{(\frac{6}{\sqrt{2}} - 3)x^2 - 2} < 1,$$

for $x^2 > \frac{3+2}{(\frac{6}{\sqrt{2}}-3)-(3-\frac{3}{\sqrt{2}})} = \frac{5\sqrt{2}}{9-6\sqrt{2}} \cong 13.7$. Thus $k < 1$ for $x \geq 4$. So this case is impossible.

2nd Row: Here $x \geq 3$ and $x^2 - 1 = 2y^2$. In particular, x is odd. Moreover, x is not a multiple of 3 if and only if y is a multiple of 3. The perimeter is $6x^2$, and the area is $6xy$. So by (3),

$$c = \frac{f(3x^2 - 3xy)}{6xy - f} = \frac{3fx(x - y)}{6xy - f}.$$

- If $f = 3$, we have $c = \frac{3x(x-y)}{2xy-1}$. As x is not a multiple of 3 if and only if y is a multiple of 3, so $2xy - 1$ is not divisible by 3. Furthermore, $2xy - 1$ is not divisible by prime divisor of x. So as $2xy - 1$ divides $3x(x - y)$, necessarily $2xy - 1$ divides $x - y$, say $x - y = k(2xy - 1)$, for some positive integer k. So $x(2ky - 1) = k - y$ and thus $x = \frac{k-y}{2ky-1} \leq \frac{k-1}{2k-1} \leq 1$, contradicting the assumption that $x \geq 3$.

- If $f = 3x^2 - 2$, the value f is odd, and not divisible by 3 or any divisor of x. Let z be a prime common divisor of f and $6xy - f$. So z divides y. But $x^2 - 1 = 2y^2$, so z divides $x^2 - 1$ and $f = 3x^2 - 2$, which is impossible as z does not divide x. Hence $\gcd(f, 6xy - f) = 1$. Thus, as c is an integer, $6xy - f$ divides $3x^2 - 3xy$. Since f is odd and not divisible by 3 or any divisor of x, so $6xy - f$ divides $x - y$, say $x - y = k(6xy - 3x^2 + 2)$, for some positive integer k. Now $2y > x$ (as $4y^2 > 2y^2 + y^2 = x^2 - 1 + y^2 \geq x^2$), so

$$k = \frac{x - y}{6xy - 3x^2 + 2} < \frac{x - y}{6xy - 3x^2} = \frac{x - y}{3x(2y - x)} \leq \frac{x - y}{3x} < 1,$$

which is impossible.

- If $f = 3x^2 - 1$, the value f is even (as x is odd), and not divisible by 3 or any divisor of x. Moreover, observing the previous equality in $\mathbb{Z}_4$ implies f is exactly divisible by 2. Let z be an odd prime common divisor of f and $6xy - f$. So z divides y. But $x^2 - 1 = 2y^2$, so z divides $x^2 - 1$ and $f = 3x^2 - 1$, which is impossible as z does not divide x. Hence $\gcd(f, 6xy - f) = 2$. Thus, as c is an integer, $3xy - f/2$ divides $3x^2 - 3xy$. Since f is not divisible by 3 or any divisor of x, neither is $3xy - f/2$, so $3xy - f/2$ divides $x - y$, say $2(x - y) = k(6xy - 3x^2 + 1)$, for some positive integer k. Now $2y > x$ so

$$\frac{2(x - y)}{6xy - 3x^2 + 1} < \frac{2(x - y)}{6xy - 3x^2} = \frac{2(x - y)}{3x(2y - x)} \leq \frac{2(x - y)}{3x} < 1,$$

which is impossible.

3rd Row: Here $x \geq 5$ and $x^2 + 2 = 3y^2$, which observed in $\mathbb{Z}_4$ gives x and y are both odd, and x is not divisible by 3. The perimeter is $4x^2 + 8$, and the area is $6xy$. So by (3),

$$c = \frac{f(2x^2 + 4 - 3xy)}{6xy - f}.$$

- $f = 4$, we have $c = \frac{2(2x^2 + 4 - 3xy)}{3xy - 2}$. As x, y are both odd, $3xy - 2$ is odd, and thus $3xy - 2$ divides $2x^2 + 4 - 3xy$, say $2x^2 + 4 - 3xy = k(3xy - 2)$, for some positive integer k. Using $y > x/\sqrt{3}$, Lemma 4.2 gives

$$\frac{2x^2 + 4 - 3xy}{3xy - 2} < \frac{(2 - \sqrt{3})x^2 + 4}{\sqrt{3}x^2 - 2} < 1,$$

  for $x^2 > \frac{4+2}{\sqrt{3}-(2-\sqrt{3})} = \frac{6}{2\sqrt{3}-2} \cong 4.1$. Thus $k < 1$ for $x \geq 3$, contradicting the assumption that k is a positive integer.

- If $f = 2x^2 + 1$, the value f is odd, divisible by 3 but not by any divisor of x. Let $z > 3$ be a prime common divisor of f and $6xy - f$. So z divides y. But $x^2 + 2 = 3y^2$, so z divides $x^2 + 2$ and $f = 2x^2 + 1$, which is impossible as z does not divide x. Hence $\gcd(f, 6xy - f) = 3$. Thus, as c is an integer, $2xy - f/3$ divides $2x^2 + 4 - 3xy$, say $3(2x^2 + 4 - 3xy) = k(6xy - 2x^2 - 1)$, for some positive integer k. Using $y > x/\sqrt{3}$, Lemma 4.2 gives

$$\frac{3(2x^2 + 4 - 3xy)}{6xy - 2x^2 - 1} < \frac{3((2 - \sqrt{3})x^2 + 4)}{(2\sqrt{3} - 2)x^2 - 1} < 1$$

  for $x^2 > \frac{12+1}{(2\sqrt{3}-2)-3(2-\sqrt{3})} = \frac{13}{5\sqrt{3}-8} \cong 19.69$. Thus $k < 1$ for $x \geq 5$, contradicting the assumption that k is a positive integer.

- If $f = 2x^2 + 3$, the value f is odd, and is not divisible by 3 (x is not divisible by 3) or any divisor of x. Let z be a prime common divisor of f and $6xy - f$. So z divides y. But $x^2 + 2 = 3y^2$, so z divides $x^2 + 2$ and $f = 2x^2 + 3$, which is impossible as z does not divide 1. Hence $\gcd(f, 6xy - f) = 1$. Thus $2xy - f/3$ divides $2x^2 + 4 - 3xy$, say $2x^2 + 4 - 3xy = k(6xy - 2x^2 - 3)$, for some positive integer k. Using $y > x/\sqrt{3}$, Lemma 4.2 gives

$$\frac{2x^2 + 4 - 3xy}{6xy - 2x^2 - 3} < \frac{(2 - \sqrt{3})x^2 + 4}{(2\sqrt{3} - 2)x^2 - 3} < 1,$$

  for $x^2 > \frac{4+3}{(2\sqrt{3}-2)-(2-\sqrt{3})} = \frac{7}{3\sqrt{3}-4} \cong 5.85$. Thus $k < 1$ for $x \geq 3$, contradicting the assumption that k is a positive integer.

4th Row: Here $x \geq 2$ and $x^2 - 1 = 3y^2$. Note x is not divisible by 3. Furthermore, x, y have opposite parity, so xy is even. The perimeter is $8x^2$, and the area is $12xy$. So by (3),

$$c = \frac{f(4x^2 - 6xy)}{12xy - f} = \frac{2fx(2x - 3y)}{12xy - f}.$$

- If $f = 4$, we have $c = \frac{2x(2x-3y)}{3xy-1}$. Now $3xy - 1$ is odd and not divisible by any prime divisor of x. So $3xy - 1$ divides $2x - 3y$, say $2x - 3y = k(3xy - 1)$, for some positive integer k. Thus $x = \frac{k-3y}{3ky-2} < \frac{k}{3k-2} \leq 1$, contradicting the assumption that $x \geq 2$.

- If $f = 4x^2 - 3$, the value f is odd and not divisible by 3 (as x is not divisible by 3) or any of the prime divisors of x. Let z be a prime common divisor of f and $12xy - f$. So z divides y. But $x^2 - 1 = 3y^2$, so z divides $x^2 - 1$ and $f = 4x^2 - 3$, which is impossible as z does not divide x. Hence $\gcd(f, 12xy - f) = 1$. Thus, as c is an integer and $12xy - f$ is odd and $\gcd(x, 12xy - f) = 1$, we have that $12xy - f$ divides $2x - 3y$, say $2x - 3y = k(12xy - 4x^2 + 3)$, for some positive integer k. Now $2y \geq x$ (as $4y^2 = 3y^2 + y^2 = x^2 - 1 + y^2 \geq x^2$), so

$$k = \frac{2x - 3}{12xy - 4x^2 + 3} < \frac{2x}{2x^2 + 3} < 1,$$

for $x \geq 1$. So this is impossible.

- If $f = 4x^2 - 1$, the value f is odd. It is divisible by 3, but not by any of the prime divisors of x. Let $z > 3$ be a prime common divisor of f and $12xy - f$. So z divides y. But $x^2 - 1 = 3y^2$, so z divides $x^2 - 1$ and $f = 4x^2 - 1$, which is impossible as z does not divide x. Hence $\gcd(f, 12xy - f) = 3$. Thus, as c is an integer and $12xy - f$ is odd and $\gcd(x, 12xy - f) = 1$, $(12xy - f)/3$ divides $2x - 3y$, say $3(2x - 3y) = k(12xy - 4x^2 + 1)$, for some positive integer k. Using $2y \geq x$ again,

$$k = \frac{3(2x - 3y)}{12xy - 4x^2 + 1} < \frac{3x}{4x^2 + 2} < 1,$$

for $x \geq 1$. So this case is also impossible.

This completes the proof of Theorem 1.2.

5. Cyclic quadrilaterals; Proof of Theorem 1.3

Consider an equable cyclic quadrilateral with integer sides $a \leq b \leq c \leq d$. By Brahmagupta's formula's,

(4) $\quad (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) = 16(a+b+c+d)^2.$

Notice each sum above is of same parity and hence even. Let $2z = -a + b + c + d, 2y = a - b + c + d, 2x = a + b - c + d, 2w = a + b + c - d$ and note by elementary geometry $0 < w \leq x \leq y \leq z$. Hence

(5) $\qquad\qquad\qquad wxyz = (w + x + y + z)^2.$

Conversely,

(6)
$$a = \frac{1}{2}(-w + x + y + z), \quad c = \frac{1}{2}(w - x + y + z),$$
$$c = \frac{1}{2}(w + x - y + z), \quad d = \frac{1}{2}(w + x + y - z).$$

Since $w \leq x \leq y \leq z$, one has $w + x + y - z = 2d > 0$, and so

(7) $$z < w + x + y.$$

From (5),

$$z = \frac{wxy - 2(w + x + y) \pm \sqrt{w^2x^2y^2 - 4wxy(w + x + y)}}{2}.$$

In particular, $w^2x^2y^2 - 4wxy(w + x + y) \geq 0$, so

(8) $$wxy \geq 4(w + x + y).$$

Suppose for the moment that z is given by the positive square root. Then as $z \leq w + x + y$, we would have

$$z = \frac{wxy - 2(w + x + y) + \sqrt{w^2x^2y^2 - 4wxy(w + x + y)}}{2} \leq w + x + y,$$

so $\sqrt{w^2x^2y^2 - 4wxy(w + x + y)} \leq 4(w + x + y) - wxy \leq 0$, by (8). So the positive square root only gives a solution when the square root is zero, and this is encompassed by the case where z is given by the negative square root. So we may assume that

(9) $$z = \frac{wxy - 2(w + x + y) - \sqrt{w^2x^2y^2 - 4wxy(w + x + y)}}{2}.$$

Then $y \leq z$ gives $\sqrt{w^2x^2y^2 - 4wxy(w + x + y)} \leq wxy - 2(w + x + 2y)$. Squaring and simplifying gives

(10) $$wxy^2 \leq (w + x + 2y)^2.$$

Consequently, as $w \leq x \leq y$, we have $wxy^2 \leq (4y)^2$, so $wx \leq 16$. Hence $w^2 \leq wx \leq 16$. In particular $w \leq 4$. Rewriting (10) and using the previous we get $wx > 4$ and

$$y \leq \frac{w + x}{\sqrt{wx} - 2} \leq \frac{4 + 16}{\sqrt{5} - 2} < 85.$$

There are 63 triples (w, x, y) satisfying $w \leq x \leq y \leq 84$ with $5 \leq wx \leq 16$. Testing each possibility by replacing it in (9), and checking if z is an integer and $y \leq z < w + x + y$, we obtain only four solutions for (w, x, y, z), namely

$$(1, 9, 10, 10), \ (2, 5, 5, 8), \ (3, 3, 6, 6), \ (4, 4, 4, 4),$$

which by (6), give the corresponding four solutions announced for (a, b, c, d):

(11) $(14, 6, 5, 5), \ (8, 5, 5, 2), \ (6, 6, 3, 3), \ (4, 4, 4, 4).$

It remains to consider the possible ordering of the sides. For this, we use the hypothesis that the quadrilateral is a LEQ. The numbers $2^2, 3^2, 4^2, 6^2, 8^2, 14^2$ cannot be written as the sum of two nonzero squares, so the corresponding sides are each either vertical or horizontal. Thus the last two cases are obviously the 3×6 rectangle and the 4×4 square, while it is easy to see that the first two cases are necessarily isosceles trapezoids, arranged either vertically

or horizontally, and hence congruent to those of Figure 4. This completes the proof of Theorem 1.3.

Remark 5.1. *To put Theorem 1.3 in context, note that there are cyclic equable quadrilaterals whose sides lengths, in cyclic order, are any of the permutations of the lists in* (11). *For example, the equable kite with side length 6,3,3,6 is cyclic, but cannot be realized as a LEQ; see Figure 2.*

6. Proof of Theorem 1.4

Let a, b, c, d denote the lengths of OA, AB, BC, CO respectively. Consider a diagonal that is internal to $OABC$. By relabelling the vertices if necessary, we may take this diagonal to be OB. Assume that $p = OB$ is rational. So, since p is the distance between lattice points, p is an integer. Moreover, the area of triangles OAB and OBC are rational, as they are given by $\frac{1}{2}\|\overrightarrow{OA} \times \overrightarrow{OB}\|$ and $\frac{1}{2}\|\overrightarrow{OB} \times \overrightarrow{OC}\|$ respectively. Hence by Lemma 4.1, both triangles have integer area. So they are Heronian triangles.

Notice that $p > 4$. Indeed, let h_A, h_C denote the respective altitudes of the triangles OAB, OBC from the side OB. So $a + b > 2h_A$ and $c + d > 2h_C$. Thus

$$a + b + c + d = K = K_A + K_C = \frac{1}{2}(h_A + h_C)p < \frac{p}{4}(a + b + c + d).$$

Now observe that one of triangles OAB, OBC is perimeter dominant, as otherwise one would have

$$a + b + c + d = K = K_A + K_C \geq (a + b + p) + (c + d + p),$$

which is impossible. Assume without loss of generality that OAB is perimeter dominant. For the reader's convenience, we recall the possibilities for perimeter dominant Heronian triangles in Table 3. We treat each row in order, and consider the possibilities given that $p > 4$. Our strategy is to show that in each case, the triangle BCO is also perimeter dominant. For this, we use the following equation,

$$(12) \qquad \Delta_C := P_C - K_C = K_A + p - a - b,$$

which holds as $P_C = p + c + d$ and $K_C = K - K_A = a + b + c + d - K_A$, since $OABC$ is equable. Then when we consider row i for OAB, we need only consider rows $\geq i$ for BCO. To avoid confusion, when we consider BCO, we relabel the variables in these rows u, v instead of x, y.

Side Lengths	Restrictions	Peri. P	Area K	$\Delta = P - K$
(3,4,5)	-	12	6	6
(5,5,6)	-	16	12	4
(5,5,8)	-	18	12	6
$(3, 3x^2 + 1, 3x^2 + 2)$	$x \geq 7, \ x^2 + 1 = 2y^2$	$6x^2 + 6$	$6xy$	$6(x^2 - xy + 1)$
$(3, 3x^2 - 2, 3x^2 - 1)$	$x \geq 3, \ x^2 - 1 = 2y^2$	$6x^2$	$6xy$	$6(x^2 - xy)$
$(4, 2x^2 + 1, 2x^2 + 3)$	$x \geq 5, \ x^2 + 2 = 3y^2$	$4x^2 + 8$	$6xy$	$4x^2 - 6xy + 8$
$(4, 4x^2 - 3, 4x^2 - 1)$	$x \geq 2, \ x^2 - 1 = 3y^2$	$8x^2$	$12xy$	$8x^2 - 12xy$

TABLE 3. Perimeter-dominant Heronian triangles

Row 1. Here $K_A = 6$ and $p = 5$ and $\{a, b\} = \{3, 4\}$. From (12),

$$\Delta_C = K_A + p - a - b = 4.$$

So BCO is perimeter dominant. Consultation of Table 3 shows that the only perimeter dominant Heronian triangle having a side of length 5 with perimeter minus area equal to 4 is the 5,5,6 triangle. Thus $OABC$ has sides 3,4,5,6, being composed of a 3,4,5 right angled triangle joined along its hypotenuse with a 5,5,6 triangle. It follows that $OABC$ is the right trapezoid LEQ shown on the left of Figure 3, which indeed has a diagonal of length 5.

Row 2. Here $K_A = 12$. If $p = 5$, then $\{a, b\} = \{5, 6\}$ and from (12),

$$\Delta_C = K_A + p - a - b = 6.$$

So BCO is perimeter dominant. From Table 3, there is only perimeter dominant Heronian triangle in rows ≥ 2 having a side of length 5 with perimeter minus area equal to 6, namely the 5,5,8 triangle. Hence $OABC$ has two consecutive sides of length 5,6, two other consecutive sides of length 5,8 and those of length 6 and 8 must be horizontal or vertical, in particular either parallel or perpendicular to one another. Consideration of the possibilities leads one to the 6,8,10 right angle triangle with a vertex in the middle of the hypotenuse; we eliminate this possibility as failing the definition of a quadrilateral.

If $p = 6$, then $\{a, b\} = \{5, 5\}$ and

$$\Delta_C = K_A + p - a - b = 8.$$

So BCO is perimeter dominant. From Table 3, there is no perimeter dominant Heronian triangle having a side of length 6 with perimeter minus area equal to 8. So this case is impossible.

Row 3. Here $K_A = 12$. If $p = 5$, then $\{a, b\} = \{5, 8\}$ and

$$\Delta_C = K_A + p - a - b = 4.$$

So BCO is perimeter dominant. But from Table 3, there is no perimeter dominant Heronian triangle in rows ≥ 3 having a side of length 5 with perimeter minus area equal to 4.

If $p = 8$, then $\{a, b\} = \{5, 5\}$ and

$$\Delta_C = K_A + p - a - b = 10.$$

So BCO is perimeter dominant. But from Table 3, there is no perimeter dominant Heronian triangle having a side of length 8 with perimeter minus area equal to 10. So this case is also impossible.

Row 4. Here $K_A = 6xy$, where $x \geq 7$ and $x^2 + 1 = 2y^2$. If $p = 3x^2 + 1$, then $\{a, b\} = \{3, 3x^2 + 2\}$ and

$$\Delta_C = K_A + p - a - b = 6xy + 3x^2 + 1 - 3 - 3x^2 - 2 = 6xy - 4.$$

So BCO is perimeter dominant. We again examine Table 3. Notice that $P_C - K_C \equiv 2 \pmod 6$, so we need only consider the bottom 2 rows. For BCO in row 6, for the perimeter minus area, $4u^2 + 8 - 6uv$, to be congruent to 2 $\pmod 6$, we would require u to be a multiple of 3, which is impossible because $u^2 + 2 = 3v^2$. It remains to consider row 7. Suppose OBC has side lengths $4, 4u^2 - 3, 4u^2 - 1$, with $u \geq 2$ and $u^2 - 1 = 3v^2$. So either $p = 3x^2 + 1 = 4u^2 - 3$ or $3x^2 + 1 = 4u^2 - 1$. But in both cases x must be even, which is impossible as $x^2 + 1 = 2y^2$.

If $p = 3x^2 + 2$, then $\{a, b\} = \{3, 3x^2 + 1\}$ and

$$\Delta_C = K_A + p - a - b = 6xy + 3x^2 + 2 - 3 - 3x^2 - 1 = 6xy - 2.$$

So BCO is perimeter dominant. Notice that now $P_C - K_C \equiv 4 \pmod 6$, so we need again only consider rows 6 and 7 of Table 3. For BCO in row 6, for the perimeter minus area, $4u^2 + 8 - 6uv$, to be congruent to 4 $\pmod 6$, we would require $4u^2 \equiv 2 \pmod 6$, or equivalently, $2u^2 \equiv 1 \pmod 3$. But this is impossible. For BCO in row 7, for the perimeter minus area, $8u^2 - 12uv$, to be congruent to 4 $\pmod 6$, we require $2u^2 \equiv 4 \pmod 6$, or equivalently, $u^2 \equiv 2 \pmod 3$. But this is again impossible.

Row 5. Here $K_A = 6xy$, where $x \geq 3$ and $x^2 - 1 = 2y^2$, in particular x is odd. If $p = 3x^2 - 2$, then $\{a, b\} = \{3, 3x^2 - 1\}$ and

$$\Delta_C = K_A + p - a - b = 6xy + 3x^2 - 2 - 3 - 3x^2 + 1 = 6xy - 4.$$

Hence BCO is perimeter dominant and $\Delta_C \equiv 2 \pmod 6$ so we consider only the two last rows. If $\Delta_C = 4u^2 - 6uv + 8 \equiv 2 \pmod 6$, then u is a multiple of 3, which is impossible because $u^2 + 2 = 3v^2$. Otherwise, either $3x^2 - 2 = 4u^2 - 3$ or $3x^2 - 2 = 4u^2 - 1$, with both $x^2 - 1 = 2y^2$ and $u^2 - 1 = 3v^2$. By substituting x and u respectively by y and v we get in the first case $y^2 = 2v^2$ a contradiction, and in the second $3y^2 - 6v^2 = 1$ another contradiction.

If $p = 3x^2 - 1$, then $\{a, b\} = \{3, 3x^2 - 2\}$ and

$$\Delta_C = K_A + p - a - b = 6xy + 3x^2 - 1 - 3 - 3x^2 + 2 = 6xy - 2.$$

Once again, BCO is perimeter dominant, $\Delta_C \equiv 4 \pmod 6$, and $4u^2 + 8 - 6uv \equiv 4 \pmod 6$ implies $2 \equiv u^2 \pmod 3$ a contradiction. Otherwise, either $3x^2 - 1 =$

$4u^2 - 3$ or $3x^2 - 1 = 4u^2 - 1$. In the first case, $3x^2 = 4u^2 - 2$ is impossible since x is odd. The second implies $3x^2 = 4u^2$, another contradiction.

Row 6. Here $K_A = 6xy$, where $x \geq 5$ and $x^2 + 2 = 3y^2$. Moreover, x is odd by observing $x^2 + 2 = 3y^2$ in $\mathbb{Z}_4$. If $p = 2x^2 + 1$, then $\{a, b\} = \{4, 2x^2 + 3\}$ and

$$\Delta_C = K_A + p - a - b = 6xy + 2x^2 + 1 - 4 - 2x^2 - 3 = 6xy - 6.$$

So BCO is perimeter dominant. Since $x^2 + 2 = 3y^2$, x is not divisible by 3, so $p = 2x^2 + 1$ is divisible by 3. We consider the rows 6 and 7 of Table 3. For BCO in row 6, $u^2 + 2 = 3v^2$ so u is not divisible by 3. Thus p cannot equal the side $2u^2 + 3$. If $p = 2u^2 + 1$, then $p = 2x^2 + 1 = 2u^2 + 1$, so $x = u$, and thus $y = v$. But then, as the perimeter minus area is $6xy - 6 = 4u^2 + 8 - 6uv$, we have $6uv - 2u^2 = 7$, which is impossible.

For BCO in row 7, as the perimeter minus area is divisible by 6, we have $8u^2 - 12uv \equiv 0 \pmod 6$, so u must be a multiple of 3. But this is impossible as $u^2 - 1 = 3v^2$.

If $p = 2x^2 + 3$, then $\{a, b\} = \{4, 2x^2 + 1\}$ and

$$\Delta_C = K_A + p - a - b = 6xy + 2x^2 + 3 - 4 - 2x^2 - 1 = 6xy - 2.$$

Once again, BCO is perimeter dominant, $\Delta_C \equiv 4 \pmod 6$, and $4u^2 + 8 - 6uv \equiv 4 \pmod 6$ implies $2 \equiv u^2 \pmod 3$ a contradiction. Otherwise, either $2x^2 + 3 = 4u^2 - 3$ or $2x^2 + 3 = 4u^2 - 1$. In the first case, $3 = 2u^2 - x^2$ is impossible since $3 \nmid ux$. The second implies $2 = 2u^2 - x^2$, another contradiction as x is odd.

Row 7. Here $K_A = 12xy$, where $x \geq 2$ and $x^2 - 1 = 3y^2$. If $p = 4x^2 - 3$, then $\{a, b\} = \{4, 4x^2 - 1\}$ and

$$\Delta_C = K_A + p - a - b = 12xy + 4x^2 - 3 - 4 - 4x^2 + 1 = 12xy - 6.$$

So BCO is perimeter dominant. For BCO in row 7, as the perimeter minus area is divisible by 6, we have $8u^2 \equiv 0 \pmod 6$, so u must be a multiple of 3. But this is impossible as $u^2 - 1 = 3v^2$.

This completes the proof of the theorem.

REFERENCES

1. Aebi, C. and Cairns, G., *Lattice equable quadrilaterals I: parallelograms*, to appear in L'Enseign. Math., *preprint available at* `https://arxiv.org/abs/2006.07566`.
2. Balmoral Software, *Equability*, `http://www.balmoralsoftware.com/equability/equability.htm`.
3. Blichfeldt, H.F., *On triangles with rational sides and having rational areas*, Ann. of Math., **11(1896/97)**, 57–60.
4. Cheney, Wm. Fitch, Jr., *Heronian triangles*, Amer. Math. Monthly, **36(1929)**, 22–28.
5. Dickson, L.E., *History of the theory of numbers. Vol. II*, Chelsea Publishing Co., New York, 1966.
6. Dolan, S., *Less than equable Heronian triangles*, Math. Gaz., **100(2016)**, 482–489.

7. Kummer, E.E., *Über die Vierecke, deren Seiten und Diagonalen rational sind*, J. Reine Angew. Math., **37(1848)**, 1–20.

8. Nagarajan, K.R. and Sridharan, R., *On Brahmagupta's and Kummer's quadrilaterals*, Elem. Math., **61(2006)**, 45–57.

9. Nelson, R.D. *Perfect quadrilaterals*, Amer. Math. Monthly, **86(1979)**, 845–847.

10. Read, E., *Heronian triangles*, Math. Gaz., **100(2016)**, 103–108.

11. Sastry, K.R.S., *Brahmagupta quadrilaterals*, Forum Geom., **2(2002)**, 167–173.

12. Sloane, N.J.A., *The On-line Encyclopedia of Integer Sequences*, `https:\oeis.org`.

13. Usiskin, Z. and Griffin, J., *The classification of quadrilaterals : a study of definition*, Research in Mathematics Education, IAP, Charlotte, 2008.

COLLÈGE CALVIN
GENEVA
SWITZERLAND 1211
E-mail address: `christian.aebi@edu.ge.ch`

DEPARTMENT OF MATHEMATICS
LA TROBE UNIVERSITY
MELBOURNE
AUSTRALIA 3086
E-mail address: `G.Cairns@latrobe.edu.au`