INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 11(2022), No. 1, 104-121

# TWO NEW PROOFS AND APPLICATIONS OF AN ACUTE TRIANGLE INEQUALITY INVOLVING MEDIANS AND SIDES

JIAN LIU

**Abstract.** Two new proofs of an acute triangle inequality involving medians and sides are obtained. Some applications of this inequality are also given.

## 1. Introduction

In 1996, in a Chinese paper the author propose the following inequality as an open problem (see the problem shc 15(h) in [5]):

**Theorem 1.1.** *Let $ABC$ be an acute triangle with side lengths $a, b, c$ and corresponding medians $m_a, m_b, m_c$. Then*

$$(m_b + m_c)^2 + (m_c + m_a)^2 + (m_a + m_b)^2 \geq (a + b + c)^2, \tag{1.1}$$

*with equality if and only if the acute triangle $ABC$ is equilateral.*

Four years later, X.G.Chu [2] proved inequality (1.1). In fact, he established the following inequality for the acute triangle $ABC$:

$$4(m_b m_c + m_c m_a + m_a m_b) \geq 5s^2 - (18 + 6\sqrt{2})Rr + (9 + 12\sqrt{2})r^2, \tag{1.2}$$

where $s, R$ and $r$ are the semiperimeter, circumradius and inradius of $\triangle ABC$, respectively. Based on (1.2), X.G.Chu further proved inequality (1.1).

By the well known identity:

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2), \tag{1.3}$$

we easily know that inequality (1.1) has the following equivalent forms:

$$(m_a + m_b + m_c)^2 \geq \frac{1}{4}(a^2 + b^2 + c^2 + 8bc + 8ca + 8ab), \tag{1.4}$$

$$m_b m_c + m_c m_a + m_a m_b \geq bc + ca + ab - \frac{1}{4}(a^2 + b^2 + c^2), \tag{1.5}$$

$$(m_b - m_c)^2 + (m_c - m_a)^2 + (m_a - m_b)^2 \leq (b - c)^2 + (c - a)^2 + (a - b)^2, \tag{1.6}$$

**Keywords and phrases:** Acute Triangle, Median, Side, Inequality
**(2021)Mathematics Subject Classification:** 51M16
Received: 25.06.2021. In revised form: 02.11.2021. Accepted: 28.09.2021.

$$(m_c+m_a)(m_a+m_b)+(m_a+m_b)(m_b+m_c) \\ +(m_b+m_c)(m_c+m_a) \geq 3(bc+ca+ab). \tag{1.7}$$

Also, from (1.4) or (1.7), it is easy to obtain the following consequence:

$$m_a+m_b+m_c \geq \frac{3}{2}\sqrt{bc+ca+ab}. \tag{1.8}$$

In fact, this weaker inequality still can not be proved easily.

The main purpose of this paper is to present two new proofs of inequality (1.1) and give some applications.

## 2. The first new proof of Theorem 1.1

In the next Lemma 2.1-2.3, we first give some identities in a triangle. For convenience, we shall use $\sum$ and $\prod$ to denote cyclic sums and products over three triples, respectively.

**Lemma 2.1.** *In $\triangle ABC$ we have*

$$\sum a^2 = 2s^2-8Rr-2r^2, \tag{2.1}$$

$$\sum a^3 = 2s^3-(12Rr+6r^2)s, \tag{2.2}$$

$$\sum a^4 = 2s^4-4(4R+3r)rs^2+2(4R+r)^2r^2, \tag{2.3}$$

$$\sum a^5 = 2s^5-20(Rr+r^2)s^3+10(2R+r)(4R+r)r^2s, \tag{2.4}$$

$$\sum a^6 = 2s^6-(24Rr+30r^2)s^4+(144R^2r^2+144Rr^3+30r^4)s^2 \\ -2(4R+r)^3r^3, \tag{2.5}$$

$$\sum a^7 = 2s^7-14(2R+3r)rs^5+14(16R^2+20Rr+5r^2)r^2s^3 \\ -14(2R+r)(4R+r)^2r^3s, \tag{2.6}$$

$$\sum a^8 = 2s^8-(32Rr+56r^2)s^6+(320R^2r^2+480Rr^3+140r^4)s^4 \\ -(1024R^3r^3+1280R^2r^4+480Rr^5+56r^6)s^2 \\ +2(4R+r)^4r^4. \tag{2.7}$$

**Lemma 2.2.** *In $\triangle ABC$ we have*

$$\sum b^3c^3 = s^6+(-12Rr+3r^2)s^4+3r^4s^2+(4R+r)^3r^3, \tag{2.8}$$

$$\sum b^4c^4 = s^8+(-16Rr+4r^2)s^6+(32R^2r^2-16Rr^3+6r^4)s^4 \\ +(16Rr^5+4r^6)s^2+r^4(4R+r)^4, \tag{2.9}$$

$$\sum b^5c^5 = s^{10}+(-20Rr+5r^2)s^8+(80R^2r^2-40Rr^3+10r^4)s^6 \\ +10r^6s^4+(80R^2r^6+40Rr^7+5r^8)s^2+r^5(4R+r)^5, \tag{2.10}$$

The identities (2.1)-(2.3) are well known (see e.g. [4, p.52-55]). The proofs of the other identities can be found in [7] and [8].

**Lemma 2.3.** *In $\triangle ABC$, we set*

$$K_1 = \sum \frac{(2a^2+ab+ca-b^2+2bc-c^2)^3}{m_b^2 m_c^2},$$

*then*

$$K_1 = \frac{64N_1}{M_1}, \tag{2.11}$$

*where*

$$M_1 = s^6 - 3(4R-11r)rs^4 - 3(20R^2+40Rr+11r^2)r^2s^2 - (4R+r)^3r^3,$$
$$N_1 = s^8 + 2(R+13r)rs^6 - 3(10R-9r)(4R-5r)r^2s^4 - 16(4R+r)(20R^2-14Rr-r^2)r^3s^2 - 16(4R+r)^4r^4.$$

**Proof.** Let

$$x_1 = 4\sum m_a^2(2a^2+ab+ca-b^2+2bc-c^2)^3.$$

Using the median formula

$$m_a^2 = \frac{1}{4}(2b^2+2c^2-a^2), \tag{2.12}$$

we get

$$x_1 = \sum(2b^2+2c^2-a^2)(2a^2+ab+ca-b^2+2bc-c^2)^3.$$

Further, it is easy to obtain

$$\begin{aligned} x_1 =& -15\sum a^8 + 6\sum a\sum a^7 - 3\sum a^2\sum a^6 - 174abc\sum a^5 \\ &- 27abc\sum a\sum a^4 + 129abc\sum a^2\sum a^3 + 144(abc)^2\sum a^2 \\ &+ 62\sum b^3c^3\sum a^2 - 90\sum b^4c^4 - 174(abc)^2\sum bc. \end{aligned} \tag{2.13}$$

Using the following basic identities

$$\sum a = 2s, \tag{2.14}$$

$$abc = 4Rrs, \tag{2.15}$$

$$\sum bc = s^2+4Rr+r^2, \tag{2.16}$$

and identities given in Lemma 2.1 and 2.2, we can obtain

$$x_1 = 16N_1. \tag{2.17}$$

Now, we set

$$x_2 = 16\prod m_a^2 m_b^2 m_c^2.$$

Using the formula (2.12), we easily get

$$4x_2 = -10\sum a^6 + 6\sum a^2\sum a^4 + 3(abc)^2. \tag{2.18}$$

Dividing both sides by 4 and using the previous identities (2.1), (2.3), (2.5) and (2.15), we further obtain

$$x_2 = M_1. \tag{2.19}$$

Finally, note that $K_1 = 4x_1/x_2$, the identity (2.11) follows from (2.17) and (2.19). Lemma 2.3 is proved.

**Lemma 2.4.** *Let $\triangle ABC$ be an acute (non-obtuse) triangle, then*

$$s^2 \geq 2R^2 + 8Rr + 3r^2, \tag{2.20}$$

*with equality if and only if $\triangle ABC$ is equilateral or right isosceles.*

Inequality (2.20) was first proposed by A.W.Walker in [12]. In the recent paper [9], the author gave a general generalization.

**Lemma 2.5.** *Let $\triangle ABC$ be an acute (non-obtuse) triangle, then*

$$s^2 \geq 16Rr - 3r^2 - \frac{4r^3}{R}, \tag{2.21}$$

*with equality if and only if $\triangle ABC$ is equilateral or right isosceles.*

Inequality (2.21) was first established by the author in [6]. Also, the author gave a direct proof in the recent paper [9].

Next, we give the first new proof of inequality (1.1).

**Proof.** According to the Radon inequality (cf.[3]), we know that for any positive real numbers $x, y, z, u, v, w$ the following inequality holds:

$$\frac{x^3}{u^2} + \frac{y^3}{v^2} + \frac{z^3}{w^2} \geq \frac{(x+y+z)^3}{(u+v+w)^2}. \tag{2.22}$$

Note that $a^2 - (b-c)^2 > 0$, we may take

$$\begin{aligned} x &= 2a^2 - b^2 - c^2 + 2bc + a(b+c), \\ y &= 2b^2 - c^2 - a^2 + 2ca + b(c+a), \\ z &= 2c^2 - a^2 - b^2 + 2ab + c(a+b). \end{aligned}$$

At the same time, we take $u = (m_b m_c)^2, v = (m_c m_a)^2, w = (m_a m_b)^2$, then

$$\sum \frac{(2a^2 - b^2 - c^2 + 2bc + ca + ab)^3}{(m_b m_c)^2} \geq \frac{64\left(\sum bc\right)^3}{\left(\sum m_b m_c\right)^2}. \tag{2.23}$$

Since inequality (1.1) is equivalent to (1.5), we only need to prove (1.5). Now by (2.23), it is sufficient to prove

$$64\left(\sum bc\right)^3 \geq \left(\sum bc - \frac{1}{4}\sum a^2\right)^2 \sum \frac{(2a^2 - b^2 - c^2 + 2bc + ca + ab)^3}{(m_b m_c)^2}.$$

Again, by Lemma 2.3, the above inequality is equivalent to

$$16M_1\left(\sum bc\right)^3 - N_1\left(4\sum bc - \sum a^2\right)^2 \geq 0. \tag{2.24}$$

In view of the previous identities (2.1) and (2.16), we have to prove the following inequality

$$Q_0 \equiv 4M_1\left(s^2 + 4Rr + r^2\right)^3 - N_1\left(s^2 + 12Rr + 3r^2\right)^2 \geq 0. \tag{2.25}$$

Substituting the expressions of $M_1$ and $N_1$ in Lemma 2.3 into (2.25), we further know that it is equivalent to

$$\begin{aligned}Q_0 \equiv &3s^{12} - 2(13R-56r)rs^{10} - 6(116R^2-15Rr-41r^2)r^2s^8\\&- 2(4R+r)(164R^2+1363Rr-280r^2)r^3s^6 + (2152R^2-4594Rr\\&+859r^2)(4R+r)^2r^4s^4 + 48(83R^2-38Rr-4r^2)(4R+r)^3r^5s^2\\&+140(4R+r)^6r^6 \geq 0. \qquad (2.26)\end{aligned}$$

According to Walker's inequality (2.20), we analyze to obtain

$$\begin{aligned}Q_0 \equiv\ &3w_0^6 + (36R^2+118rR+166r^2)w_0^5 + (180R^4+1180R^3r\\&+2804R^2r^2+6340Rr^3+2331r^4)w_0^4 + (480R^6+4720rR^5\\&+15792r^2R^4+41216r^3R^3+95704r^4R^2+72846r^5R\\&+15212r^6)w_0^3 + (720R^8+9440rR^7+40736r^2R^6+95136r^3R^5\\&+305160r^4R^4+633684r^5R^3+784944r^6R^2+360832r^7R\\&+53068r^8)w_0^2 + (576R^{10}+9440rR^9+49888r^2R^8+88832r^3R^7\\&+146784r^4R^6+846504r^5R^5+1173936r^6R^4+2904352r^7R^3\\&+2534192r^8R^2+844696r^9R+96384r^{10})w_0\\&+16(R-2r)(12R^{11}+260rR^{10}+1996r^2R^9+5580r^3R^8\\&-1261r^4R^7-18707r^5R^6-2502r^6R^5-125960r^7R^4\\&-191317r^8R^3-105063r^9R^2-25196r^{10}R\\&-2242r^{11}) \geq 0, \qquad (2.27)\end{aligned}$$

where $w_0 = s^2-2R^2-8Rr-3r^2 \geq 0$ (follows from Lemma 2.4). Thus, to prove $Q_0 \geq 0$ we only need to show that

$$\begin{aligned}Q_1 \equiv &(576R^{10}+9440rR^9+49888r^2R^8+88832r^3R^7+146784r^4R^6\\&+846504r^5R^5+1173936r^6R^4+2904352r^7R^3+2534192r^8R^2\\&+844696r^9R+96384r^{10})w_0+16(R-2r)(12R^{11}+260rR^{10}\\&+1996r^2R^9+5580r^3R^8-1261r^4R^7-18707r^5R^6-2502r^6R^5\\&-125960r^7R^4-191317r^8R^3-105063r^9R^2-25196r^{10}R\\&-2242r^{11}) \geq 0. \qquad (2.28)\end{aligned}$$

We consider the following two cases to finish the proof of inequality $Q_1 \geq 0$.

**Case 1.** $R$ and $r$ satisfy $R > \frac{12}{5}r$.

In this case, by Euler's inequality

$$R \geq 2r, \qquad (2.29)$$

which is valid for any $\triangle ABC$. We only need to prove

$$\begin{aligned}&12R^{11}+260rR^{10}+1996r^2R^9+5580r^3R^8-1261r^4R^7-18707r^5R^6\\&-2502r^6R^5-125960r^7R^4-191317r^8R^3-105063r^9R^2\\&-25196r^{10}R-2242r^{11} > 0. \qquad (2.30)\end{aligned}$$

By the hypothesis, we may assume that $R = \frac{12}{5}r + p(p \geq 0)$ and substitute it into (2.30). Then, all the terms of the left hand of (2.30) are non-negative after expanding. Hence inequality (2.30) holds.

**Case 2.** $R$ and $r$ satisfy $2r \leq R \leq \frac{12}{5}r$.

Substituting $w_0 = s^2 - 2R^2 - 8Rr - 3r^2$ into (2.28), expanding and collecting like terms gives

$$\begin{aligned} Q_1 \equiv &(576R^{10} + 9440R^9r + 49888R^8r^2 + 88832R^7r^3 + 146784R^6r^4 \\ &+ 846504R^5r^5 + 1173936R^4r^6 + 2904352R^3r^7 + 2534192R^2r^8 \\ &+ 844696Rr^9 + 96384r^{10})s^2 - 960R^{12} - 19712R^{11}r - 153408R^{10}r^2 \\ &- 579680R^9r^3 - 1352624R^8r^4 - 3392736R^7r^5 - 9001664R^6r^6 \\ &- 19675000R^5r^7 - 30855360R^4r^8 - 26234848R^3r^9 \\ &- 11594032R^2r^{10} - 2534760Rr^{11} - 217408r^{12} \geq 0. \end{aligned} \tag{2.31}$$

According to Lemma 2.5, to prove the above inequality we need to prove

$$\begin{aligned} Q_1 \equiv &(576R^{10} + 9440R^9r + 49888R^8r^2 + 88832R^7r^3 + 146784R^6r^4 \\ &+ 846504R^5r^5 + 1173936R^4r^6 + 2904352R^3r^7 + 2534192R^2r^8 \\ &+ 844696Rr^9 + 96384r^{10})\left(16Rr - 3r^2 - \frac{4r^3}{R}\right) - 960R^{12} \\ &- 19712R^{11}r - 153408R^{10}r^2 - 579680R^9r^3 - 1352624R^8r^4 \\ &- 3392736R^7r^5 - 9001664R^6r^6 - 19675000R^5r^7 - 30855360R^4r^8 \\ &- 26234848R^3r^9 - 11594032R^2r^{10} - 2534760Rr^{11} - 217408r^{12} \geq 0. \end{aligned}$$

Further, it is easily known that this inequality is equivalent to

$$Q_1 \equiv \frac{16(R-2r)}{R}Q_2 \geq 0, \tag{2.32}$$

where

$$\begin{aligned} Q_2 = &- 60R^{12} - 776R^{11}r - 1808R^{10}r^2 + 8128R^9r^3 + 8835R^8r^4 \\ &- 76720R^7r^5 + 80730R^6r^6 - 89707R^5r^7 + 364739R^4r^8 \\ &+ 785942R^3r^9 + 490704R^2r^{10} + 127441Rr^{11} + 12048r^{12}. \end{aligned}$$

Under the hypothesis, we may put $r = \frac{5}{11}R + q(q \geq 0)$ and substitute it into $Q_2$. Then all the terms are non-negative after expanding. So, we have inequality $Q_2 > 0$. And inequality $Q_1 \geq 0$ follows from (2.32) and Euler's inequality $R \geq 2r$.

Finally, combining the arguments of the above two cases, we conclude that inequality $Q_1 \geq 0$ holds for any acute $\triangle ABC$. Therefore, inequality (1.1) is proved. Also, it is to see that the equality in (1.1) holds if and only if $a = b = c$. This completes the proof of Theorem 1.1.

## 3. The second new proof of Theorem 1.1

Before giving the second new proof of inequality (1.1), we first give several lemmas.

**Lemma 3.1.** *Let $\triangle ABC$ be an acute triangle. Then we have*

$$\begin{aligned} P_0 \equiv\, & 768a^5 + (256b + 256c)a^4 + (1225b^2 - 1682bc + 1225c^2)a^3 \\ & + (b + c)(1085b^2 - 1914bc + 1085c^2)a^2 + (-1221b^4 + 2634b^2c^2 \\ & - 1221c^4)a - (b + c)(33b^2 + 8bc - 33c^2)(33b^2 - 8bc - 33c^2) > 0. \end{aligned} \tag{3.1}$$

**Proof.** Putting $(b+c-a)/2 = x, (c+a-b)/2 = y, (a+b-c)/2 = z$, then $a = y + z, b = z + x, c = x + y$. Substituting them into $P_0$, we easily get

$$P_0 = \frac{1}{8}F(x, y, z), \tag{3.2}$$

where

$$\begin{aligned} F(x, y, z) =\, & 16x^5 + 64(y + z)x^4 - (945y^2 - 2498yz + 945z^2)x^3 \\ & - 4(y + z)(505y^2 - 1258yz + 505z^2)x^2 - (964y^4 - 896y^3z \\ & - 3736y^2z^2 - 896yz^3 + 964z^4)x + 32(y + z)(y + 2z)^2(2y + z)^2. \end{aligned}$$

Since $\triangle ABC$ is an acute triangle, thus the positive real numbers $x, y, z$ satisfy the following three relations:

$$(z + x)^2 + (x + y)^2 - (y + z)^2 > 0, \tag{3.3}$$

$$(x + y)^2 + (y + z)^2 - (z + x)^2 > 0, \tag{3.4}$$

$$(y + z)^2 + (z + x)^2 - (x + y)^2 > 0. \tag{3.5}$$

It remains to prove the strict inequality $F(x, y, z) > 0$ under the conditions (3.3)-(3.5).

Note that $y$ and $z$ are symmetric with respect to $F(x, y, z)$. To prove $F(x, y, z) > 0$ we may assume that $y \le z$ and consider the following three cases to complete the proof of this inequality.

**Case 1.** The positive real numbers $x, y$ and $z$ satisfy $x \le y \le z$.

In this case, we may assume that $y = x+m, z = x+m+n (m \ge 0, n \ge 0)$. Substituting them into $F(x, y, z)$, we get

$$\begin{aligned} & F(x, x + m, x + m + n) \\ & = 11520x^5 + (47616m + 23808n)x^4 + (80000m^2 + 80000mn \\ & \quad + 9319n^2)x^3 + 4(2m + n)(8528m^2 + 8528mn + 267n^2)x^2 \\ & \quad + (29520m^4 + 59040m^3n + 38656m^2n^2 + 9136mn^3 + 444n^4)x \\ & \quad + 32(2m + n)(3m + 2n)^2(3m + n)^2. \end{aligned} \tag{3.6}$$

As $x > 0, m \ge 0$ and $n \ge 0$, it is obvious that $F(x, x + m, x + m + n) > 0$.

**Case 2.** The positive real numbers $x, y$ and $z$ satisfy $y \le z \le x$.

In this case we may assume that $z = y+m, x = y+m+n (m \ge 0, n \ge 0)$. Substituting them into (3.4), we easily get $4y^2 - 2m^2 - 2mn > 0$. If we let

$$y_0 = 2y^2 - m^2 - mn, \tag{3.7}$$

then $y_0 > 0$. It is easy to obtain

$$\begin{aligned}
&F(y+m+n,x,y+m)\\
&= 11520y^5 + (33792m + 9984n)y^4 + (29287m^2 + 24704mn\\
&\quad + 4736n^2)y^3 + (1280n^3 - 759m^3 + 5917m^2n + 9024mn^2)y^2\\
&\quad + (-11367m^4 + 1696mn^3 + 208n^4 - 15342m^3n - 2043m^2n^2)y\\
&\quad + (-61m^2 + 4mn + n^2)(61m^3 + 127m^2n + 80mn^2 + 16n^3). \qquad (3.8)
\end{aligned}$$

Also, it is easy to verify the following identity:

$$\begin{aligned}
&4F(y+m+n,x,y+m)\\
&= y_0(7850y + 7442m + 3843n)(2y^2 + 4ym + 2m^2 + mn) + 14680y^5\\
&\quad + (42600m + 24564n)y^4 + (41912m^2 + 68072mn + 18944n^2)y^3\\
&\quad + (13480m^3 + 47382m^2n + 36096mn^2 + 5120n^3)y^2\\
&\quad + 2(3661m^3 + 7525m^2n + 3392mn^2 + 416n^3)yn\\
&\quad + (1727m^3 + 1727m^2n + 576mn^2 + 64n^3)n^2. \qquad (3.9)
\end{aligned}$$

Thus, by $y_0 > 0, m \geq 0, n \geq 0$ and $y > 0$ we have $F(y+m+n, x, y+m) > 0$.

**Case 3.** The positive real numbers $x, y$ and $z$ satisfy $y \leq x \leq z$.

We may assume that $x = y + m, z = y + m + n (m \geq 0, n \geq 0)$. It is easy to get

$$\begin{aligned}
&F(y+m,y,y+m+n)\\
&= 11520y^5 + (33792m + 23808n)y^4 + (29287m^2 + 33870mn\\
&\quad + 9319n^2)y^3 + (-759m^3 - 8194m^2n - 5087mn^2 + 1068n^3)y^2\\
&\quad + (-11367m^4 - 30126m^3n - 24219m^2n^2 - 5224mn^3 + 444n^4)y\\
&\quad - (61m^2 + 126mn + 64n^2)(61m^3 + 56m^2n + 9mn^2 - 2n^3). \qquad (3.10)
\end{aligned}$$

On the other hand, substituting $x = y + m$ and $z = y + m + n$ into (3.4), inequality $4y^2 - 2m^2 - 2mn > 0$ follows immediately. So, inequality $y_0 > 0$ still hold under the third case. Again, it is easy to verify the following identity:

$$\begin{aligned}
&2F(y+m,y,y+m+n)\\
&= y_0\left[11520y^3 + (33792m + 23808n)y^2 + (35047m^2 + 39630mn\right.\\
&\quad \left. + 9319n^2)y + 16137m^3 + 20606m^2n + 6817mn^2 + 1068n^3\right]\\
&\quad + (511m^2 - 1129mn + 888n^2)yn^2 + (4405m^2 - 1307mn\\
&\quad + 420n^2)mn^2 + (12313m + 14425n)ym^3 + 8695m^5\\
&\quad + 14539m^4n + 256n^5. \qquad (3.11)
\end{aligned}$$

Note that $y_0 > 0, m \geq 0, n \geq 0, 511m^2 - 1129mn + 888n^2 > 0$ and $4405m^2 - 1307mn + 420n^2 > 0$. One sees that $F(y+m, y, y+m+n) > 0$ holds.

Finally, combining the arguments of the above three cases, we conclude that the strict inequality $F(x, y, z) > 0$ holds under the conditions (3.3)-(3.5). This completes the proof of Lemma 3.1.

**Lemma 3.2.** *Let $\triangle ABC$ be an acute triangle. Then we have*

$$(m_b+m_c)^2 \geq \frac{N_2}{M_2}, \tag{3.12}$$

*where*

$$\begin{aligned}M_2 =&64(2a^2+bc),\\ N_2 =&256a^4+(67b^2+122bc+67c^2)a^2-2(b+c)(b-c)^2a\\ &-(33b^2-82bc+33c^2)(b+c)^2.\end{aligned}$$

*Equality in* (3.12) *holds if and only if $b=c$.*

**Proof.** Inequality (3.12) is equivalent to

$$2m_bm_c \geq \frac{N_2-M_2(m_b^2+m_c^2)}{M_2}.$$

Using the median formula (2.12), we further know that the above inequality is equivalent to

$$2m_bm_c \geq \frac{G_0}{M_2}, \tag{3.13}$$

where

$$\begin{aligned}G_0 =&128a^4+(35b^2+58bc+35c^2)a^2-2(b+c)(b-c)^2a\\ &-33b^4+98b^2c^2-33c^4.\end{aligned}$$

To prove inequality (3.13) we first show that

$$G_0>0. \tag{3.14}$$

Putting $(b+c-a)/2=x, (c+a-b)/2=y, (a+b-c)/2=z$, then we have $a=y+z, b=z+x, c=x+y$. Substituting them into (3.14), one knows that it is equivalent to

$$\begin{aligned}&32x^4+64(y+z)x^3+(28y^2+648yz+28z^2)x^2\\ &-8(y+z)(y^2-74yz+z^2)x+32(y+2z)^2(2y+z)^2>0.\end{aligned} \tag{3.15}$$

Thus, we only need to prove

$$\begin{aligned}&(28y^2+648yz+28z^2)x^2-8(y+z)(y^2-74yz+z^2)x\\ &+32(y+2z)^2(2y+z)^2>0.\end{aligned} \tag{3.16}$$

The left is a quadratic function of $x$. A simple computation gives its discriminant $F_x$ as follows:

$$\begin{aligned}F_x =&-(14272y^6+412800zy^5+1459776z^2y^4+2198272y^3z^3\\ &+1459776z^4y^2+412800z^5y+14272z^6).\end{aligned}$$

Clearly, $F_x<0$ holds, so (3.16) holds true. Inequality (3.14) is proved.

Now, in view of inequality (3.14), to prove inequality (3.13) we need to prove

$$4m_b^2m_c^2M_2^2-G_0^2 \geq 0. \tag{3.17}$$

Using the median formula (2.12) for $m_b$ and $m_c$, we easily obtain the following identity:

$$4m_b^2m_c^2M_2^2-G_0^2=(b+c-a)(b-c)^2P_0, \tag{3.18}$$

where $P_0$ is the same as in Lemma 3.1. Hence, by Lemma 3.1 and (3.18) one sees that inequality (3.17) holds. Thus, inequality (3.13) is proved. Also, from (3.18) we can conclude that the equality in (3.12) holds if and only if $b = c$. This completes the proof of Lemma 3.2.

**Lemma 3.3.** *In $\triangle ABC$, we have*

$$\sum \frac{N_2}{M_2} = \frac{N_3}{16M_3}, \tag{3.19}$$

*where $M_2$ and $N_2$ are the same as in Lemma 3.2 and*

$$\begin{aligned} M_3 =& s^6 - (8R-3r)rs^4 + 3(2R-r)^2r^2s^2 + (4R+r)^3r^3, \\ N_3 =& 88s^8 - (1060R-459r)rs^6 + (3248R^2 - 3014Rr + 317r^2)r^2s^4 \\ & + (4R+r)(524R^2 + 1594Rr - 127r^2)r^3s^2 - 73(4R+r)^4r^4. \end{aligned}$$

**Proof.** Putting

$$y_1 = \prod(2a^2 + bc)$$

and then expanding gives

$$y_1 = 4\sum b^3c^3 + 9(abc)^2 + 2abc\sum a^3. \tag{3.20}$$

Further, using the previous identities (2.2), (2.8) and (2.15), we obtain

$$y_1 = 4M_3. \tag{3.21}$$

Now, putting

$$\begin{aligned} y_2 =& \sum(2b^2+ca)(2c^2+ab)\left[256a^4 + (67b^2+122bc+67c^2)a^2\right. \\ & \left.-2(b+c)(b-c)^2a - (33b^2 - 82bc + 33c^2)(b+c)^2\right], \end{aligned}$$

we easily obtain the following identity:

$$\begin{aligned} y_2 =& 202\sum a^8 - 66\sum a\sum a^7 - 136\sum a^2\sum a^6 + 161abc\sum a^5 \\ & + 159abc\sum a\sum a^4 + 392\sum b^4c^4 + 710\sum a^2\sum b^3c^3 \\ & + 1722(abc)^2\sum a^2 + 283abc\sum a\sum b^2c^2 - 143(abc)^2\sum bc. \end{aligned} \tag{3.22}$$

Using the previous identities, we get

$$y_2 = 16N_3. \tag{3.23}$$

Finally, note that $\sum \frac{N_2}{M_2} = \frac{y_2}{64y_1}$, we immediately obtain (3.19) from (3.21) and (3.23). This completes the proof of Lemma 3.3.

**Lemma 3.4.** *Let $\triangle ABC$ be an acute (non-obtuse) triangle, then*

$$s \geq 2R + r, \tag{3.24}$$

*with equality if and only if $\triangle ABC$ is a right triangle.*

Inequality (3.24) is a fundamental inequality for acute triangles, which was first appeared in Ciamberlini's article [1].

Next, we give the second new proof of inequality (1.1).

**Proof.** According to Lemma 3.2 and Lemma 3.3, we have

$$\sum(m_b + m_c)^2 \geq \frac{N_3}{16M_3}. \tag{3.25}$$

Thus, to prove inequality (1.1) we only need to prove

$$\frac{N_3}{16M_3} - 4s^2 \geq 0,$$

i.e.,

$$W_0 \equiv N_3 - 64M_3s^2 \geq 0.$$

Using the expressions of $M_3$ and $N_3$ in Lemma 3.3, we obtain

$$\begin{aligned} W_0 =& 24s^8 - (548R - 267r)rs^6 + (2480R^2 - 2246Rr + 125r^2)r^2s^4 \\ & - (4R + r)(500R^2 - 1082Rr + 191r^2)r^3s^2 - 73(4R + r)^4r^4, \end{aligned} \tag{3.26}$$

which can be rewritten as

$$W_0 = w_0^2W_1 + W_2, \tag{3.27}$$

where

$$\begin{aligned} w_0 =& s^2 - 2R^2 - 8Rr - 3r^2, \\ W_1 =& 24s^4 + (96R^2 - 164Rr + 411r^2)s^2 + 288R^4 + 112R^3r + 252R^2r^2 \\ & + 2194Rr^3 + 2375r^4, \\ W_2 =& (768R^6 + 2640rR^5 + 836r^2R^4 + 6184R^3r^3 + 25716R^2r^4 \\ & + 33230Rr^5 + 10360r^6)s^2 - (1152R^8 + 9664R^7r + 26480R^6r^2 \\ & + 39176R^5r^3 + 125516R^4r^4 + 274536R^3r^5 + 295088R^2r^6 \\ & + 134914Rr^7 + 21448r^8). \end{aligned}$$

By Euler's inequality $R \geq 2r$ we have $W_1 > 0$. So, it remains to show that

$$W_2 \geq 0 \tag{3.28}$$

holds for the acute triangle $ABC$. We shall consider two cases to complete the proof of this inequality.

**Case 1.** $R$ and $r$ satisfy $R^2 - 2Rr - r^2 \geq 0$.

In this case, by Ciamberlini's inequality (3.24), we only need to show

$$\begin{aligned} & (768R^6 + 2640rR^5 + 836r^2R^4 + 6184R^3r^3 + 25716R^2r^4 \\ & + 33230Rr^5 + 10360r^6)(2R + r)^2 - (1152R^8 + 9664R^7r \\ & + 26480R^6r^2 + 39176R^5r^3 + 125516R^4r^4 + 274536R^3r^5 \\ & + 295088R^2r^6 + 134914Rr^7 + 21448r^8) > 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & (R^2 - 2Rr - r^2)(1920R^6 + 7808rR^5 + 5728r^2R^4 + 10808R^3r^3 \\ & + 30264R^2r^4 + 38768Rr^5 + 12788r^6) + 100(41R + 17r)r^7 > 0. \end{aligned}$$

This inequality holds true under the hypothesis. Hence, inequality (3.28) is proved under Case 1.

**Case 2.** $R$ and $r$ satisfy $R^2 - 2Rr - r^2 < 0$.

In this case, by Lemma 2.5, to prove $W_2 \geq 0$ we need to prove

$$(768R^6 + 2640rR^5 + 836r^2R^4 + 6184R^3r^3 + 25716R^2r^4 + 33230Rr^5 + 10360r^6)\left(16Rr - 3r^2 - \frac{4r^3}{R}\right) - (1152R^8 + 9664R^7r + 26480R^6r^2 + 39176R^5r^3 + 125516R^4r^4 + 274536R^3r^5 + 295088R^2r^6 + 134914Rr^7 + 21448r^8) \geq 0,$$

which is equivalent to

$$\frac{R-2r}{R}H_0 \geq 0, \tag{3.29}$$

where

$$H_0 = -288R^8 + 80R^7r + 3524R^6r^2 - 2150R^5r^3 - 14210R^4r^4 + 336R^3r^5 + 34349R^2r^6 + 25771Rr^7 + 5180r^8.$$

By Euler's inequality $R \geq 2r$, we only need to show $H_0 > 0$. But, it is easy to obtain

$$H_0 = -(R^2 - 2Rr - r^2)(288e^6 + 3952e^5r + 19996e^4r^2 + 46126e^3r^3 + 52174e^2r^4 + 48578er^5 + 48269r^6) + 25(239R + 99r)r^7, \tag{3.30}$$

where $e = R - 2r \geq 0$. So, by the hypothesis, one sees that $H_0 > 0$ holds strictly. Thus, inequality (3.28) is proved under Case 2.

Combining the arguments of the above two cases, we conclude that inequality (3.28) holds for any acute $\triangle ABC$. And, inequality (1.1) is proved. Also, it is easily seen that the equality in (1.1) holds if and only if $\triangle ABC$ is equilateral. This completes the proof of Theorem 1.1.

**Remark 3.1.** *In the acute $\triangle ABC$, we have the following inequality similar to* (3.12)*:*

$$(m_b + m_c)^2 \geq \frac{N_0}{M_0}, \tag{3.31}$$

*where*

$$M_0 = 64(2a^2 + bc),$$
$$N_0 = 256a^4 + (65b^2 + 126bc + 65c^2)a^2 - 33b^4 + 14b^3c + 102b^2c^2 + 14bc^3 - 33c^4.$$

*Inequality* (3.31) *could be proved by using the method to prove the inequality of Lemma 3.2. Furthermore, inequality* (1.1) *could be proved by applying inequality* (3.31)*. Therefore, we can obtain actually the third new proof of inequality* (1.1)*.*

## 4. Applications of Theorem 1.1

Both inequalities (1.4) and (1.5) are equivalent to (1.1) given in Theorem 1.1. In this section, we shall discuss applications of these two inequalities.

Firstly, we use inequality (1.4) to establish the following linear inequality:

**Corollary 4.1.** *In the acute* $\triangle ABC$*, we have*

$$m_a + m_b + m_c \ge \frac{5}{2}R + 4r. \tag{4.1}$$

**Proof.** According to inequality (1.4), we only need to prove

$$\frac{1}{4}\left(\sum a^2 + 8\sum bc\right) - \left(\frac{5}{2}R + 4r\right)^2 \ge 0.$$

By using the previous identities (2.1) and (2.16), it is easily known that the above inequality is equivalent to

$$\frac{1}{4}\left(10s^2 - 25R^2 - 56Rr - 58r^2\right) \ge 0,$$

i.e.,

$$10s^2 \ge 25R^2 + 56Rr + 58r^2. \tag{4.2}$$

If $5R > 12r$, by Ciamberlini's inequality (3.24) we have

$$\begin{aligned}
&10s^2 - (25R^2 + 56Rr + 58r^2)\\
&> 10(2R + r)^2 - (25R^2 + 56Rr + 58r^2)\\
&= (3R + 4r)(5R - 12r) > 0.
\end{aligned}$$

If $5R \le 12r$, by inequality (2.21) we have

$$\begin{aligned}
&10s^2 - (25R^2 + 56Rr + 58r^2)\\
&\ge 10\left(16Rr - 3r^2 - \frac{4r^3}{R}\right) - (25R^2 + 56Rr + 58r^2)\\
&= \frac{R - 2r}{R}\left(20r^2 + 54Rr - 25R^2\right)\\
&\ge \frac{R - 2r}{R}\left(20 \cdot \frac{25R^2}{144} + 54R \cdot \frac{5R}{12} - 25R^2\right)\\
&= \frac{35R(R - 2r)}{36} \ge 0.
\end{aligned}$$

Therefore, inequality (4.2) is valid for all acute triangles. Hence, inequality (4.1) is proved.

**Remark 4.1.** *In fact, the acute triangle inequality* (4.2) *similar to Walker's inequality* (2.20) *can be improved to*

$$10s^2 \ge 25R^2 + 57Rr + 56r^2. \tag{4.3}$$

**Corollary 4.2.** *In the acute* $\triangle ABC$*, we have*

$$m_b m_c + m_c m_a + m_a m_b \ge \frac{27}{2}Rr. \tag{4.4}$$

**Proof.** By the following known identity

$$\sum \frac{1}{bc} = \frac{1}{2Rr}, \tag{4.5}$$

one sees that (4.4) is equivalent to

$$\sum m_b m_c \sum \frac{1}{bc} \ge \frac{27}{4}. \tag{4.6}$$

According to inequality (1.5) we only to prove

$$\left(\sum bc - \frac{1}{4}\sum a^2\right)\sum \frac{1}{bc} \geq \frac{27}{4},$$

which is equivalent to

$$\frac{D_0}{4abc} \geq 0,$$

where

$$D_0 = \sum a(b^2+c^2) - \sum a^3 - 15abc.$$

Note that

$$D_0 = \sum a(b-c)^2 + 2\sum a(b^2+c^2) - \sum a^3 - 9abc,$$

to prove $D_0 \geq 0$ we only need to show that

$$2\sum a(b^2+c^2) - \sum a^3 - 9abc \geq 0. \tag{4.7}$$

If we take $x = b+c-a, y = c+a-b, z = a+b-c$ in the the following special case of Schur's inequality (cf.[3]):

$$\sum x(x-y)(x-z) \geq 0, \tag{4.8}$$

where $x, y, z$ are arbitrary positive real numbers, then inequality (4.7) is obtained immediately. Hence, inequality (4.4) is proved.

From (4.6), using the algebraic inequality $(\sum x)^2 \geq 3\sum yz$, we get

**Corollary 4.3.** *In the acute $\triangle ABC$, we have*

$$(m_a+m_b+m_c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{9\sqrt{3}}{2}. \tag{4.9}$$

In fact, we have the following conclusion which is stronger than Corollary 4.2:

**Corollary 4.4.** *In the acute $\triangle ABC$, we have*

$$m_bm_c + m_cm_a + m_am_b \geq \frac{9\sqrt{3}abc}{4\sqrt{bc+ca+ab}}. \tag{4.10}$$

**Proof.** According to inequality (1.5), we only to prove that

$$16\sum bc\left(\sum bc - \frac{1}{4}\sum a^2\right) - (9\sqrt{3}abc)^2 \geq 0.$$

Using the previous identities (2.1), (2.15) and (2.16), we easily know that the above inequality is equivalent to

$$4s^6+28(4R+r)rs^4-12(244R^2-40Rr-5r^2)r^2s^2+36(4R+r)^3r^3 \geq 0. \tag{4.11}$$

One can rewrite it as follows:

$$\begin{aligned} &4g_1^3 + 16r(19R-2r)g_1^2 + 16r^2(233R^2-104Rr+5r^2)g_1 \\ &+ 16r^3(32R-r)(R-2r)^2 \geq 0, \end{aligned} \tag{4.12}$$

where $g_1 = s^2 - 16Rr + 5r^2$. By Euler's inequality $R \geq 2r$ and Gerretsen's inequality (see [4, p.45]):

$$s^2 \geq 16Rr - 5r^2, \tag{4.13}$$

which is valid for any triangle $ABC$, one sees that (4.12) holds. Hence inequality (4.10) is proved.

Using $(a+b+c)^2 \geq 3(bc+ca+ab)$ and $abc = 4Rrs$, we obtain (4.4) from (4.10). Thus, inequality (4.10) is stronger than (4.4).

It follows from (1.8) and (4.10) that

$$(m_a + m_b + m_c)(m_b m_c + m_c m_a + m_a m_b) \geq \frac{27\sqrt{3}}{8} abc. \tag{4.14}$$

Again, note the following known algebraic inequality

$$\prod (y+z) \geq \frac{8}{9} \sum x \sum yz \tag{4.15}$$

where $x, y, z > 0$, we obtain

**Corollary 4.5.** *In the acute $\triangle ABC$, we have*

$$(m_b + m_c)(m_c + m_a)(m_a + m_b) \geq 3\sqrt{3} abc. \tag{4.16}$$

Using the arithmetic-geometric mean inequality, one can obtain from the above inequality that

$$\frac{m_b + m_c}{a} + \frac{m_c + m_a}{b} + \frac{m_a + m_b}{c} \geq 3\sqrt{3}. \tag{4.17}$$

Incidentally, the author has proved the following better inequality in the recent paper [11]:

$$\frac{a}{m_b + m_c} + \frac{b}{m_c + m_a} + \frac{c}{m_a + m_b} \leq \sqrt{3}. \tag{4.18}$$

Next, we prove an inequality which is stronger than inequality (1.8).

**Corollary 4.6.** *In the acute $\triangle ABC$, we have*

$$(m_a + m_b + m_c)^2 \geq \frac{\sqrt{3}(a+b+c)^3}{4\sqrt{a^2+b^2+c^2}}. \tag{4.19}$$

**Proof.** To prove (4.19), we need to show by (1.4) that

$$\left(\sum a^2 + 8\sum bc\right)^2 \sum a^2 - 3\left(\sum a\right)^6 \geq 0.$$

By expanding, it is easy to check the following identity:

$$\begin{aligned}&\left(\sum a^2 + 8\sum bc\right)\sum a^2 - 3\left(\sum a\right)^6 \\ &= \sum (b-c)^2 \left[10\sum b^2c^2 - \sum a^4 + 22abc\sum a - 2\sum a(b^3+c^3)\right].\end{aligned}$$

Thus, we have to show that

$$10\sum b^2c^2 - \sum a^4 + 22abc\sum a - 2\sum a(b^3+c^3) > 0.$$

Since

$$2\sum b^2c^2 - \sum a^4 = \sum a \prod (b+c-a) > 0,$$

we only need to prove

$$8\sum b^2c^2 - 2\sum a(b^3+c^3) > 0.$$

We consider to prove the following stronger inequality:

$$D_1 \equiv 3\sum b^2c^2 - \sum a(b^3+c^3) > 0. \tag{4.20}$$

Putting $b+c-a=2x, c+a-b=2, a+b-c=2c$, then we have $a=y+z, b=z+x, c=x+y(x,y,z>0)$. Substituting them into $D_1$ gives

$$D_1=\sum x^4+3\sum y^2z^2+12xyz\sum x. \tag{4.21}$$

So $D_1>0$. This completes the proof of inequality (4.19).

**Remark 4.2.** *We can prove that inequality* (4.19) *is stronger than* (1.8) *as follows: We need to prove*

$$\frac{\sqrt{3}\left(\sum a\right)^3}{4\sqrt{\sum a^2}}\geq\frac{9}{4}\sum bc,$$

*which is equivalent to*

$$\left(\sum a\right)^6\geq 27\sum a^2\left(\sum bc\right)^2. \tag{4.22}$$

*Further, we can write the above inequality as follows:*

$$1-\frac{9\left(\sum bc\right)^2}{\left(\sum a\right)^4}\geq 1-\frac{\left(\sum a\right)^2}{3\sum a^2},$$

*that is*

$$\frac{\left[\left(\sum a\right)^2-3\sum bc\right]\left[\left(\sum a\right)^2+3\sum bc\right]}{\left(\sum a\right)^4}\geq\frac{3\sum a^2-\left(\sum a\right)^2}{3\sum a^2}.$$

*Since*

$$\left(\sum a\right)^2-3\sum bc=\frac{3}{2}\sum a^2-\frac{1}{2}\left(\sum a\right)^2,$$

*we need to show that*

$$\frac{\left(\sum a\right)^2+3\sum bc}{2\left(\sum a\right)^4}\geq\frac{1}{3\sum a^2}. \tag{4.23}$$

*Again, it is easy to check the following identity:*

$$3\sum a^2\left[\left(\sum a\right)^2+3\sum bc\right]-2\left(\sum a\right)^4=\frac{1}{2}\sum(b-c)^2\left(\sum a^2+8\sum bc\right).$$

*So, inequality* (4.23) *holds and* (4.22) *is proved.*

Next, we use (1.5) to prove an inequality involving the medians $m_a, m_b, m_c$ and the bisectors $w_a, w_b, w_c$ of the acute triangle $ABC$.

**Corollary 4.7.** *In the acute* $\triangle ABC$*, we have*

$$3(m_bm_c+m_cm_a+m_am_b)\geq(w_a+w_b+w_c)^2. \tag{4.24}$$

**Proof.** According to the following known formula:

$$w_a=\frac{2}{b+c}\sqrt{s(s-a)bc} \tag{4.25}$$

and the simplest arithmetic-geometric mean inequality, we get

$$2w_a \le \frac{\sqrt{3}\,bc}{b+c} + \frac{4\sqrt{3}\,s(s-a)}{3(b+c)}. \tag{4.26}$$

Using $s = (a+b+c)/2$, we further obtain

$$w_a \le \frac{\sqrt{3}(b^2+5bc+c^2-a^2)}{6(b+c)}. \tag{4.27}$$

Now, by the acute triangle inequality (1.5), to prove (4.24) we only need to show that

$$I_0 \equiv \sum bc - \frac{1}{4}\sum a^2 - \frac{1}{36}\left(\sum \frac{b^2+5bc+c^2-a^2}{b+c}\right)^2 \ge 0$$

holds for any triangle $ABC$. It is not difficult to check the following identity:

$$I_0 = \frac{I_1 \sum (b-c)^2}{72\prod (b+c)^2}, \tag{4.28}$$

where

$$I_1 = \sum a^4(a-b)(a-c) + 2\sum a^5(b+c-a) + 11\sum b^3c^3 + 59abc\sum a(b^2+c^2) + 6\sum a^2(b^4+c^4) + 12abc\sum a^3 + 111(abc)^2.$$

By the special case of Schur's inequality:

$$\sum x^5(x-y)(x-z) \ge 0 \tag{4.29}$$

(where $x, y, z > 0$), we see that $I_1 > 0$. Hence $I_0 \ge 0$ and inequality (4.24) are proved.

Finally, we use (1.8) to prove a conjecture proposed by the author in [11], namely, the following inequality:

**Corollary 4.8.** *In the acute $\triangle ABC$, we have*

$$m_a + m_b + m_c \ge \frac{3\sqrt{3}}{4}a + \frac{3}{2}w_a. \tag{4.30}$$

**Proof.** To prove (4.30) we need to show

$$(m_a+m_b+m_c)^2 \ge \frac{27}{16}a^2 + \frac{9}{4}w_a^2 + \frac{9\sqrt{3}}{4}aw_a.$$

By formula (4.25), inequalities (1.8) and (4.27) , we only need to prove that

$$\frac{9}{4}(bc+ca+ab) \ge \frac{27}{16}a^2 + \frac{9(b+c-a)(a+b+c)}{4(b+c)^2} + \frac{9a(b^2+5bc+c^2-a^2)}{8(b+c)}.$$

Further, we easily know that the above inequality is equivalent to

$$\frac{9a\left[2(b+c)a^2 - (3b^2+2bc+3c^2)a + 2(b+c)(b^2-bc+c^2)\right]}{16(b+c)^2} \ge 0.$$

Thus, we have to show

$$2(b+c)a^2 - (3b^2+2bc+3c^2)a + 2(b+c)(b^2-bc+c^2) \ge 0. \tag{4.31}$$

The left hand side is a quadratic function in $a$. A simple computation gives its discriminant $F_a$ as follows:

$$F_a = -(7b^2 + 18bc + 7c^2)(b - c)^2 \le 0.$$

Hence inequality (5.3) holds and inequality (4.30) is proved.

## 5. A conjecture

In this section, we shall introduce a conjecture as an open problem.

The author finds that the following linear inequality:

$$m_a + m_b + m_c > 4R \tag{5.1}$$

holds for the acute triangle $ABC$ and it can be easily proved by using the following known inequality:

$$\frac{m_a}{h_a} \ge \frac{b^2 + c^2}{2bc}, \tag{5.2}$$

which is valid for any triangle $ABC$ (see [4, p.223]). The author conjectures that the following stronger inequality holds:

**Conjecture 5.1.** *In the acute triangle $ABC$, we have*

$$m_a + m_b + m_c \ge 4R + \frac{32r^6}{R^5}. \tag{5.3}$$

## References

[1] Ciamberlini, C., *Sulla condizione necessaria e sufficiente affinche un triangolo sia acutangolo o ottusangolo*. Boll.Unione Mat.Ital., **5(1943)**, 37–41.

[2] Chu X.G., *Several inequalities related to the medians of an acute triangle*, Research in Inequalities, Tibet People's Press, Lhasa, 2000. (in Chinese)

[3] Hardy, G.H., Littlewood, J.E. and Polya, G., *Inequalities*, London, Cambridge University Press, 1952.

[4] Mitrinović, D.S., Pečarić, J.E. and Volenec, V., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1989.

[5] Liu. J., *One hundred unsolved triangle inequalities*, Geometric Inequalities in China. Jiangsu Educational Publishing House, Nanjing, 1996. (in Chinese)

[6] Liu, J., *An inequality involving geometric elements R, r and s in non-obtuse triangles*, Teaching Mon., **7(2010)**, 51–53. (in Chinese)

[7] Liu, J., *A refinement of an equivalent form of a Gerretsen inequality*, J.Geom., **106(3)(2015)**, 605–615.

[8] Liu, J., *Two new Weighted Erdös-Mordell inequalities*, Discrete.Comput.Geom., **59(3)(2018)**, 707–724.

[9] Liu, J., *Further generalization of Walker's inequality in acute triangles and its applications*, Aims Mathematics., **5(6)(2020)**, 6657–6672.

[10] Liu, J., *Two new refinements of a linear geometric inequality*, Int.J.Geom., **9(2)(2020)**, 81–92.

[11] Liu, J., *An inequality involving medians and sides of an acute triangle*, Int.J.Open Problems Compt.Math., **14(3)(2021)**, 49–63.

[12] Walker, A.W., *Problem E2388*, Am. Math. Mon., **79(1972)**, 1135.

EAST CHINA JIAOTONG UNIVERSITY
JIANGXI PROVINCE NANCHANG CITY, 330013 CHINA
*E-mail address*: China99jian@163.com