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ORTHOLOGIC PEDAL TRIANGLES AND THEIR RELATIONS WITH THE TRIANGLES OF RESIDUAL CENTROIDS

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Abstract. The pedal triangle of an arbitrary point is orthologic to the reference triangle, so it is also orthologic to the medial triangle of the reference triangle. But the median triangle is the pedal triangle of the circumcenter, so the pedal triangle of the circumcenter is orthologic to the pedal triangle of any other point in the plane of the reference triangle. In this paper we investigate the necessary and sufficient condition for the orthology of the pedal triangles of two different points. Moreover we prove that the triangle formed by the centroids of the residual triangles of a pedal triangle is orthologic both to the reference triangle and to the pedal triangle. This property gives a nontrivial example of an infinite set of pairwise orthologic triangles.

1. Introduction

Two triangles ABC and $A'B'C'$ are said to be *orthologic* if perpendiculars from A, B, C to $B'C'$, $C'A'$, $A'B'$ are concurrent (see [5], pp 55). The point of concurrency is known as the *orthologic center* of ΔABC with respect to $\Delta A'B'C'$ (Figure 1).

The problem of orthologic triangles was posed by Jacob Steiner in 1827 (see [13], problem 54) and he gave the following result in [12]:

Theorem 1.1. *If A', B', C' are arbitrary points on the sides BC, CA and AB of the triangle ABC, the perpendiculars from A', B' and C' to BC, CA and AB respectively are concurrent if and only if*

$$A'B^2 + B'C^2 + C'A^2 = AB'^2 + BC'^2 + CA'^2.$$

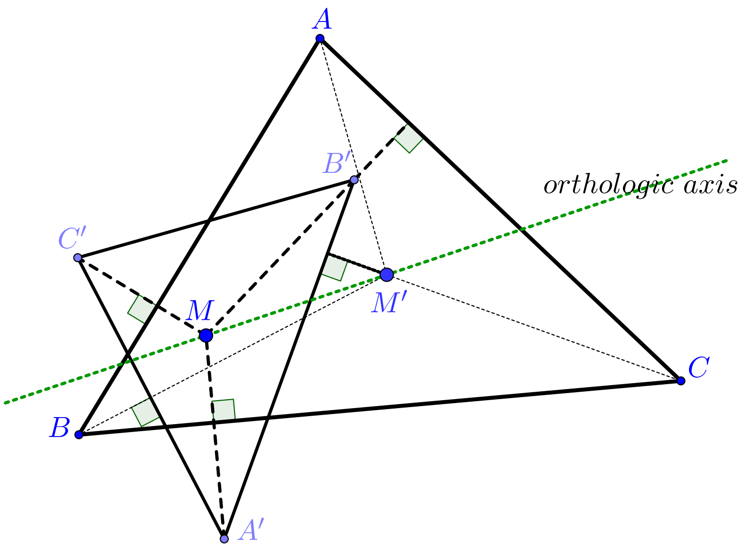

Figure 1. Orthology centers and the orthologic axis

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Some authors refer to this theorem as Carnot's theorem (see [10] page 26. or [1]). In [5] (page 55, problem 82) the following general case was included

Theorem 1.2. *If A', B', C' are arbitrary points in the plane of the triangle ABC, the perpendiculars from A', B' and C' to BC, CA and AB respectively are concurrent if and only if*

$$A'B^2 + B'C^2 + C'A^2 = AB'^2 + BC'^2 + CA'^2. \tag{1}$$

Jacob Steiner proved that if the triangles ABC and $A'B'C'$ are orthologic, then the perpendiculars from A', B', C' to BC, CA, AB are concurrent, too ([12]). Consequently two orthologic triangles in general have two different orthologic centers which determine a line named the *orthologic axis* of these triangles (Figure 1). In what follows if T_i and T_j are two triangles, we will denote by C_{ij} the common point of the perpendiculars from the vertices of T_i to the sides of T_j.

In the last two centuries the pairs of orthologic triangles were studied by several mathematicians (see [14], [4], [10]). If we denote by D, E and F the perpendicular projections of an arbitrary point M to the sides of the reference triangle ABC ($D \in BC, E \in CA,\ \ F \in AB$), the triangle DEF is called the *pedal triangle* (see [5], Chapter V.) of the point M with respect to the triangle ABC (Figure 2).

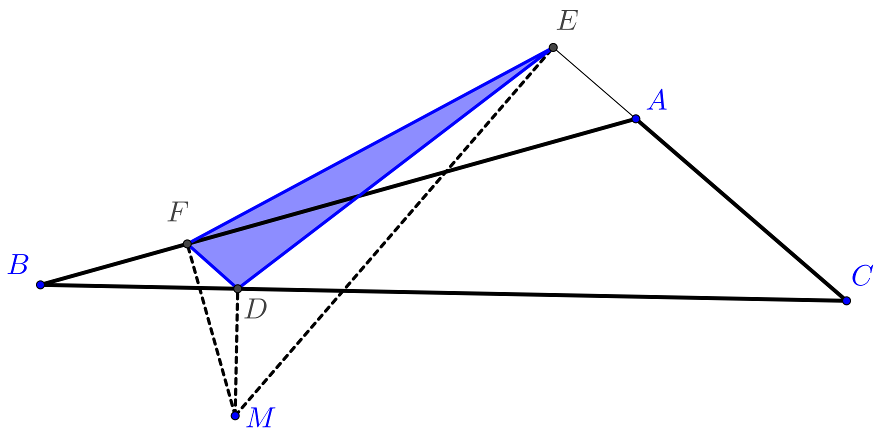

FIGURE 2. The pedal triangle DEF of the point M with respect to the triangle ABC

In this paper we investigate the orthology of two different pedal triangles with respect to a given triangle and the orthology of the triangle formed by the centroids of residual triangles and the reference triangle. The first question is motivated by the existence of several special cases of such orthologic triangle pairs. The most known are the medial triangle with the orthic triangle or the Gergonne triangle with the medial triangle (see [8]). The later appeared also in the Crux Mathematicorum ([7]) and was proposed to the International Mathematical Olympiad by Bulgaria in 1984 (see [3], page 170). The second question is motivated by some recent papers dealing with the properties of residual triangles and the intrinsec connection to the results concerning the first question.

Throughout the paper we use the barycentric coordinates (see [15], [9]) with respect to the triangle ABC: $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. As usually for the normalized barycentric coordinates we use the notation (x, y, z), while for general barycentric coordinates we use the notation $(u : v : w)$. In the previous case the sum of coordinates is 1, while in the later case we do not have such a relation. Let a, b, c be the length of sides BC, CA, AB and O the circumcenter of the triangle ABC. In the plan of the triangle ABC we consider an arbitrary point M, with barycentric coordinates $M = (u : v : w)$ so that $uvw \neq 0$. Denote with μ the sum of coordinates of M ($\mu = u + v + w \neq 0$) Let s be the semiperimeter ($2s = a + b + c$), σ the area, $S = 2\sigma = \frac{abc}{2R}$ the twice of area of ABC. In general for any angle φ the Conway triangle notation S_φ refers to $S_\varphi = S\cot\varphi$. In particular

$$S_A = bc\cos A = \frac{1}{2}\left(-a^2+b^2+c^2\right),$$

$$S_B = ca\cos B = \frac{1}{2}\left(a^2-b^2+c^2\right),$$

$$S_C = ab\cos C = \frac{1}{2}\left(a^2+b^2-c^2\right).$$

In the same way $S_{\varphi\theta} = S_\varphi \cdot S_\theta$ for any angles φ, θ. The Conway notations are used to simplify the barycentric coordinates of some triangle centers. For example the barycentric coordinates of the circumcenter are $O = \left(a^2S_A : b^2S_B : c^2S_C\right)$ (see [9]).

In order to calculate the barycentric coordinates of the vertices of the pedal triangle (the points D, E and F) with respect to a given triangle ABC we need the following property:

Lemma 1.1. *The equation of the line, which passes through the point $(x' : y' : z')$ and is perpendicular to the line $lx+my+nz=0$ is*

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ \mu_a & \mu_b & \mu_c \end{vmatrix} = 0,$$

where $\mu_a = la^2 - mS_C - nS_B$, $\mu_b = mb^2 - nS_A - lS_C$ and $\mu_c = nc^2 - lS_B - mS_A$.

Due to this lemma the equations of the line MD, ME and MF are

$$\begin{vmatrix} x & y & z \\ u & v & w \\ -a^2 & S_C & S_B \end{vmatrix} = 0, \quad \begin{vmatrix} x & y & z \\ u & v & w \\ S_C & -b^2 & S_A \end{vmatrix} = 0, \quad \begin{vmatrix} x & y & z \\ u & v & w \\ S_B & S_A & -c^2 \end{vmatrix} = 0.$$

These can be written in the following equivalent forms:

$$\left(S_Bv - S_Cw\right)x - \left(wa^2 + S_Bu\right)y + \left(va^2 + S_Cu\right)z = 0,$$

$$\left(wb^2 + S_Av\right)x + \left(S_Cw - S_Au\right)y - \left(ub^2 + S_Cv\right)z = 0,$$

$$-\left(vc^2 + S_Aw\right)x + \left(uc^2 + S_Bw\right)y + \left(S_Au - S_Bv\right)z = 0.$$

Consequently the barycentric, respectively the absolute barycentric coordinates of the points D, E, F can be expressed as (see [15]):

$$D = \left(0 : a^2v + S_Cu : a^2w + S_Bu\right) = \left(0, \frac{a^2v + S_Cu}{a^2\mu}, \frac{a^2w + S_Bu}{a^2\mu}\right),$$

$$E = \left(b^2u + S_Cv : 0 : b^2w + S_Av\right) = \left(\frac{b^2u + S_Cv}{b^2\mu}, 0, \frac{b^2w + S_Av}{b^2\mu}\right),$$

$$F = \left(c^2u + S_Bw : c^2v + S_Aw : 0\right) = \left(\frac{c^2u + S_Bw}{c^2\mu}, \frac{c^2v + S_Aw}{c^2\mu}, 0\right).$$

It is clear that the triangles ABC and DEF are orthologic because $DM \perp BC$, $EM \perp CA$, $FM \perp AB$ and one of the orthology centers is the point M. The following result shows the connection between the two orthology centers.

Theorem 1.3. *The orthology centers of the pedal triangles DEF and ABC are isogonally conjugated with respect to the reference triangle ABC (see* [11]*).*

Proof. From the construction one of the orthology centers of DEF and ABC is the point M. If we denote by M^* the second orthology center, the quadrilateral $AEMF$ is cyclic, so

$$m(\widehat{BAM^*}) = 90^\circ - m(\widehat{AFE}) = 90^\circ - m(\widehat{AME}) = m(\widehat{EAM}) = m(\widehat{CAM}).$$

This equality implies that AM and AM^* are symmetric with respect to the angle bisector of the angle A. Repeating this argument for the vertices B and C we obtain that M and M^* are isogonally conjugated (Figure 3).

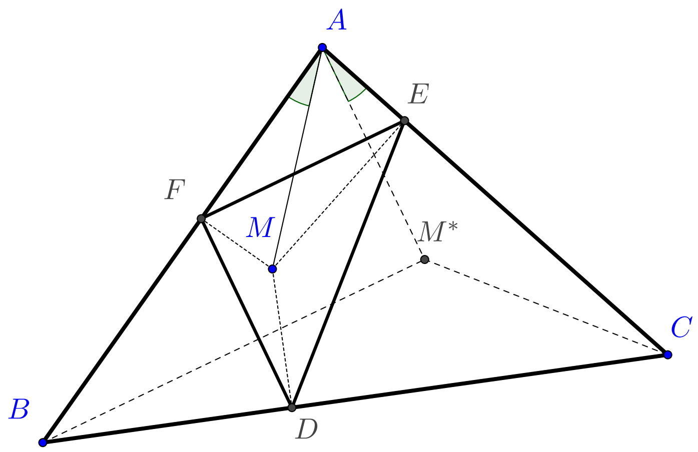

FIGURE 3. The two orthology centers are isogonally conjugated

Remark 1.1. *The barycentric coordinates of M^* can be expressed as (see [11])*

$$M^* = \left(a^2vw : b^2wu : c^2uv\right).$$

2. Condition of orthology of two pedal triangles

The main result of this section is a necessary and sufficient condition for the orthology of two pedal triangles. This theorem generalizes all the previously known results of this type.

Theorem 2.1. *If we denote by DEF and $D'E'F'$ the pedal triangles of the points M and M' respectively then the triangles DEF and $D'E'F'$ are orthologic if and only if M, M' and the circumcenter O of the triangle ABC are collinear (Figure 4).*

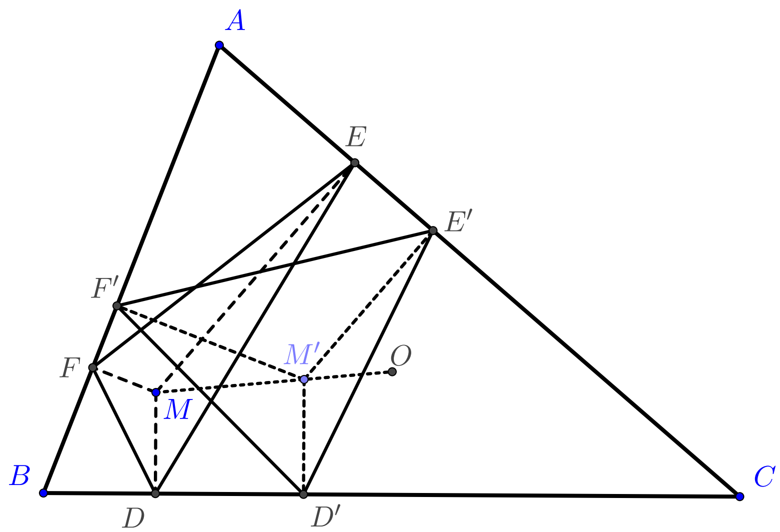

FIGURE 4. Orthology of two pedal triangles

Proof. If $M' = (u' : v' : w')$, $\mu' = u' + v' + w' \neq 0$, then the normalized barycentric coordinates of its projections are

$$D' = \left(0, \frac{a^2v' + S_Cu'}{a^2\mu'}, \frac{a^2w' + S_Bu'}{a^2\mu'}\right),$$

$$E' = \left(\frac{b^2u' + S_Cv'}{b^2\mu'}, 0, \frac{b^2w' + S_Av'}{b^2\mu'} \right),$$

$$F' = \left(\frac{c^2u' + S_Bw'}{c^2\mu'}, \frac{c^2v' + S_Aw'}{c^2\mu'}, 0 \right).$$

Due to Theorem 1. the triangles DEF and $D'E'F'$ are orthologic if and only if

$$DE'^2 - DF'^2 + EF'^2 - ED'^2 + FD'^2 - FE'^2 = 0.$$

This is equivalent to the following

$$\begin{aligned} &(x_D - x_{E'})^2 S_A + (y_D - y_{E'})^2 S_B + (z_D - z_{E'})^2 S_C \\ &-(x_D - x_{F'})^2 S_A - (y_D - y_{F'})^2 S_B - (z_D - z_{F'})^2 S_C \\ &+(x_E - x_{F'})^2 S_A + (y_E - y_{F'})^2 S_B + (z_E - z_{F'})^2 S_C \\ &-(x_E - x_{D'})^2 S_A - (y_E - y_{D'})^2 S_B - (z_E - z_{D'})^2 S_C \\ &+(x_F - x_{D'})^2 S_A + (y_F - y_{D'})^2 S_B + (z_F - z_{D'})^2 S_C \\ &-(x_F - x_{E'})^2 S_A - (y_F - y_{E'})^2 S_B - (z_F - z_{E'})^2 S_C = 0 \end{aligned}$$

Regrouping the terms and using equivalent transformations we obtain

$$(-x_E x_{F'} + x_{E'} x_F) S_A + (-y_F y_{D'} + y_{F'} y_D) S_B + (-z_D z_{E'} + z_{D'} z_E) S_C = 0$$

$$\begin{aligned} &\left(-\frac{b^2u + S_Cv}{b^2\mu} \cdot \frac{c^2u' + S_Bw'}{c^2\mu'} + \frac{b^2u' + S_Cv'}{b^2\mu'} \cdot \frac{c^2u + S_Bw}{c^2\mu} \right) S_A \\ &+ \left(-\frac{c^2v + S_Aw}{c^2\mu} \cdot \frac{a^2v' + S_Cu'}{a^2\mu'} + \frac{c^2v' + S_Aw'}{c^2\mu'} \cdot \frac{a^2v + S_Cu}{a^2\mu} \right) S_B \\ &+ \left(-\frac{a^2w + S_Bu}{a^2\mu} \cdot \frac{b^2w' + S_Av'}{b^2\mu'} + \frac{a^2w' + S_Bu'}{a^2\mu'} \cdot \frac{b^2w + S_Av}{b^2\mu} \right) S_C = 0 \end{aligned}$$

$$\begin{aligned} &\left[(wu' - w'u) b^2 S_B + (uv' - v'u) c^2 S_C - (vw' - v'w) S_{BC} \right] a^2 S_A \\ &+ \left[-(wu' - w'u) S_{CA} + (uv' - v'u) c^2 S_C + (vw' - v'w) a^2 S_A \right] b^2 S_B \\ &+ \left[(wu' - w'u) b^2 S_B - (uv' - v'u) S_{AB} + (vw' - v'w) a^2 S_A \right] c^2 S_C = 0 \end{aligned}$$

$$\begin{aligned} &(vw' - v'w)\left(b^2 S_B + c^2 S_C - S_{BC}\right) a^2 S_A \\ &+ (wu' - w'u)\left(c^2 S_C + a^2 S_A - S_{CA}\right) b^2 S_B \\ &+ (uv' - v'u)\left(a^2 S_A + b^2 S_B - S_{AB}\right) c^2 S_C = 0 \end{aligned}$$

$$(vw' - v'w) S^2 a^2 S_A + (wu' - w'u) S^2 b^2 S_B + (uv' - v'u) S^2 c^2 S_C = 0$$

$$(vw' - v'w) a^2 S_A + (wu' - w'u) b^2 S_B + (uv' - v'u) c^2 S_C = 0$$

$$\begin{vmatrix} u & v & w \\ u' & v' & w' \\ a^2 S_A & b^2 S_B & c^2 S_C \end{vmatrix} = 0.$$

The last equality is the necessary and sufficient condition for the collinearity of M, M' and O.

Remark 2.1. *Since the circumcenter, the centroid and the orthocenter are situated on the Euler-line, the corresponding pedal triangle of the centroid and of the orthocenter are orthologic. More exactly the medial triangle, the orthic triangle and the pedal triangle of the centroid are pairwise orthologic. In a similar way the incenter I and the circumcenter are situated on a line passing through the circumcenter, so the corresponding pedal triangle are orthologic. This means that tha Gergonne triangle (or intouch triangle) and the medial triangle are orthologic. It is clear that the medial triangle is orthologic with any other pedal triangle. So our theorem generalizes the corresponding result from* [8].

3. Triangles of residual centroids for pedal triangles

If we denote by G_A, G_B and G_C the centroids of the triangle AEF, BDF and CED respectively, then the triangle $G_AG_BG_C$ is called the *triangle of residual centroids*. We will denote by T_0 the reference triangle ABC, by T_1 the pedal triangle DEF and by T_2 the triangle of residual centroids.

Theorem 3.1. *The triangles $G_AG_BG_C$ and DEF are orthologic (Figure 5).*

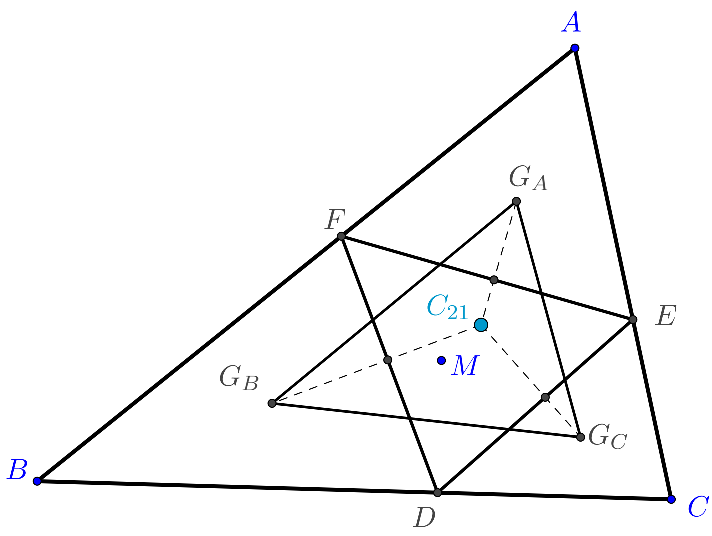

FIGURE 5. The orthology of the pedal triangle and the triangle of residual centroids

Proof. We use relation (1) for the characterization of orthogonality and the length of medians:

$$G_AF^2 - G_AE^2 = \frac{4}{9}\left(\frac{2(AF^2+EF^2)-AE^2}{4} - \frac{2(AE^2+EF^2)-AF^2}{4}\right)$$

$$G_AF^2 - G_AE^2 = \frac{1}{3}(AF^2 - AE^2).$$

Using similar relations for G_B and G_C we have

$$G_AF^2 - G_AE^2 + G_CE^2 - G_CD^2 + G_BD^2 - G_BF^2 =$$
$$= \frac{1}{3}(AF^2 - AE^2 + EC^2 - DC^2 + BD^2 - BF^2).$$

This relation and Theorem 1.2. imply that the triangles $G_AG_BG_C$ and DEF are orthologic.

Remark 3.1. *The above argument can be used also to prove the converse of the theorem: If $G_AG_BG_C$ and DEF are orthologic and D, E, F are arbitrary points on the sides BC, CA, AB, then ABC and DEF are also orthologic.*

In the following proof we will use a particular case of the Huygens-Leibniz identity (see [6], page 57) or equivalently a particular case of Lagrange's second identity (in some cases these relations are refered as Leibniz's relation):

Theorem 3.2. *If P is a point in the plan of the triangle UVW, G is the centroid of the triangle UVW, then*

$$\begin{aligned} PU^2 + PV^2 + PW^2 &= 3PG^2 + GU^2 + GV^2 + GW^2 \\ &= 3PG^2 + \frac{UV^2 + VW^2 + WU^2}{3}. \end{aligned} \tag{2}$$

It is interesting that this property in a more general setting (for a system of n points) was used several times by Lazare Carnot in his works as a known property ("propriétés connues"), see for example [2].

Theorem 3.3. *The triangles $G_AG_BG_C$ and ABC are orthologic (Figure 6).*

Proof. We use Theorem 1.2. for the characterization of orthologic triangles and the Leibniz relation to calculate the necessary distances. From the Leibniz relation we have

$$BF^2 + BE^2 + BA^2 = 3BG_A^2 + \frac{AF^2 + AE^2 + EF^2}{3} \quad \text{and}$$

$$CE^2 + CF^2 + CA^2 = 3CG_A^2 + \frac{AF^2 + AE^2 + EF^2}{3}, \quad \text{so}$$

$$G_AB^2 - G_AC^2 = \frac{1}{3}(BF^2 + BE^2 + BA^2 - CE^2 - CF^2 - CA^2).$$

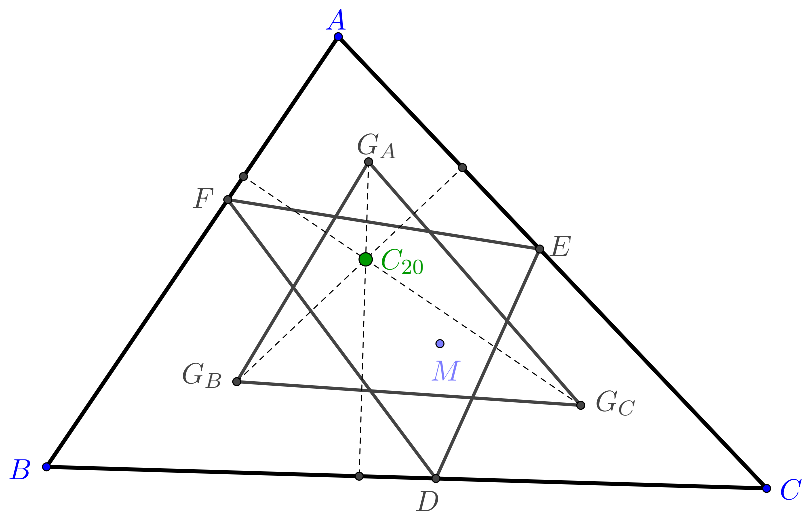

FIGURE 6. The orthology of the reference triangle and the triangle of residual centroids

Using similar relations for G_B and G_C we obtain:

$$G_AB^2 - G_AC^2 + G_BC^2 - G_BA^2 + G_CA^2 - G_CB^2$$

$$= \frac{1}{3}(BF^2 - AF^2 + AE^2 - EC^2 + CD^2 - BD^2).$$

But the pedal triangle DEF is orthologic to the reference triangle ABC, so based on Theorem 1.2. the last expression is 0. So using once again the characterization of the orthologic triangles we deduce that $G_AG_BG_C$ and ABC are orthologic.

Remark 3.2. *The previous proof can be modified to obtain also the following converse of the theorem: If D, E and F are arbitrary points on the sides BC, CA and AB respectively and the triangles $G_AG_BG_C$ and ABC, where G_A, G_B, G_C are the centroids of the residual triangles AFE, BDF and CED are orthologic, then the triangles ABC and DEF are orthologic.*

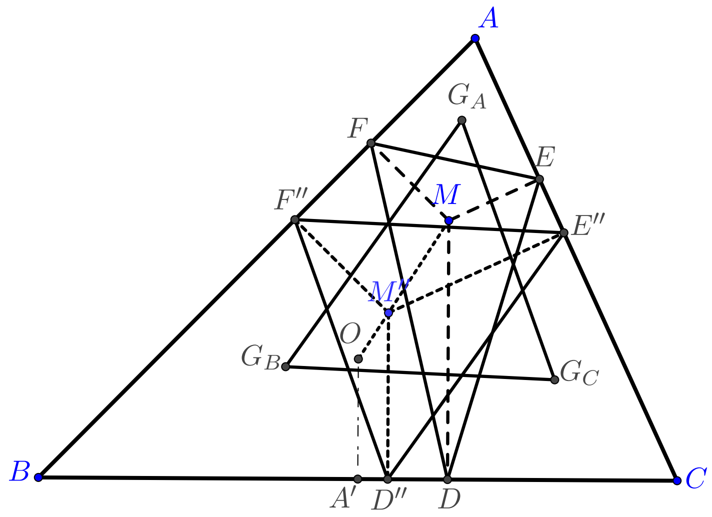

FIGURE 7. The triangle of residual centroids is congruent to the pedal triangle of M''

Theorem 3.4. *If $M'' \in OM$ and $OM = 3OM''$, $D''E''F''$ is the pedal triangle of M'', then the triangle $G_AG_BG_C$ of residual centroids of the point M is congruent to the triangle $D''E''F''$, moreover the corresponding sides of these two triangles are parallel (Figure 7).*

Proof. Denote by A', B', C' the midpoints of BC, CA and AB respectivelly. So the barycentric coordonates of these points are $A' = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $B' = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $C' = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Since $M''D''||MD$ and $\frac{MM''}{M''O} = 2$ in the trapezoid $MDA'O$ we have $\frac{DD''}{D''A'} = 2$, so $D'' = \frac{1}{3}(2A' + D)$. In a similar manner we obtain $E'' = \frac{1}{3}(2B' + E)$ and $F'' = \frac{1}{3}(2C' + F)$. On the other hand we have

$$G = \frac{1}{3}\left(A + 2A'\right) = \frac{1}{3}\left(B + 2B'\right) = \frac{1}{3}\left(C + 2C'\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right),$$

where G is the centroid of the triangle ABC, so we can easily prove that the quadrilateral $G_BG_CE''F''$ is a paralellogram. For this it is sufficient to prove that the midpoints of the segments G_BE'' and G_CF'' coincide. But the relation $G_B + E'' = G_C + F''$ is equivalent to the following relations:

$$\frac{1}{3}(B + F + D) + \frac{1}{3}\left(2B' + E\right) = \frac{1}{3}(C + D + E) + \frac{1}{3}\left(2C' + F\right)$$
$$\frac{1}{3}\left(B + 2B'\right) = \frac{1}{3}\left(C + 2C'\right)$$

and this true, so we obtain $G_BG_C = E''F''$ and $G_BG_C||E''F''$. Using a similar reasoning we can prove that $G_CG_AF''D''$ and $G_AG_BD''E''$ are also paralellograms and this concludes the proof.

Corollary 3.1. *The triangles $G_AG_BG_C$ and $D''E''F''$ are orthologic and its orthologic centers are the orthocenters of these triangles.*

Proof. Indeed the perpendiculars from G_A, G_B, G_C to the sides of the triangle $E''F''$, $F''D''$, $D''E''$ are the altitudes of the triangle $G_AG_BG_C$, so they meet in the orthocentre of the triangle $G_AG_BG_C$.

With the notations of the previous theorem the segments G_AD'', G_BE'' and G_CF'' have a common midpoint denoted by G_M. The following theorem shows an additional property of this point.

Theorem 3.5. *a) The points G^*, G^g, G_M, G'' and G are collinear, where G_M is defined as in the previouse paragraph, G^*, G^g, G'' and G are the centroids of the triangles DEF, $G_AG_BG_C$, $D''E''F''$ and ABC respectively. Moreover G_M is the midpoint of the segments G^*G and G^gG'' and $\frac{G_MG^g}{G_MG^*} = \frac{G_MG''}{G_MG} = \frac{1}{3}$ (Figure 8).*

b) The normalized barycentric coordinates $(x_{G_M}, y_{G_M}, z_{G_M})$ *of* G_M *are*

$$x_{G_M} = \frac{u}{2\mu} + \frac{c^2(b^2+S_C)v + b^2\left(c^2+S_B\right)w}{6b^2c^2\mu},$$

$$y_{G_M} = \frac{v}{2\mu} + \frac{a^2\left(c^2+S_A\right)w + c^2(a^2+S_C)u}{6c^2a^2\mu},$$

$$z_{G_M} = \frac{w}{2\mu} + \frac{b^2\left(a^2+S_B\right)u + a^2(b^2+S_A)v}{6a^2b^2\mu}.$$

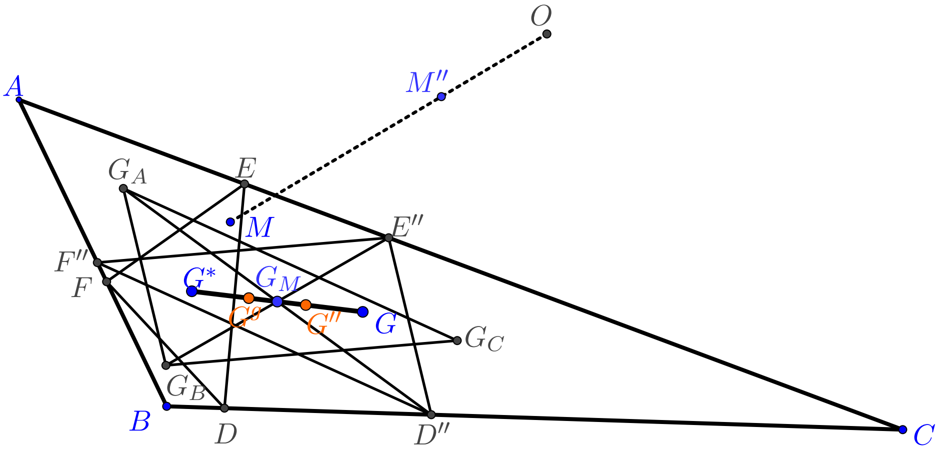

Figure 8. Collinear centroids

Proof. a) Using the previous theorem we have

$$G_M = \frac{1}{2}\left(G_A + D''\right) = \frac{1}{6}\left(A+E+F+2A'+D\right) = \frac{1}{2}\left(G+G^*\right).$$

Moreover the triangles $G_AG_BG_C$ and $D''E''F''$ are symmetric with respect to G_M, so $G_M = \dfrac{G^g+G''}{2}$. On the other hand

$$2G_M + G = G_A + D'' + \frac{A+B+C}{3} = \frac{A+E+F}{3} + \frac{B+C+D}{3} + \frac{A+B+C}{3}$$

and

$$3G'' = D'' + E'' + F'' = \frac{2A+2B+2C+D+E+F}{3},$$

so we have

$$G'' = \frac{2G_M + G}{3}.$$

This completes the proof of the first property.

b) In order to calculate the barycentric coordinates first we calculate the coordinates of the points G and G^* by using the coordinates of the vertices A, B, C and D, E, F :

$$G = \frac{1}{3}(A+B+C) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and } G^* = \frac{1}{3}\left(D+E+F\right),$$

so

$$\begin{aligned} x_{G^*} &= \frac{1}{3}\left(x_D + x_E + x_F\right) \\ &= \frac{1}{3}\left(\frac{b^2u + S_Cv}{b^2\mu} + \frac{c^2u + S_Bw}{c^2\mu}\right) \\ &= \frac{2b^2c^2u + c^2S_Cv + b^2S_Bw}{3\mu b^2c^2} \end{aligned}$$

Using a similar calculation or simply by simultaneous circular permutation of the variables (a, b, c), (u, v, w), (A, B, C) we can calculate y_{G^*} and z_{G^*}.

Since $G_M = \frac{1}{2}(G + G^*)$ we have

$$x_{G_M} = \frac{1}{2}(x_G + x_{G^*}) = \frac{u}{2\mu} + \frac{c^2(b^2 + S_C)v + b^2\left(c^2 + S_B\right)w}{6b^2c^2\mu}$$

and the similar relations for y_{G_M}, z_{G_M}.

Theorem 3.6. *a) If we consider the orthocenters H'' and H^g of the triangles $D''E''F''$ and $G_AG_BG_C$ and the circumcenters O'' and O^g of the same triangles, then O^g is the midpoint of G^*H'' and O'' is the midpoint of the segment GH^g (Figure 9).*

b) If we consider the Nagel points N'' and N^g of the triangles $D''E''F''$ and $G_AG_BG_C$ and the incenters I'' and I^g of the same triangles, then I^g is the midpoint of G^*N'' and I'' is the midpoint of the segment GN^g (Figure 11).

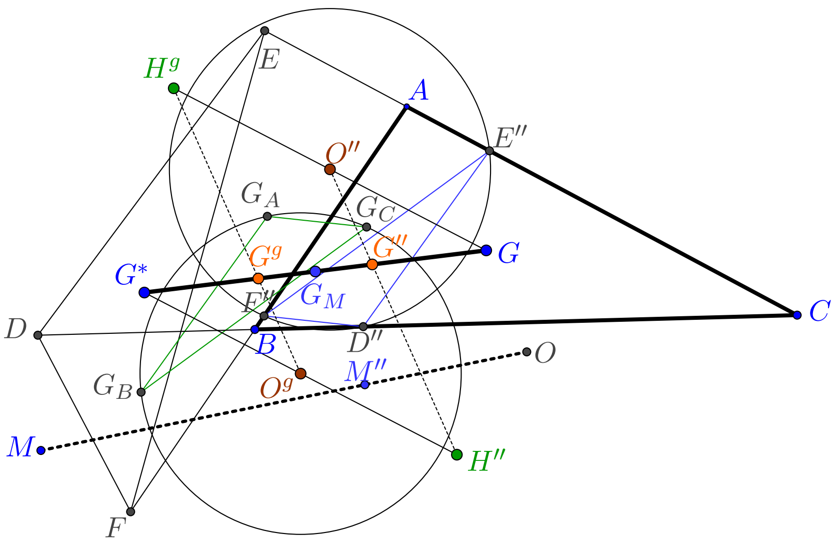

FIGURE 9. O^g and O'' are the midpoints of G^*H'' and H^gG respectively

Proof. a) Due to Theorem 3.4. the triangles $D''E''F''$ and $G_AG_BG_C$ are symmetric with respect to G_M, so the quadrilateral $O^gH''O''H^g$ is a paralellogram with center G_M. In this parallelogram G^g and G'' are situated on opposite sides and we have $\frac{O^gG^g}{O^gH^g} = \frac{O''G''}{O''H''} = \frac{1}{3}$. So if we consider the points $\{\overline{G}\} = H^gO'' \cap G^gG''$ and $\{\overline{G^*}\} = G^gG'' \cap O^gH''$, then we

$$\frac{\overline{G^*}O^g}{\overline{G^*}H''} = \frac{G^gO^g}{G''H''} = \frac{1}{2} \quad \text{and} \quad \frac{\overline{G}O''}{\overline{G}H^g} = \frac{G''O''}{G^gH^g} = \frac{1}{2},$$

so O^g and O'' are the midpoints of the segments $\overline{G^*}H''$ and $H^g\overline{G}$ respectivelly (Figure 10). But this implies that in the triangle $\overline{G}H^gH''$ the segments $H''O''$ and $\overline{G}G_M$ are medians, so G'' is the centroid of the triangle $H''H^g\overline{G}$. This implies $\frac{G_MG''}{G_M\overline{G}} = \frac{1}{3}$, so $\overline{G} = G$. By a similar reasoning we deduce $\overline{G^*} = G^*$, so the proof is complete.

b) Since I'', G'' and N'' are on the Nagel line of the triangle $D''E''F''$ and I^g, G^g, N^g are on the Nagel line of the triangle $G_AG_BG_C$ and $\frac{I''G''}{I''N''} = \frac{I^gG^g}{I^gN^g}$, a similar argument can be applied to the parallelogram $I''N''I^gN^g$.

Theorem 3.1. and Theorem 3.3. give an example of three pairwise orthologic triangles. The orthology relation is not transitive in general, but it is an interesting question to give a necessary and sufficient condition in order to assure the transitivity. Based on Theorem 2.1., Theorem 3.1., Theorem 3.4 and Theorem 3.3. we can state the following result.

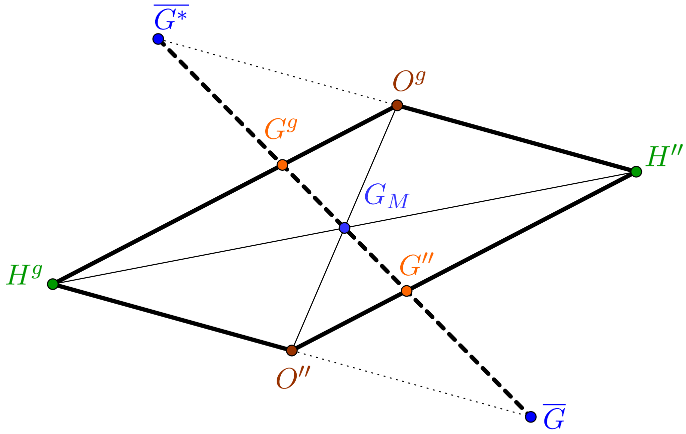

FIGURE 10. G^* and G on the sides of the parallelogram $O^gH''O''H^g$

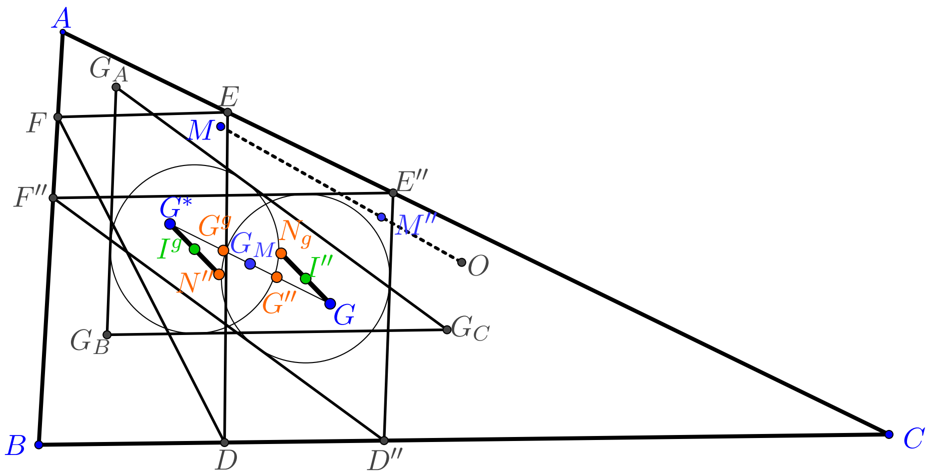

FIGURE 11. I'' and I^g are the midpoints of the segments GN^g and G^*N'' respectivelly

Theorem 3.7. *Consider an arbitrary M point in the plane of the triangle ABC and denote by O the circumcenter of the triangle ABC, by DEF the pedal triangle of M and G_A, G_B, G_C the centroids of the residuals triangles AFE, BDF and CED. If $M' \in OM$ and $D'E'F'$ is the pedal triangle of the point M' with respect to the triangle ABC and G'_A, G'_B and G'_C are the centroids of the residual triangles $A'F'E'$, $B'D'F'$ and $C'E'D'$, then the triangle pairs $(G_AG_BG_C, G'_AG'_BG'_C)$, $(DEF, D'E'F')$, $(DEF, G'_AG'_BG'_C)$, $(D'E'F', G_AG_BG_C)$ are orthologic.*

Proof. Due to Theorem 3.4 the sides of the triangles $G_AG_BG_C$ and $G'_AG'_BG'_C$ are parallel to the sides of the pedal triangles corresponding to some points on OM, so they are orthologic based on Theorem 2.1. The same argument can be applied to the pairs $(DEF, D'E'F')$, $(DEF, G'_AG'_BG'_C)$, $(D'E'F', G_AG_BG_C)$, so all these pairs are orthologic.

Remark 3.3. *Due to the previous theorem for each point M and arbitrary $\lambda \in \mathbb{R}$ we can consider the point $M(\lambda)$ with the property $\overrightarrow{OM(\lambda)} = \lambda \cdot \overrightarrow{OM}$ and construct the corresponding pedal triangle $D'(\lambda)E'(\lambda)F'(\lambda)$ and the triangle determined by the centroids of the corresponding residual triangles $G'_A(\lambda)G'_B(\lambda)G'_C(\lambda)$. If we consider the set of all these triangles:*

$$\mathcal{T} = \{D'(\lambda)E'(\lambda)F'(\lambda) | \lambda \in \mathbb{R}\} \cup \{G'_A(\lambda)G'_B(\lambda)G'_C(\lambda) | \lambda \in \mathbb{R}\},$$

then the elements of the set $\mathcal{T}$ are orthologic to the reference triangle and any two elements of $\mathcal{T}$ are orthologic. So there exists an infinite set of triangles with the property that any two of them are orthologic.

4. Further questions

Let's define a series five of triangles as follows: $T_0 = ABC, T_1 = DEF, T_2 = G_AG_BG_C$, $T_3 = D'E'F'$ $T_4 = G'_AG'_BG'_C$. All possible pairs formed by these triangles are orthologic, so we can denote by C_{ij} the meeting point of the perpendiculars from the vertices of the T_i to the sides of the T_j. For the previous 5 triangles there exists 20 such orthology centers. We propose as a further research to establish some properties among these centers and the geometric loci of these points when M describes some particular line passing through O. We assert without proof the following properties of these points:

Theorem 4.1. *a) The points C_{02}, C_{20} and C_{12} are collinear and $\overrightarrow{C_{12}C_{02}} = 3 \cdot \overrightarrow{C_{20}C_{02}}$ (Figure 12.).*

b) The centers C_{01}, C_{21} and C_{10} are collinear and $\overrightarrow{C_{01}C_{10}} = 3 \cdot \overrightarrow{C_{01}C_{21}}$ (Figure 13.).

c) The lines $C_{01}C_{02}, C_{21}C_{20}$ and $C_{10}C_{12}$ are parallel (Figure 14.).

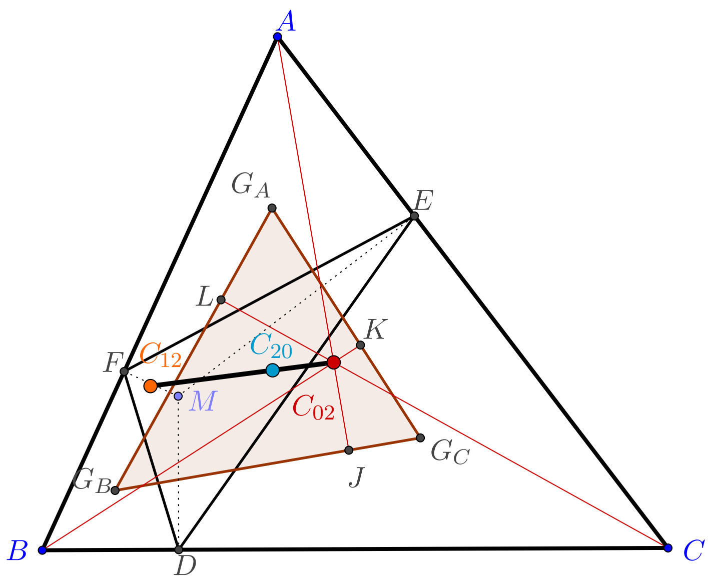

Figure 12. C_{12}, C_{02} and C_{20} are collinear

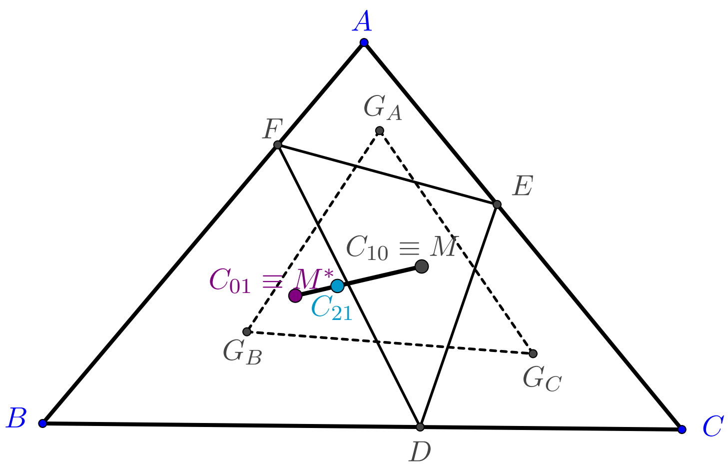

Figure 13. C_{21}, C_{10}, C_{01} are collinear

Remark 4.1. *If M is the circumcenter or the orthocenter of the triangle ABC, then b) will reduce to the Euler line.*

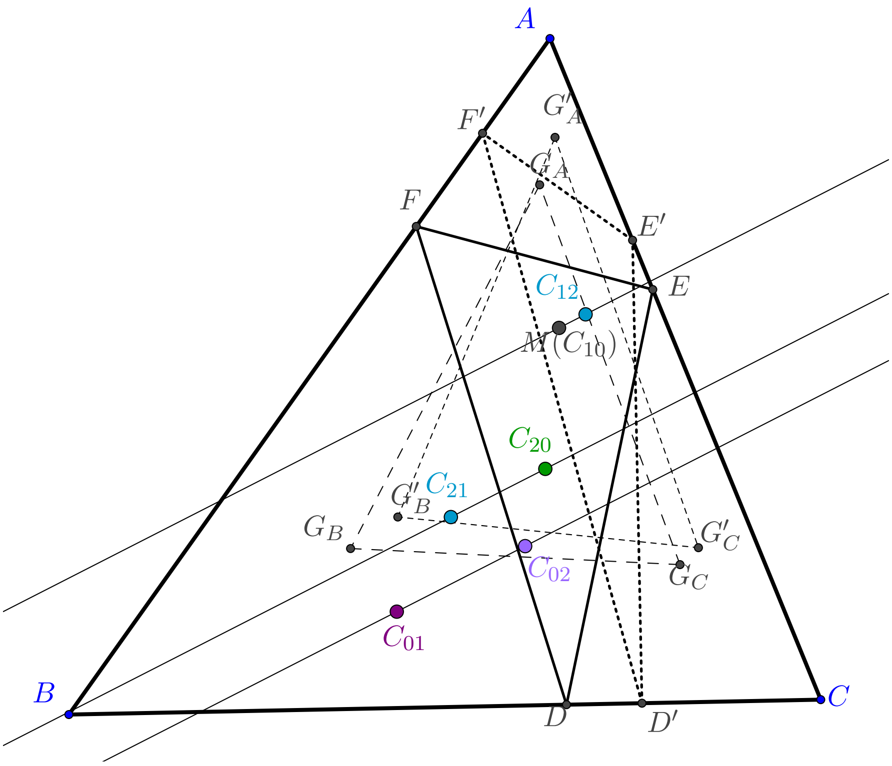

FIGURE 14. $C_{10}C_{12}$, $C_{21}C_{20}$ and $C_{01}C_{02}$ are parallel

Since there are no known results for the case when we consider other triangle centers (such as incenters, Gergonne points, etc.) instead of the centroids it would be interesting to find a general characterization in terms of the triangle centers we construct in the residual triangles.

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