INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 11 (2022), No. 1, 78 - 89

MUTUAL JACOBI TRIANGLES AND MORLEY'S TRIANGLE

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Abstract. Given a triangle ABC there is a 1-parameter set of angles (α, β, γ) which, by a Jacobi-construction, gives the reciprocal triangle $A'B'C'$ for which there also exists a Jacobi-construction that produces the original triangle ABC. It is shown here that if the set of angles (α, β, γ) is fixed then there are exactly two triangles, $A_1B_1C_1$ and $A_2B_2C_2$, with this property. Furthermore the two reciprocal triangles are similar and so can be arranged to coincide. Also if the two possible mutual triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent then their common reciprocal becomes a line segment and there are three distinct Jacobi centres which form an equilateral triangle with sides parallel to the principal Morley's triangle.

1. Reciprocal Jacobi Triangles

With ABC being any triangle, construct the points A', B', C' so that

$$\angle C'AB = \angle B'AC = \alpha,\ \angle A'BC = \angle C'BA = \beta \text{ and } \angle B'CA = \angle A'CB = \gamma.$$

These points form a *Jacobi triangle* for ABC and Jacobi's theorem states that the lines AA', BB' and CC' are concurrent (at the point K), see Figure 1.

Many proofs of this result are available, e.g. [3] and [1]. This result may be expressed as the triangle $A'B'C'$ is Jacobi-related to the triangle ABC by the angles α, β, γ. In [5] it is shown that if $A'B'C'$ is Jacobi-related to ABC with angles α, β, γ and if ABC is simultaneously Jacobi-related to $A'B'C'$ then

$$\frac{\sin(A+2\alpha)}{\sin A} = \frac{\sin(B+2\beta)}{\sin B} = \frac{\sin(C+2\gamma)}{\sin C} = \mu. \tag{1}$$

The triangles ABC and $A'B'C'$ are said to be *reciprocal* and this is expressed symbolically by

$$ABC \frac{\alpha, \beta, \gamma \ \rightarrow}{\leftarrow \ \alpha', \beta', \gamma'} A'B'C'$$

Keywords and phrases: Jacobi's theorem; Morley's triangle; triangle; reciprocal.
(2020)Mathematics Subject Classification: 51M04, 51M15
Received: 14.06.2021. In revised form: 25.11.2021. Accepted: 09.09.2021.

so that ABC is Jacobi-related to $A'B'C'$ with angles α', β', γ'. A typical case is illustrated in Figure 2.

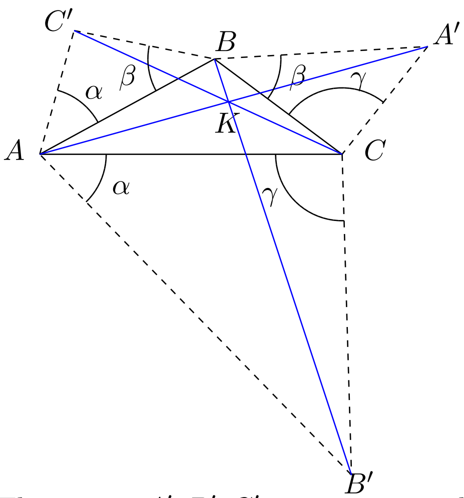

FIGURE 1. The points A', B', C' are constructed on a base triangle ABC with pairs of equal angles as shown. Jacobi's theorem states that AA', BB', CC' are concurrent and K will be used for their common point.

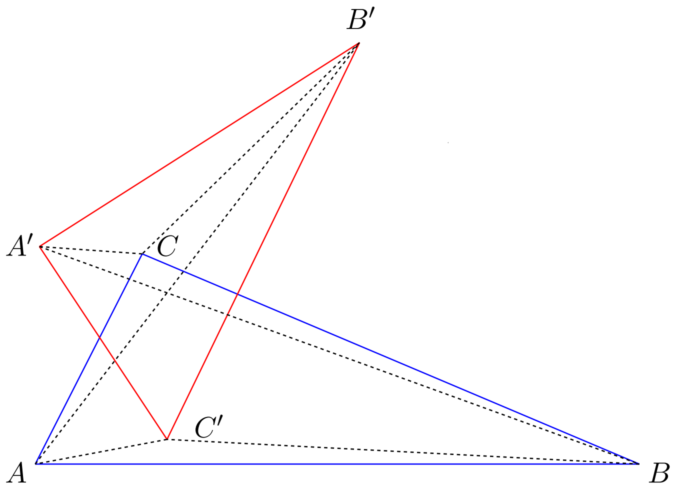

FIGURE 2. ABC and $A'B'C'$ are reciprocal Jacobi triangles. There are six pairs of equal angles, e.g. $\angle BAC' = \angle CAB'$.

In [4] it is further shown that the condition (1) is sufficient for the triangles to be reciprocal. In the Appendix, we give an outline of the proof of this result. The amount of algebra is reduced if the parameter λ is used (rather than μ) where

$$\lambda = \frac{\mu+1}{\mu-1} = \frac{\tan(A+\alpha)}{\tan\alpha} = \frac{\tan(B+\beta)}{\tan\beta} = \frac{\tan(C+\gamma)}{\tan\gamma}. \tag{2}$$

The following results from [5] will be used later:

$$\alpha' = A + \alpha - \beta - \gamma,\ \beta' = B - \alpha + \beta - \gamma,\ \gamma' = C - \alpha - \beta + \gamma$$

and the angles of the triangle $A'B'C'$ are

(3)
$$A' = \pi - 2A - 2\alpha + \beta + \gamma,\ B' = \pi - 2B + \alpha - 2\beta + \gamma,\ C' = \pi - 2C + \alpha + \beta - 2\gamma.$$

2. Mutual Jacobi Triangles

For any triangle ABC there is a 1-parameter family of triangles reciprocal to it and the members of this family are conveniently parametrized by λ. Suppose that instead of fixing the triangle ABC it is the angles α, β, γ which are considered as given. From equation (2) we have

$$\tan A = \frac{(\lambda - 1)\tan\alpha}{1 + \lambda \tan^2\alpha} \tag{4}$$

with two other similar results. The condition that the angles A, B, C form a triangle gives

$$\lambda^2 p(q-1) - \lambda(p - rq) + (r - p) = 0 \tag{5}$$

where

$$\begin{aligned} p &= \tan\alpha\tan\beta\tan\gamma, \\ q &= \tan\beta\tan\gamma + \tan\gamma\tan\alpha + \tan\alpha\tan\beta, \\ r &= \tan\alpha + \tan\beta + \tan\gamma. \end{aligned}$$

Thus for any set of α, β, γ there are just two values for λ, which will be denoted by λ_1 and λ_2. These values, using equation(4), give values A_1, A_2 and similarly B_1, B_2 and C_1, C_2. Since each of these six angles is found from its tangent, we may insist that each lies in $[0, \pi)$. But the values of $(A_1 + B_1 + C_1)$ and $(A_2 + B_2 + C_2)$ may be π or 2π. If the latter value is obtained then in order to construct a proper pair of triangles (that is ABC and $A_iB_iC_i,\ i = 1$ or 2) it would be necessary to change the signs of α, β, γ to preserve the relationship (1). But the values of α, β, γ are fixed. Thus for the set α, β, γ to produce two pairs of real triangles, equation (5) must have real roots and each of the sums $(A_1 + B_1 + C_1)$ and $(A_2 + B_2 + C_2)$ must equal π. Figure 3 shows these permissible values of (α, β) for a particular value of γ. For any permissible set the situation is succinctly described by

$$A_1B_1C_1 \underset{\leftarrow\ \alpha_1', \beta_1', \gamma_1'}{\overset{\alpha, \beta, \gamma\ \rightarrow}{\rule{2cm}{0.4pt}}} A_1'B_1'C_1' \text{ and } A_2B_2C_2 \underset{\leftarrow\ \alpha_2', \beta_2', \gamma_2'}{\overset{\alpha, \beta, \gamma\ \rightarrow}{\rule{2cm}{0.4pt}}} A_2'B_2'C_2'.$$

Theorem 2.1. *The triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$ are similar.*

Proof. Now

$$\tan A_i = \frac{(\lambda_i - 1)t}{1 + \lambda_i t^2}\ (i = 1, 2; t = \tan\alpha)$$

and (3) implies

$$A_1' + A_2' = 2\pi - 2(A_1 + A_2 + 2\alpha - \beta - \gamma)$$

which give

$$\tan(A_1 + A_2) = \frac{t[(\lambda_1 + \lambda_2)(1 - t^2) + 2\lambda_1\lambda_2 t^2 - 2]}{1 - t^2 + 2t^2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2(t^4 - t^2)}.$$

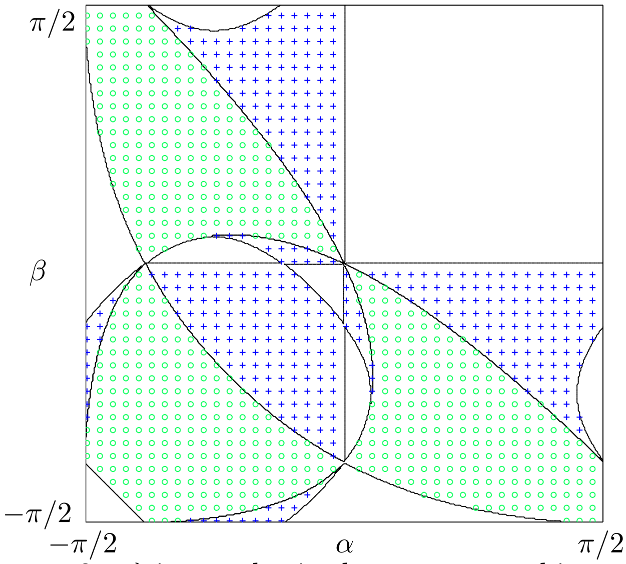

FIGURE 3. λ is complex in the areas covered in green small circles, $A+B+C=2\pi$ in the blank areas and the solutions are acceptable in the areas covered in blue plus signs. The figure is drawn for the case $\gamma = 1.2$

Equation (5) gives expressions for $(\lambda_1+\lambda_2)$ and $\lambda_1\lambda_2$ and the result is

$$\tan(A_1+A_2) = \frac{t[3p-rq-2pq+t^2(2r+rq-3p)]}{p(q-1)+t^2(4p-pq-2rq-r)+t^4(r-p)}.$$

An equivalent expression is obtained when $\tan(-2\alpha+\beta+\gamma)$ is expanded and $\tan\beta$ and $\tan\gamma$ are eliminated in favour of p, q, r. Thus $A_1'+A_2'$ is a multiple of 2π as required. □

The value of $(A_1'+B_1'+C_1')$ or $(A_2'+B_2'+C_2')$ may be 2π rather than π. Thus the signs of $\alpha_1', \beta_1', \gamma_1'$ or $\alpha_2', \beta_2', \gamma_2'$ may have to be changed but this does not affect the correctness of the geometrical construction.

The scale and orientation of the triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$ are arbitrary and so they can be made to coincide. In this case each of these triangles is re-labelled as $A'B'C'$ so that

$$A_1B_1C_1 \frac{\alpha,\beta,\gamma\ \rightarrow}{\leftarrow\ \alpha_1',\beta_1',\gamma_1'} A'B'C' \frac{\leftarrow\ \alpha,\beta,\gamma}{\alpha_2',\beta_2',\gamma_2'\ \rightarrow} A_2B_2C_2.$$

Definition 2.1. *The triangles $A_1B_1C_1$ and $A_2B_2C_2$ constructed as above are called Mutual Jacobi triangles. Each is a reciprocal Jacobi triangle to $A'B'C'$.*

Figure 4 shows a typical configuration.

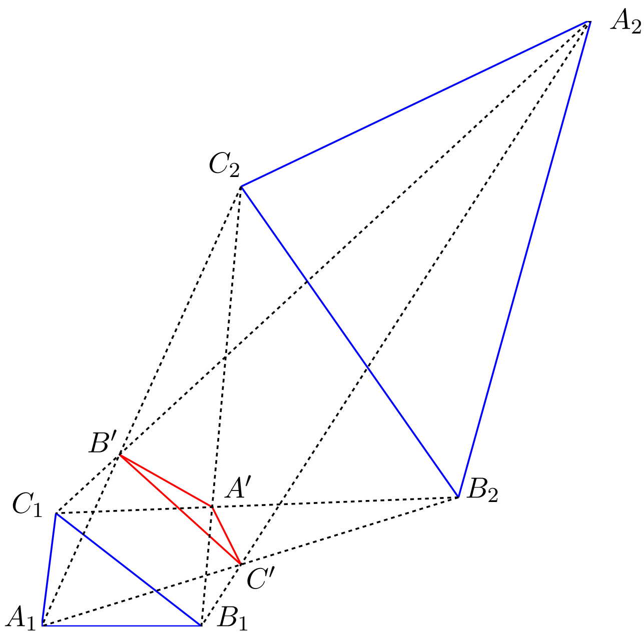

FIGURE 4. $A_1B_1C_1$ and $A_2B_2C_2$ are mutual Jacobi triangles.

3. Further Results

Here several results are given without proof. They were discovered by a combination of algebraic and trigonometrical manipulation.

$$\frac{\sin A'}{x} = \frac{\sin B'}{y} = \frac{\sin C'}{z} = \frac{\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}}{2xyz}$$

where $x = \sin(\beta+\gamma),\ y = \sin(\gamma+\alpha),\ z = \sin(\alpha+\beta)$.

$$\frac{\sin A_i}{\sin(\beta'_i+\gamma'_i)} = \frac{\sin B_i}{\sin(\gamma'_i+\alpha'_i)} = \frac{\sin C_i}{\sin(\alpha'_i+\beta'_i)} = \frac{\lambda_i - 1}{\lambda_i + 1}\ (i = 1, 2)$$

$$\frac{\tan\alpha'_i}{\tan\alpha} = \frac{\tan\beta'_i}{\tan\beta} = \frac{\tan\gamma'_i}{\tan\gamma} = -\frac{(\lambda_1\lambda_2)}{\lambda_i}\ (i = 1, 2)$$

$$\frac{\tan\alpha'_1}{\tan\alpha'_2} = \frac{\tan\beta'_1}{\tan\beta'_2} = \frac{\tan\gamma'_1}{\tan\gamma'_2} = \frac{\lambda_2}{\lambda_1}$$

$$\lambda_1\lambda_2 = -\frac{\tan(\alpha+\beta+\gamma)}{\tan\alpha\tan\beta\tan\gamma}$$

$$\frac{\tan(A'_i+\alpha'_1)}{\tan\alpha'_i} = \frac{\tan(B'_i+\beta'_1)}{\tan\beta'_i} = \frac{\tan(C'_i+\gamma'_1)}{\tan\gamma'_i} = \lambda'_i\ (i = 1, 2)$$

$$\lambda'_1\lambda'_2 = 1$$

Also, if the circumradii of $A_1B_1C_1$ and $A_2B_2C_2$ are R_1 and R_2, respectively, then

$$\frac{R_1}{R_2} = \left|\frac{\lambda_2 - 1}{\lambda_1 - 1}\right|.$$

4. Repeated Roots and Morley's Triangle

The equation (5) has repeated roots when

$$(p - rq)^2 = 4p(q-1)(r-p),$$

which implies

$$r = \frac{p(2Q+1)}{(1+Q)^2} \text{ and } \lambda = \frac{-1}{1+Q} \text{ where } Q = \pm\sqrt{1-q}. \tag{6}$$

Theorem 4.1. *When equation(5) has a repeated root, the points A', B', C' are collinear.*

Proof. Let the lengths BC, AC, BC of the sides of the triangle ABC be a, b, c, respectively. For an arbitrary Jacobi triangle construction, the barycentric coordinates of P, Q, R (see Fig. 1) are, respectively $(-a, bZ, cY)$, $(aZ, -b, cX)$, $(aY, bX, -c)$, where

$$X = \frac{\sin(A+\alpha)}{\sin\alpha}, \; Y = \frac{\sin(B+\beta)}{\sin\beta}, \; Z = \frac{\sin(C+\gamma)}{\sin\gamma}.$$

Thus the condition that P, Q, R be collinear is

$$0 = \begin{vmatrix} -a & bZ & cY \\ aZ & -b & cX \\ aY & bX & -c \end{vmatrix} = abc(X^2 + Y^2 + Z^2 + 2XYZ - 1). \tag{7}$$

For brevity, let

$$t_1 = \tan\alpha, \; t_2 = \tan\beta, \; t_3 = \tan\gamma,$$

then

$$X^2 = \frac{\sin^2(A+\alpha)}{\sin^2\alpha} = \frac{\lambda^2 + \lambda^2 t_1^2}{1 + \lambda^2 t_1^2}$$

and so

$$X^2 + Y^2 + Z^2 = \lambda^2\left[\frac{1+t_1^2}{1+\lambda^2 t_1^2} + \frac{1+t_2^2}{1+\lambda^2 t_2^2} + \frac{1+t_3^2}{1+\lambda^2 t_3^2}\right]$$

$$= \lambda^2 \frac{[3 + (1+2\lambda^2)(r^2-2q) + \lambda^2(2+\lambda^2)(q^2-2pr) + 3\lambda^4 p^2]}{1 + \lambda^2(r^2-2q) + \lambda^4(q^2-2pr) + \lambda^6 p^2} = \frac{Q^2+Q+1}{(Q+1)^2}$$

where the results of (6) have been used. In a similar fashion it is found that

$$(XYZ)^2 = \lambda^6 \frac{[(1-q)^2 + (r-p)^2]}{(1-\lambda^2 q)^2 + \lambda^2(r-\lambda^2 p)^2} = \frac{Q^2}{4(Q+1)^4}$$

and so (7) is satisfied. □

Since it is now known that the points A', B', C' are collinear, we may choose the corresponding angles as follows:

$$A' = \pi,\ B' = 0,\ C' = 0$$

also A will be used for A_1; likewise for B and C. Hence, using (3),

$$\begin{cases} \pi &= \pi - 2A - 2\alpha + \beta + \gamma, \\ 0 &= \pi - 2B + \alpha - 2\beta + \gamma, \\ 0 &= \pi - 2C + \alpha + \beta - 2\gamma, \end{cases}$$

It follows that

$$\begin{cases} \beta &= \pi + \alpha - \frac{2}{3}(2B + C), \\ \gamma &= \pi + \alpha - \frac{2}{3}(B + 2C). \end{cases}$$

Also

$$\frac{\sin(A + 2\alpha)}{\sin A} = \frac{\sin(B + 2\beta)}{\sin B} = \frac{\sin[2\alpha - \frac{1}{3}(5B + 4C)]}{\sin B}$$

and this gives α in terms of the angles A, B, C. The neatest way found of expressing this relationship is

$$\tan(A + 2\alpha) = \frac{\sin A}{2\cos\frac{1}{3}(B - C) - \cos A}.$$

This gives two values for α which differ by $\pi/2$. A cyclic change of A, B, C results in three pairs of solutions for α, β, γ. Typical solutions for an arbitrary triangle are depicted in Figure 5.

4.1. **The Position of K.** The common point K of the lines AA', BB', CC' is the same for each of the three pairs of solutions just found. Thus there are just three K points which may be labelled as K_A, K_B, K_C depending upon which of A', B' or C' has been chosen to equal π. With O as the circumcentre of the triangle ABC, the coordinates of O, A, K are

$$O(\sin 2A, \sin 2B, \sin 2C),\ A(1, 0, 0),\ K(\sin 2\alpha, \sin 2\beta, \sin 2\gamma)$$

and so all the sides of the triangle OAK may be determined. With the help of Maple it was found that

- $OA = OK$
- $\angle AOK_A = \pi \pm \frac{2}{3}(B - C)$
- the angle between AC and K_AK_B is $\frac{1}{3}(\pi + A - C)$.

It follows that the points K_A, K_B, K_C lie on the circumcircle of the triangle ABC and form an equilateral triangle. For reciprocal Jacobi triangles we know that the Jacobi centre, K, lies on the McCay cubic of ABC and it is known (see [2], example $K003$) that the common points of the circumcircle and the McCay cubic (other than A, B, C) form an equilateral triangle. Furthermore the triangle $K_AK_BK_C$ has its sides parallel to the Morley triangle formed by the trisectors of A, B, C. Figure 6 shows the McCay cubic and the circumcircle of ABC. The Jacobi points K_A, K_B, K_C are at the intersections of these two curves.

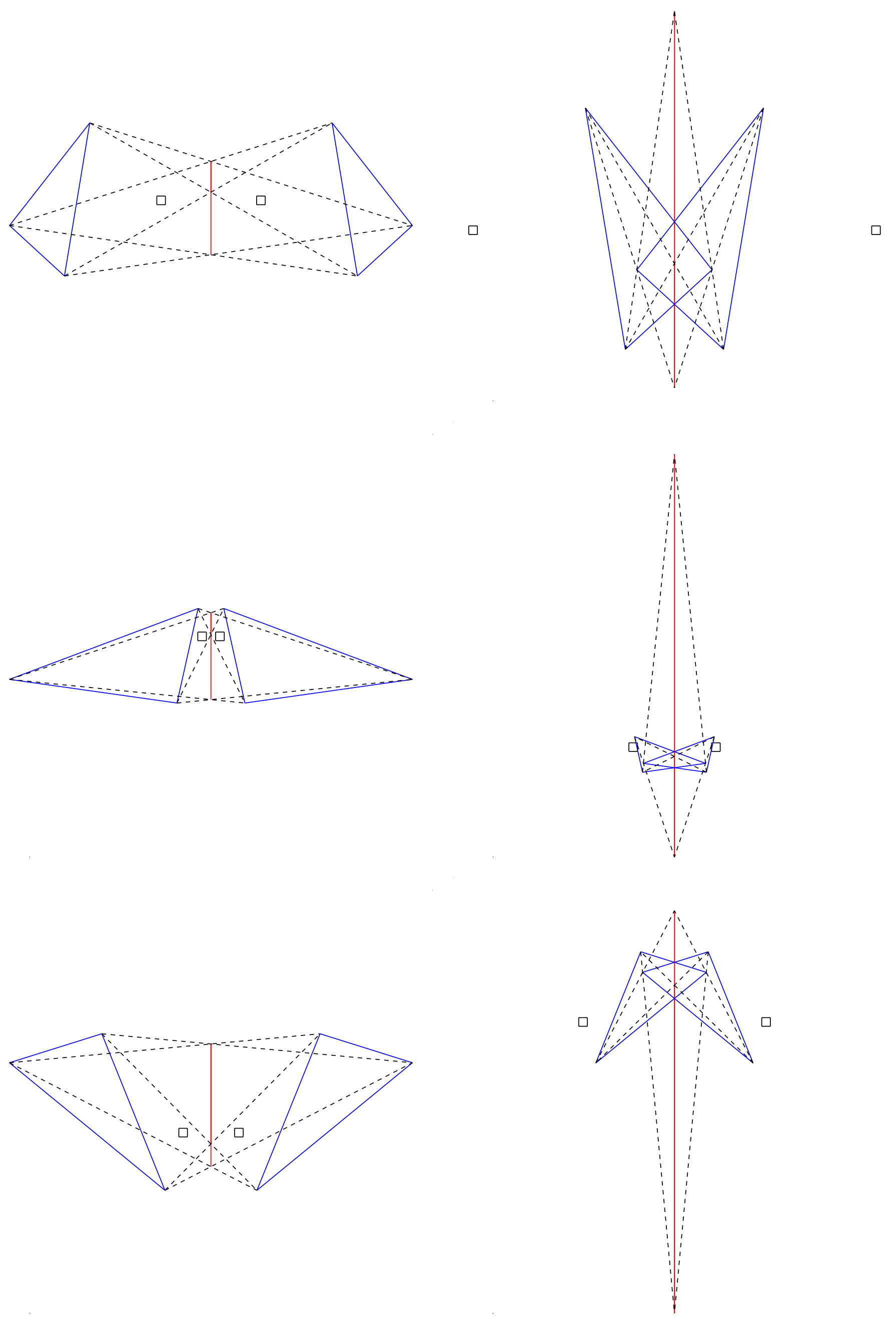

FIGURE 5. The three pairs of mutual Jacobi triangles when there are repeated roots. The common mutual triangle has collapsed to a line segment, shown in red. The Jacobi centres are shown as squares. In adjacent diagrams, the corresponding values of α, β, γ differ by $\pi/2$.

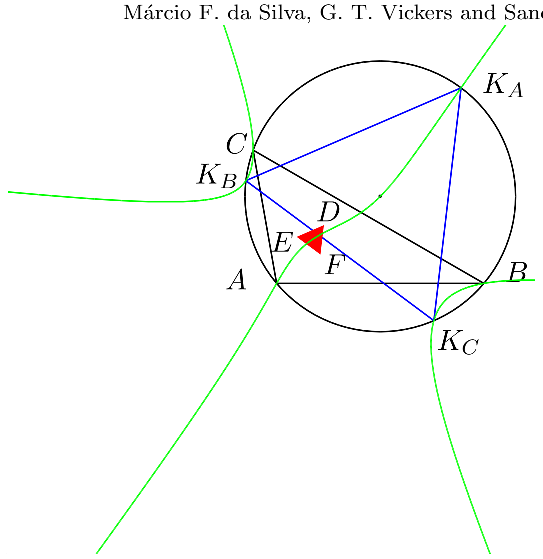

FIGURE 6. Shows the McCay cubic (green) and the circumcircle of ABC. The Jacobi points K_A, K_B, K_C are at the intersection of these two curves and form an equilateral triangle with sides parallel to the Morley triangle DEF.

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APPENDIX

Here we give an outline of the proof that the condition (1) is sufficient for the triangles to be reciprocal. More precisely, *if ABC is any triangle and $A'B'C'$ its Jacobi triangle with angles α, β, γ constructed on A, B, C, respectively, such that*

$$\frac{sin(\angle BAC + 2\alpha)}{sin(\angle BAC)} = \frac{sin(\angle ABC + 2\beta)}{sin(\angle ABC)} = \frac{sin(\angle ACB + 2\gamma)}{sin(\angle ACB)} = \mu,$$

where μ is a real constant, then ABC and $A'B'C'$ are reciprocal Jacobi triangles.

Outline of the proof. Let $\angle CA'B' = \alpha', \angle BA'C' = \theta, \angle AB'C' = \beta', \angle CB'A' = \phi, \angle BC'A' = \gamma'$ and $\angle AC'B' = \psi$, as illustrated in Figure 7.

By the law of sines for triangles ABC' and $AB'C'$ we have

$$\frac{AC'}{\sin\beta} = \frac{BC'}{\sin\alpha} = \frac{AB}{\sin(\pi - \alpha - \beta)} = \frac{AB}{\sin(\alpha+\beta)} \text{ and } \frac{AC'}{\sin\beta'} = \frac{AB'}{\sin\psi}.$$

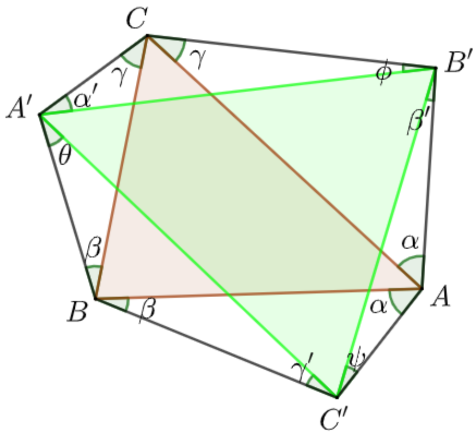

FIGURE 7. We need to prove that $\theta = \alpha', \phi = \beta'$ and $\psi = \gamma'$.

Consequently,

$$\frac{AB \cdot \sin\beta}{\sin(\alpha+\beta)} \cdot \frac{1}{\sin\beta'} = \frac{AB'}{\sin\psi}. \tag{8}$$

From the law of sines for triangle $AB'C$ it follows that

$$AB' = \frac{AC \cdot \sin\gamma}{\sin(\alpha+\gamma)}. \tag{9}$$

From (8) and (9) we get

$$\frac{AB \cdot \sin\beta}{\sin(\alpha+\beta)} \cdot \frac{1}{\sin\beta'} = \frac{AC \cdot \sin\gamma}{\sin(\alpha+\gamma)} \cdot \frac{1}{\sin\psi}.$$

From triangle $AB'C'$ we have $\psi = \pi - \angle BAC - 2\alpha - \beta'$. Then

$$\cot\beta' = \frac{AC \cdot \sin\gamma \cdot \sin(\alpha+\beta)}{AB \cdot \sin\beta \cdot \sin(\alpha+\gamma) \cdot \sin(\angle BAC + 2\alpha)} - \cot(\angle BAC + 2\alpha).$$

Since

$$\frac{\sin(\angle BAC + 2\alpha)}{\sin(\angle BAC)} = \mu,$$

we obtain

$$\mu \cdot \cot\beta' = \frac{AC \cdot \sin\gamma \cdot \sin(\alpha+\beta)}{AB \cdot \sin\beta \cdot \sin(\alpha+\gamma) \cdot \sin(\angle BAC)} - \frac{\cos(\angle BAC + 2\alpha)}{\sin(\angle BAC)}. \tag{10}$$

From triangle ABC,

$$\frac{AC}{AB} = \frac{\sin(\angle ABC)}{\sin(\angle ACB)}$$

and $\angle ABC = \pi - \angle ACB - \angle BAC$. Thus

$$\frac{AC}{AB} = \sin(\angle BAC) \cdot \cot(\angle ACB) + \cos(\angle BAC). \tag{11}$$

From (10) and (11) it follows that

$$\mu \cdot \cot\beta' = (\cot(\angle ACB) + \cot(\angle BAC)) \cdot \frac{\sin\gamma \cdot \sin(\alpha+\beta)}{\sin\beta \cdot \sin(\alpha+\gamma)} - \frac{\cos(\angle BAC + 2\alpha)}{\sin(\angle BAC)}. \tag{12}$$

But

$$-\frac{\cos(\angle BAC + 2\alpha)}{\sin(\angle BAC)} = \frac{-\cos(\angle BAC)\cdot\cos 2\alpha + \sin(\angle BAC)\cdot\sin 2\alpha}{\sin(\angle BAC)} \Longrightarrow$$

$$(13)\quad -\frac{\cos(\angle BAC + 2\alpha)}{\sin(\angle BAC)} = \frac{1}{\sin 2\alpha}\cdot(1-\cos 2\alpha\cdot\mu) = -\frac{(\mu\cdot\cos 2\alpha - 1)}{\sin 2\alpha}.$$

By replacing (13) in (12) we get

(14)

$$\mu\cdot\cot\beta' = (\cot(\angle ACB)+\cot(\angle BAC))\cdot\frac{\sin\gamma\cdot\sin(\alpha+\beta)}{\sin\beta\cdot\sin(\alpha+\gamma)} - \frac{(\mu\cdot\cos 2\alpha - 1)}{\sin 2\alpha}.$$

Analogously

(15)

$$\mu\cdot\cot\phi = (\cot(\angle ACB)+\cot(\angle BAC))\cdot\frac{\sin\alpha\cdot\sin(\gamma+\beta)}{\sin\beta\cdot\sin(\alpha+\gamma)} - \frac{(\mu\cdot\cos 2\gamma - 1)}{\sin 2\gamma}.$$

If $\alpha = \gamma$ then by (14) and (15) we have $\mu\cdot\cot\beta' = \mu\cdot\cot\phi$. Therefore $\beta' = \phi$ for $\beta', \phi \in (0,\pi)$.

We now suppose that $\alpha \neq \gamma$. Since

$$\mu = \frac{\sin(\angle BAC + 2\alpha)}{\sin(\angle BAC)} = \cos 2\alpha + \sin 2\alpha\cdot\cot(\angle BAC),$$

thus

$$(16)\quad \cot(\angle BAC) = \frac{\mu}{\sin 2\alpha} - \cot 2\alpha.$$

In a similar way,

$$(17)\quad \cot(\angle ACB) = \frac{\mu}{\sin 2\gamma} - \cot 2\gamma.$$

From (16) and (17) we get

(18)

$$\sin 2\alpha\cdot\sin 2\gamma\cdot(\cot(\angle BAC)+\cot(\angle ACB)) = \mu\cdot(\sin 2\alpha+\sin 2\gamma)-\sin(2\alpha+2\gamma).$$

But $\cot\beta' = \cot\phi \Longleftrightarrow \mu\cdot\cot\beta' = \mu\cdot\cot\phi \Longleftrightarrow$

$$(\cot(\angle ACB)+\cot(\angle BAC))\cdot\sin 2\alpha\cdot\sin 2\gamma\cdot\frac{\sin(\alpha-\gamma)}{\sin(\alpha+\gamma)} =$$

$$= \mu\cdot 2\sin(\alpha-\gamma)\cdot\cos(\alpha-\gamma) - 2\cdot\sin(\alpha-\gamma)\cdot\cos(\alpha+\gamma).$$

Since $\sin(\alpha-\gamma)\neq 0$ for $\alpha\neq\gamma$, we have that $\mu\cdot\cot\beta' = \mu\cdot\cot\phi \Longleftrightarrow$

$$(\cot(\angle ACB)+\cot(\angle BAC))\cdot\sin 2\alpha\cdot\sin 2\gamma =$$

$$= -\sin(2\alpha+2\gamma)+2\mu\cdot(\sin\alpha\cdot\cos\alpha\cdot(\sin^2\gamma+\cos^2\gamma)+\cos\gamma\cdot\sin\gamma\cdot(\sin^2\alpha+\cos^2\alpha).$$

Therefore $\mu\cdot\cot\beta' = \mu\cdot\cot\phi \Longleftrightarrow$

$$(\cot(\angle ACB)+\cot(\angle BAC))\cdot\sin 2\alpha\cdot\sin 2\gamma = -\sin(2\alpha+2\gamma)+\mu\cdot(\sin 2\alpha+\sin 2\gamma),$$

which is true by (18). We then get $\beta' = \phi$ for $\beta', \phi \in (0,\pi)$.

Analogously we can conclude that $\alpha' = \theta$ and $\gamma' = \psi$ so that ABC and $A'B'C'$ are reciprocal Jacobi triangles.

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