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On the distance between the incenter and the circumcenter of a triangle

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Abstract. In this paper, using mainly the distance in the title, we present an alternative proof of Feuerbach's theorem, some inequalities and remarks.

1. Introduction

K. W. Feuerbach has proved in 1822 that the nine point circle of a triangle is tangent to the incircle and to the excircles of the triangle. There are many proofs on Feuerbach's theorem but our proof in this note is easy and simple. We shall confine our attention to acute triangles and incircles because our theorem can be proved in the same manner in other cases. This theorem is trivial for the equilateral triangle. We remove here this triangle.

Given a triangle ABC, denote by O the cicumcenter, H the orthocenter, I the incenter, R the circumradius, r the inradius of ABC, and $\mathcal{C}(U, V)$ the circle with the center U and radius V.

2. Theorems

Theorem 2.1. *The nine-point circle of a triangle is tangent to each of the incircle and excircles of the triangle.*

Proof. Let N and A' be the midpoints of segments OH and BC, respectively, and J is the reflection of the point I with respect to the point N. Points K and L are the midpoints of the segments JO and IH, respectively.

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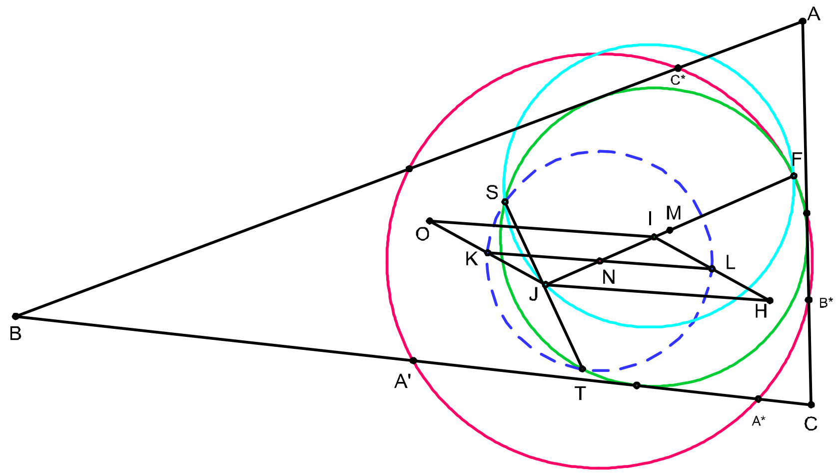

Figure 1

Since N is the midpoint of both lines OH and JI, the quadrilateral $JOIH$ is a parallelogram. From this and two midpoints theorem, the segments OI, KL and JH are parallel and N is on the line KL.

Let the line ℓ be perpendicular to the line JI at N and let S (over N) and T be the intersection points of ℓ and the circle $\mathcal{C}(N, KN)$. Then $SN = NT = KN = OI/2$. The point F is the intersection point of the line NI on I side and the nine point circle $\gamma = \mathcal{C}(N, NA')$ where $NA' = R/2$.

Let α be the circumcircle of $\triangle SJF$ and let M be the midpoint of line JF. Since the line JF is the perpendicular bisector of the segment ST, a quadrilateral $SJTF$ is symmetric with respect to the line JF and hence three circles $\alpha = \mathcal{C}(M, MF), \beta = \mathcal{C}(N, KN)$ and $\gamma = \mathcal{C}(N, NA' = R/2)$ are also symmetry with respect to the line JF.

Moreover, we have

$$2MF = JF = JN + NF = NI + NF < 2NF = R.$$

The circle α is tangent internaly at F to γ by $MF < R/2$. We obtain

$$(*) \qquad IN = R/2 - r$$

from the next using [1, p.85, Euler's Theorem],

$$(R/2) \cdot NI = NF \cdot JN = SN^2 = KN^2 = (IO/2)^2 = (R/2)(R/2 - r).$$

□

Theorem 2.2. *The following relations hold:*

$$OH^2 = R^2 - 4\rho R, \quad (1)$$

$$IH^2 = 2r^2 - 2\rho R, \quad (2)$$

$$R > 2r, \quad (3)$$

$$r > 2\rho, \quad (4)$$

$$KN > NI, \quad (5)$$

$$JF > OI, \quad (6)$$

$$OI > JI, \quad (7)$$

$$\angle OIH > \frac{\pi}{2}, \quad (8)$$

where ρ is the inradius of the orthic triangle of ABC.

Proof. (1) Since the pedal triangle $\triangle A^*B^*C^*$ of the orthocenter of triangle $\triangle ABC$ has the incircle $\mathcal{C}(H,\rho)$ and the circumcircle $\gamma = \mathcal{C}(N, R/2)$, we get

$$OH^2 = 4 \cdot NH^2 = 4 \cdot R/2 \cdot (R/2 - 2\rho) = R^2 - 4\rho R.$$

(2) If we use the median theorem in triangle OIH, then we get

$$4 \cdot IN^2 = 2(IO^2 + IH^2) - OH^2,$$

and it follows that

$$2 \cdot IH^2 = 4 \cdot IN^2 + OH^2 - 2OI^2$$

Now, using formulas $(*)$, (1) and Euler's relation we obtain

$$IH^2 = 2r^2 - 2\rho R.$$

(3) Using the fact that $NI > 0$, we obtain Euler's inequality.
(4) From the relations (2) and (3) we get

$$r^2 > R\rho > 2r\rho$$

and hence, $r > 2\rho$.
(5) We have

$$KN^2 = R/2 \cdot (R/2 - r) > (R/2 - r)^2 = NI^2.$$

(6) We have

$$JF^2 = (R - r)^2 = R(R - 2r) + r^2 = OI^2 + r^2 > OI^2.$$

(7) Because $OI = 2KN$ from the relation (5) we get

$$OI > 2NI = JI.$$

(8) We use the Law of Cosines in the triangle IOH and obtain

$$\cos \widehat{IOH} = \frac{OI^2 + IH^2 - OH^2}{2IO \cdot IH}$$

$$= \frac{(R^2 - 2Rr) + (2r^2 - 2\rho R) - (R^2 - 4\rho R)}{2\sqrt{R^2 - 2Rr} \cdot \sqrt{2r^2 - 2\rho R}} = \frac{r^2 + \rho R - Rr}{\sqrt{R^2 - 2Rr} \cdot \sqrt{2r^2 - 2\rho R}}.$$

It is clear that we have $r^2 + \rho R < r^2 + \frac{r}{2}R = r\left(r + \frac{R}{2}\right) < rR$, so we obtain that $\cos\widehat{IOH} < 0$, i.e. $\angle OIH > \frac{\pi}{2}$. □

3. Remarks

Remark 3.1. *It is essential to construct the right triangle $\triangle SJF$ with $SN^2 = NF \cdot JN, NF = R/2$ and points N, I on JF as the Figure 1 by the next reason. Since $\triangle FSN$ and $\triangle SJN$ are similar, we have*

$$\frac{SN}{FN} = \frac{JN}{SN},$$

and hence we obtain

$$NI = \frac{R}{2} - r$$

by the last equation in the proof of Theorem 2.1.

Remark 3.2. *Since midpoints A', B', C' of three sides BC, AC, AB and midpoints A'', B'', C'' of AH, BH, CH are on the nine point circle γ, we can see that $\triangle ABC$, $\triangle A'B'C'$ and $\triangle A''B''C''$ are positions of similarity with the centers; centroid and orthocenter H with a ratio $2 : -1 : 1$, respectively.*

Remark 3.3. *We give the segment JF, the mid point M of the segment JF, the point I on the line JF with small segment $JI < JM$ on side M and the mid point N of JI. We draw circles $\alpha := \mathcal{C}(M, MF)$ and $\gamma := \mathcal{C}(N, NF)$. Let S over N and T be the intersecrion points of α and of the perpendicular line ℓ to the line JF at N. We set $\beta := \mathcal{C}(N, SN)$, $R := 2NF$, $r := IF$, $\theta := \mathcal{C}(I, r)$ and $\sigma := \mathcal{C}(O, R)$ where $OI := 2NT$ on lower JF and left NT. From equations*

$$OI^2 = (2TN)^2 = 4NF \cdot NI = R(R - 2r)$$

using

$$NI + r = NI + IF = NF = R/2$$

and [1, p.86, 155. Theorem], we draw two tangents to the incircle θ from any point A on the circumcircle σ, we obtain $\triangle ABC$ having the same circumcircle σ and incircle θ, where BC is the opposite side of A. Please take notice positions of points J, N, I, F and O with sizes of JF, OI and NF for containing these, σ, at least $\triangle ABC$.

References

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