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GEOMETRIC ASPECTS OF MÖBIUS TRANSFORMATIONS

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Abstract In this article we study a geometric method to construct the image of a point under a generic Möbius transformation. In addition, using this construction, we derive geometrically some of the well known main properties of these transformations and prove their decomposition in a glide-reflection and an inversion, both completely determined by the characteristic parallelogram of the transformation.

1 Introduction

Geometric aspects of Möbius transformations have been thoroughly studied in the past and there are several interesting textbooks on this subject ([6], [10], [11], [12]), its applications, and more general on the geometry of complex numbers ([1], [7], [9], [13]). A Möbius transformation $w = f(z)$ is defined by a simple complex function of the form

$$w = f(z) = \frac{az+b}{cz+d} \quad \text{with complex numbers satisfying} \quad ad - bc \neq 0\,. \tag{1}$$

Its *"characteristic parallelogram"* ([6, p.147], [12, p.69]) is defined by four distinguished complex numbers $\{\gamma_1, \gamma_2, z_i, w_i\}$. The two first are the *"fixed"* or *"double"* points of the transformation satisfying the equation

$$z = \frac{az+b}{cz+d} \quad \Leftrightarrow \quad cz^2 - (a-d)z - b = 0\,. \tag{2}$$

The other two are the *"pole"* z_i send by f to infinity and the pole w_i of the inverse transformation $z = f^{-1}(w)$ which is also a Möbius transformation:

$$z_i = -\frac{d}{c}, \quad w_i = \frac{a}{c} \quad \text{since} \quad z = f^{-1}(w) = \frac{dw-b}{-cw+a}. \tag{3}$$

It is known and trivial to verify, that for a generic Möbius transformation these points are opposite vertices of a parallelogram, the *"characteristic parallelogram"* of the transformation (see Figure 1). The parallelogram with the prescribed meaning of its vertices uniquely determines the transformation $f(z)$. For convenience of reference I formulate this well known result ([6, p.148], [12, p.65]) as a theorem.

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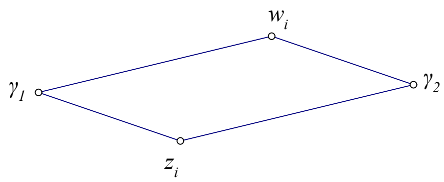

Figure 1: Characteristic parallelogram

Theorem 1. *If $\{\gamma_1, \gamma_2\}$ are fixed complex numbers and the complex variables $\{z, w\}$ have a constant cross ratio $(wz; \gamma_1\gamma_2) = k$, then they define implicitly a Möbius transformation $w = f(z)$.*

$$w = f(z) \quad \Leftrightarrow \quad (wz; \gamma_1\gamma_2) = \frac{w-\gamma_1}{w-\gamma_2} : \frac{z-\gamma_1}{z-\gamma_2} = k \,. \tag{4}$$

The number k is called *"characteristic constant"* of f and is determined if we know a particular point z_0 and the corresponding value $w_0 = f(z_0)$. F.e. taking $\{z_0 = \infty, w_0 = w_i\}$:

$$(w_i\infty; \gamma_1\gamma_2) = k \quad \Leftrightarrow \quad \frac{w_i-\gamma_1}{w_i-\gamma_2} = k \quad \Leftrightarrow \quad \frac{a-c\gamma_1}{a-c\gamma_2} = k \,. \tag{5}$$

Next theorem initiates our discussion by establishing a geometric construction of the image-point $W(w)$ of $Z(z)$ of the Möbius transformation $w = f(z)$ (see Figure 2). While there are well known geometric interpretations of the various kinds of Möbius transformations, parabolic, hyperbolic, elliptic ([6, p.150]), it seems that the following simple interpretation for the *generic* Möbius transformation, i.e. the transformation with a genuine finite characteristic parallelogram has not been yet noticed.

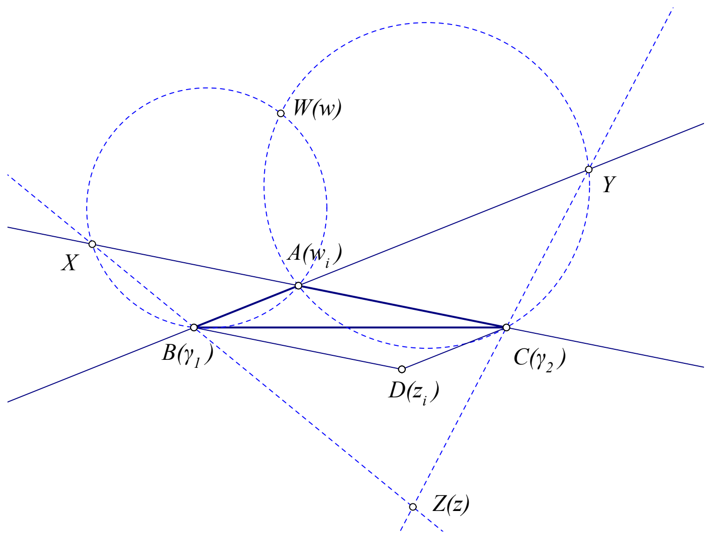

Figure 2: Geometric construction of $w = f(z)$

Theorem 2. *For a given triangle ABC and a point Z of the plane we consider the intersections of the lines $\{X = ZB \cap AC \,,\ Y = ZC \cap AB\}$ and the second intersection W of the circles $\{(ABX)\,,\ (ACY)\}$. The map $W = F(Z)$ represents geometrically a Möbius transformation $w = f(z)$ and every generic Möbius transformation can be represented this way.*

The triangle ABC defines a corresponding parallelogram $ABDC$ coinciding with the characteristic parallelogram of the Möbius transformation f, the vertices $\{B, C\}$ representing the fixed points $\{\gamma_1, \gamma_2\}$ and points $\{A, D\}$ representing the poles $\{w_i, z_i\}$ of f.

Proof. The proof uses theorem 1 and the well known complex number involution theorem for complete quadrilaterals ([6, p.162]), according to which, *"the pairs of opposite vertices of a complete quadrilateral are related by a Möbius involution"*. This applies to the pairs $\{(A,Z),(B,C),(X,Y)\}$ of the complete quadrilateral in figure 2 and guarantees the existence of a Möbius involution $g(z) = z'$ interchanging the members of these pairs. Since Möbius transformations preserve cross ratios we have

$$(wz;\gamma_1\gamma_2) \;=\; (w'z';\gamma_1'\gamma_2') \;=\; (w'w_i;\gamma_2\gamma_1) = \frac{w'-\gamma_2}{w'-\gamma_1} : \frac{w_i-\gamma_2}{w_i-\gamma_1}\,. \tag{6}$$

Now we show that $w' = g(w)$ is the point at infinity. In fact, if it were a finite point, then since g preserves the set of circles+lines and maps $\{Y \mapsto X, C \mapsto B, A \mapsto Z\}$, it maps circle $(YAC) \ni W$ to line BX and W to a point $W' \in BX$. Analogously considering the circle (XAB) and applying to it g we see that W maps to a point $W' \in CY$. Thus, W' would coincide with Z implying that $F(Z) = W = g(W') = g(Z) = A$, which is not true for F and a finite point Z but is true for $Z = \infty : F(\infty) = A$ (see Figure 3). Thus, $W'(w')$ is

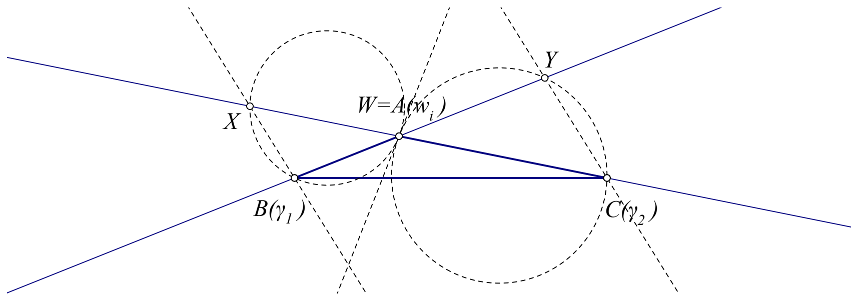

Figure 3: $W = F(Z) = A$ only for $Z = \infty$

the point at infinity and the ratio on the right side of equation (6) becomes

$$(wz;\gamma_1\gamma_2) \;=\; \frac{w'-\gamma_2}{w'-\gamma_1} : \frac{w_i-\gamma_2}{w_i-\gamma_1} \;=\; \frac{w_i-\gamma_1}{w_i-\gamma_2} \;=\; k,$$

a constant, showing that the geometrically constructed map $W = F(Z)$ defines indeed a Möbius transformation f.

For the converse, we assume that $ABDC$ is the characteristic parallelogram of a Möbius transformation f' with the prescribed meaning of the vertices and show that it coincides with the Möbius transformation f defined in the first part of the theorem. For this it suffices to see that the two maps coincide at three points, which is trivially verified for the points $\{B,C,D\}$. □

2 A line correspondence

Next theorem uses *"line homographies"* or *"projective transformations between lines"* ([8, p.5], [2, I, p.122]), which for projective coordinates on real lines have the same formal representation $w = (az+b)/(cz+d)$ as the Möbius transformations, with the only difference the reality of all entities appearing there.

Theorem 3. *Let ε be a line through D and Z a point moving on it. The following are valid properties (see Figure 4).*

1. *The corresponding points $\{X = ZB \cap AC, Y = ZC \cap AB\}$ are related with a line homography, which for appropriate projective coordinates $\{x,y\}$ respectively on lines $\{AC,AB\}$ has the representation $y = kx$ with k a constant.*

2. *The constant* $= BB'/CA = BA/CC'$ *where* $\{B' = \varepsilon \cap AB, C' = \varepsilon \cap AC\}$.
3. *The point* $W = F(Z)$ *describes a fixed line* ε' *through* A *as* Z *varies on line* ε.

Proof. *Nr-1* follows from the fact that the map $g : X \mapsto Y$ is composition of two perspectivities $g = g_2 \circ g_1$. The perspectivity g_1 is centered at B and maps $X \in AC$ to $Z \in \varepsilon$. The perspectivity g_2 is centered at C and maps $Z \in \varepsilon$ to $Y \in AB$. Since perspectivities are homographies and homographies are closed under composition, g is a homography of the form $y = g(y) = (ax+b)/(cx+d)$ for projective coordinates $\{x, y\}$ respectively along the lines $\{AC, AB\}$. Changing the coordinates to signed distances respectively from $\{C, B\}$ the homography has the same typical representation $y' = (a'x' + b')/(c'x' + d')$. It is though easily seen, that when Z obtains the position of $H = BC \cap \varepsilon$ the two circles

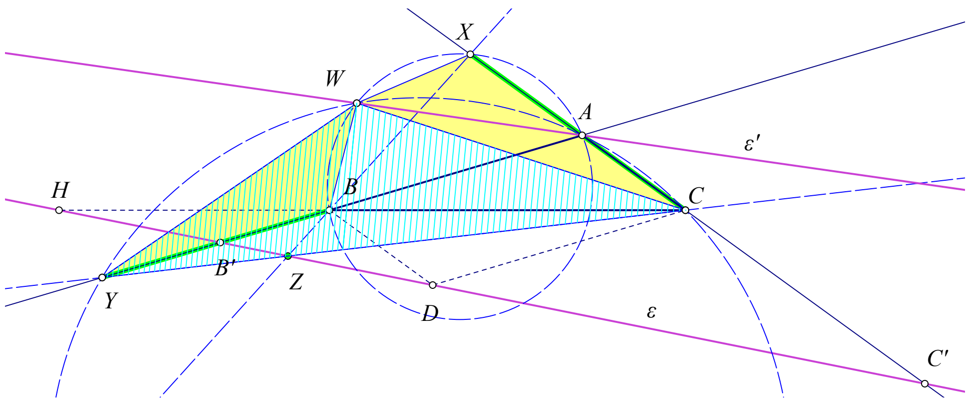

Figure 4: Lines ε through D and their $F-$images ε' through A

$\{(ABX), (ACY)\}$ coincide with the circumcircle of the triangle ABC and then $x' = y' = 0$. This implies $b' = 0$. On the other hand, when Z tends to D then $\{x', y'\}$ tend simultaneously to infinity, implying $c' = 0$ hence $y' = (a'/d')x' = kx'$ as claimed.

Nr-2 follows by observing $\{x, y\}$ for two special positions of Z on line ε. When $Z = B'$ then we have $\{X = A, Y = B'\}$ and $BB' = kCA$. When $Z = C'$ then we have the equalities $\{X = C', Y = A\}$ and $BA = kCC'$.

Nr-3. By its definition, point $W = F(Z)$ is the second intersection of the two variable circles $\{(ABX), (ACY)\}$. The triangles $\{WBY, WXC\}$ however are similar. This because $\widehat{WBY} = \widehat{WXC}$ and $\widehat{WYB} = \widehat{WCA}$. This implies that $WY/CW = BY/CX = k$ is constant. Also the angle $\widehat{YWC} = \widehat{YAC} = \widehat{A}$ is constant. Thus, triangle YWC has constant similarity type and, while its vertex C remains constant, the vertex Y varies on the fixed line AB. By a well known theorem ([3, p.49]) the third vertex W describes then a line ε'. From the cyclic quadrangle $AWYC$ we have $\widehat{WAB} = \widehat{WCY}$ which is a constant angle. This shows that W describes a fixed line through A. □

Corollary 1. *With the notation adopted so far and with* $J = \varepsilon \cap \varepsilon'$, *the angles* $\widehat{JAB'} = \widehat{AC'B'}$ *and the lines* $\{\varepsilon, \varepsilon'\}$ *are equally inclined to the bisector* AA' *of* $\widehat{A}$.

Proof. In fact, consider Z going to infinity on line ε. Then lines $\{ZB, ZC\}$ become parallel to ε and intersect $\{AC, AB\}$ respectively at points $\{X_0, Y_0\}$ (see Figure 5). The circles $\{(ABX_0), (ACY_0)\}$ are tangent at A and $W = F(Z)$ coincides with A. This implies the claimed equality of angles. The second claim is an immediate consequence, since the triangle AJA' is seen to be isosceles: $\widehat{JAA'} = \widehat{JA'A} = \delta + \alpha/2$ with $\{\delta = \widehat{JAB}, \alpha = \widehat{BAC}\}$. □

Corollary 2. *With the notation adopted so far, the triangles* $\{WBX, CDC'\}$ *are similar, the triangles* $\{AWC, DCZ\}$ *are similar and* $DZ \cdot AW = AB \cdot AC$ *(see Figure 6).*

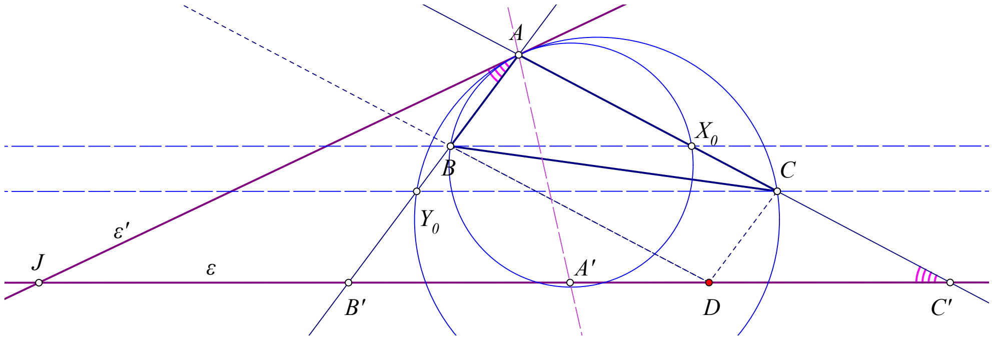

Figure 5: Lines ε through D and ε' equal inclined to AA'

Proof. The similarity of the first pair of triangles follows immediately from the preceding corollary. For the similarity of the second pair notice that $\widehat{DCZ} = \widehat{B'YZ} = \widehat{AYC} = \widehat{AWC}$. Also $\widehat{ZDC} = \widehat{WAC}$ since both angles are supplementary to equal angles $\widehat{CDC'} = \widehat{WBX}$. The stated relation follows from the last similarity and the equality $CD = AB$. □

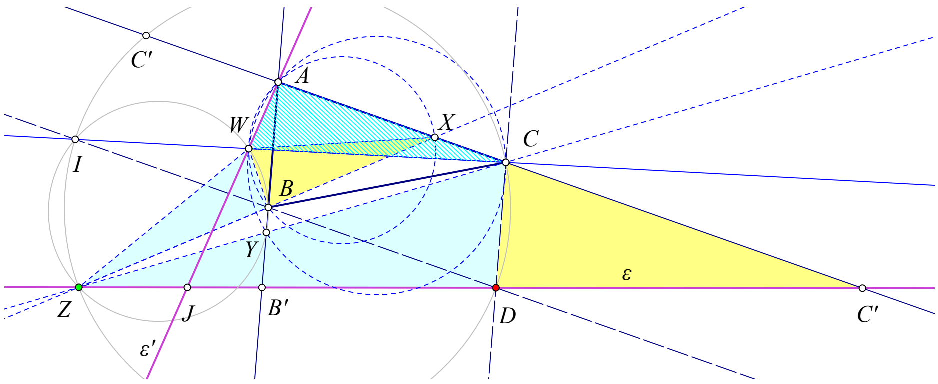

Figure 6: The relation $DZ \cdot AW = AB \cdot AC$

Corollary 3. *Lines $\{BD, CW\}$ intersect at a point I on the circle (CDZ) and triangle ZWB is similar to ZCD (see Figure 6).*

Proof. Let the circle (CDZ) intersect line BD in point I and AC in point C'. The parallels $\{AC, BD\}$ intersect from circle (CDZ) equal arcs $\overset{\frown}{C'I} = \overset{\frown}{DC}$. Hence the angles $\widehat{C'CI} = \widehat{DZC}$ and since $\widehat{ACW} = \widehat{DZC}$ point W is on CI as claimed.

It follows that the quadrangle $ZBWI$ is cyclic, since $\widehat{ZIW} = \widehat{ZIC} = \widehat{CDC'} = \widehat{WBX}$. Hence we have the equal angles $\widehat{BZW} = \widehat{BIW} = \widehat{DZC}$. Also $\widehat{ZBW} = \widehat{ZDC}$ since these have equal supplementary angles. □

Remark 1. The similarity of triangles $\{ZDC, ZBW\}$ can be used to find quickly the image $W = F(Z)$ for a given Z once we know the characteristic parallelogram ([12, p.182]).

Corollary 4. *The angles of two lines $\{\varepsilon_1, \varepsilon_2\}$ through D and their $F-$images $\{\varepsilon'_1, \varepsilon'_2\}$ through the point A are equal in measure and opposite in orientation.*

Proof. The equality of angles follows immediately from corollary 1, since two corresponding lines $\{\varepsilon, \varepsilon'\}$ are lateral sides of an isosceles triangle AJA' (see Figure 5). □

Corollary 2 proves half of a well known proposition usually coupled with the appearance of an equilateral hyperbola ([6, p.156]). Next theorem supplies the other half defining this hyperbola.

Theorem 4. *The lines* $\{\varepsilon = DZ\,,\ \varepsilon' = AW = F(\varepsilon)\}$ *corresponding under the transformation* F *intersect at a point* J*, which, as* ε *rotates about* D*, describes a rectangular hyperbola passing through the points* $\{A, B, C, D\}$*. The center of the hyperbola is the middle* M *of* BC *and its asymptotes are parallel to the bisectors of the angle* $\widehat{BAC}$ *(see Figure 7).*

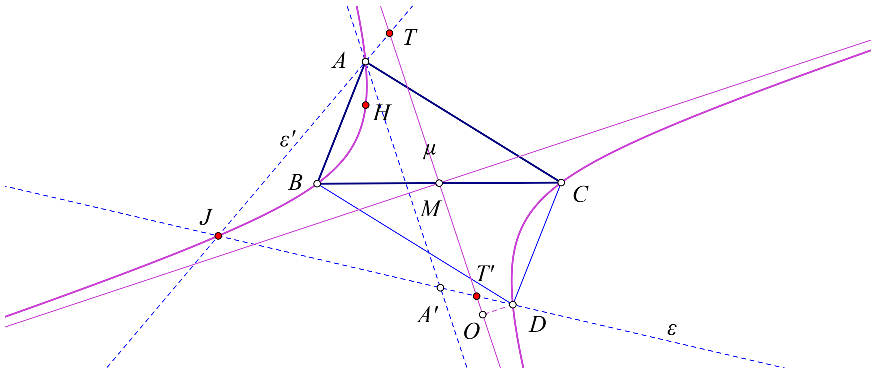

Figure 7: The rectangular hyperbola, locus of intersections $\{J = \varepsilon \cap \varepsilon'\}$

Proof. In fact, consider the line μ, which is parallel to the bisector AA' and passes through M and a line coordinate with origin O, the projection of D on it. We see easily that the parameters $\{T(t') = \mu \cap \varepsilon\,,\ T(t) = \mu \cap \varepsilon'\}$ of the lines $\{\varepsilon, \varepsilon'\}$ satisfy a linear relation $t' = at + b$. Thus, applying the Chasles-Steiner theorem ([5, p.77]), we see that their intersection J describes a conic. It is then trivially seen that the point J obtains the positions of the vertices of the parallelogram $ABDC$ when the lines $\{\varepsilon, \varepsilon'\}$ obtain the positions of the sides or diagonals of the parallelogram. The symmetry of the parallelogram about M shows that this is the center of the conic. Finally, when the positions of $\{\varepsilon, \varepsilon'\}$ become parallel to the bisectors of $\widehat{A}$, the corresponding point J goes to infinity, showing that these are the directions of the asymptotes of the conic. □

3 The images of lines and circles

The preceding discussion suggests another way to obtain $W = F(Z)$ from Z. A point $Z \neq D$ determines the line $\varepsilon = DZ$ and the equally inclined to the bisector δ of $\widehat{A}$ line

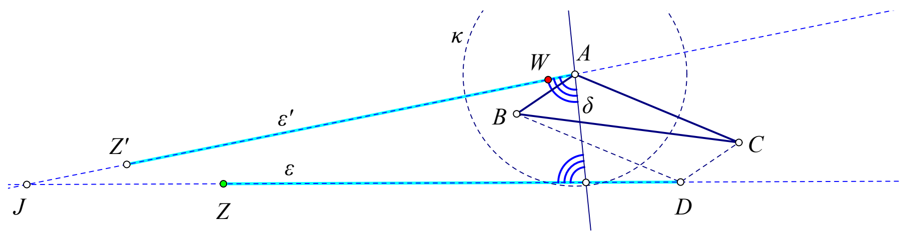

Figure 8: Another way to determine $W = F(Z)$

$\varepsilon' = F(\varepsilon)$ through A intersecting ε at J (see Figure 8). Transfer DZ to an equal segment AZ' on ε' and take the inverse W of Z' w.r.t. the circle $\kappa(A, r)$ with $r = \sqrt{AB \cdot AC}$. By corollary 2 we have $W = F(Z)$. Next theorem shows that the first step in this construction defining the map $Z' = H(Z)$ is an isometry and leads to a decomposition of the Möbius transformation in simpler transformations.

Theorem 5. *With the notation adopted so far the map $Z' = H(Z)$ is an isometry of the plane. More precisely it is a glide-reflection with axis ζ the parallel to the external bisector of $\widehat{A}$ through the middle of BC and translation DA_D, parallel to this axis, where A_D the projection of D on the bisector δ of $\widehat{A}$ (see Figure 9).*

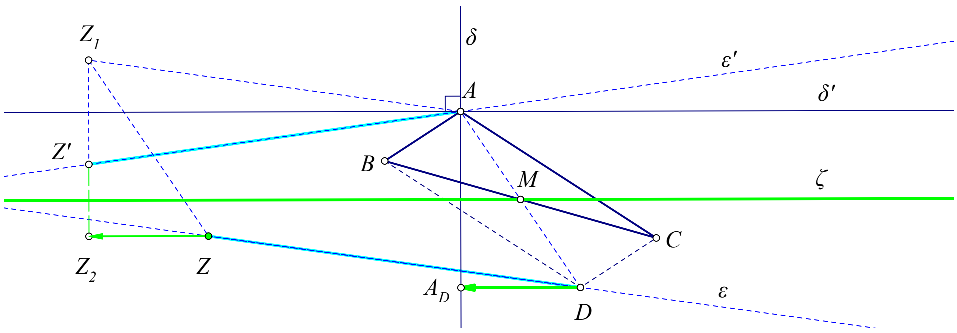

Figure 9: The map $Z' = H(Z)$ is a glide-reflection

Proof. By figure. Point Z' is found by translating Z parallel to DA in Z_1 and then reflecting this on the exterior bisector δ' of $\widehat{A}$. Such a composition of isometries: translation followed by a reflection, defines always a *"glide reflection"*, whose typical definition is ([4, p.43]): *"a composition of a reflection and a translation parallel to the reflection-axis"*. In the figure the translation vector DA is decomposed in components: parallel and orthogonal to δ'. The translation by the component orthogonal to δ' and the reflection in δ' compose to a reflection in ζ and the whole composition reduces to the translation T by DA_D and reflection R_ζ in $\zeta : H = R_\zeta \circ T = T \circ R_\zeta$. □

Corollary 5. *Every Möbius transformation is a composition $F = I_A \circ H$ of a glide reflection H and an inversion I_A, both components determined uniquely from the characteristic parallelogram of the transformation.*

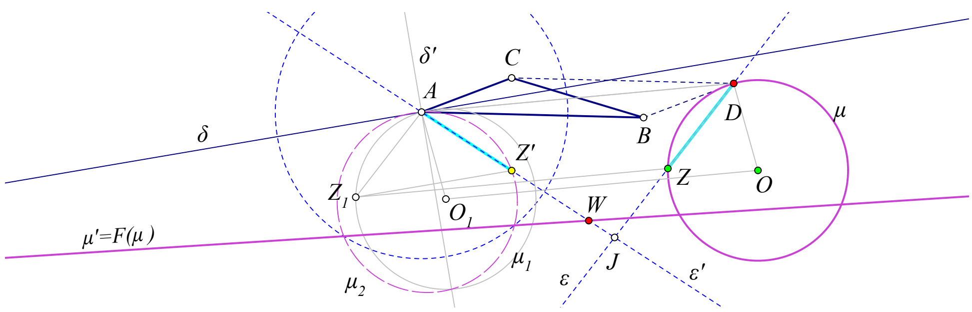

Figure 10: The image $\mu' = F(\mu)$ of a circle μ passing through D

Since isometries and inversions preserve both angles and the set Σ of lines+circles, we see that generic Möbius transformations preserve also angles and the set Σ. By the preceding discussion we saw that lines through D map to lines through A. On the ground of this decomposition we see also easily that circles through D map to lines through A, circles not containing D map to circles not containing A and lines not containing D map to circles through A. Figure 10 shows characteristically the three steps of construction of the image $\mu' = F(\mu)$ of a circle passing through D. The circle μ_1 is the translation of μ by DA. The circle μ_2 is the reflection of μ_1 in δ' and also the image of μ under the aforementioned glide reflection H. The line $\mu' = F(\mu)$ results by applying the inversion I_A to μ_2.

From these remarks results the following geometric characterization of the pole $D(z_i)$ of a generic Möbius transformation ([6, p.146]).

Corollary 6. *The pencil D^* of lines through the pole $D(z_i)$ is the only one transformed by the generic Möbius transformation to a pencil of lines. The transformed pencil is A^*, centered at the pole $A(w_i)$ of the inverse Möbius transformation.*

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